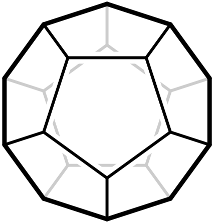
Some modern open problems on convex polytopes with symmetries and regularities

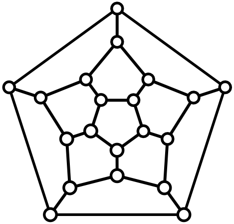
Martin Winter

01. December, 2021

Convex polytopes

$$P := \mathrm{conv}\{v_1, ..., v_n\} \subset \mathbb{R}^d$$





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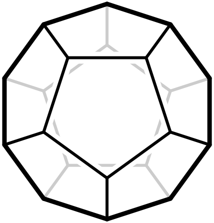
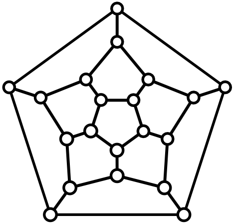
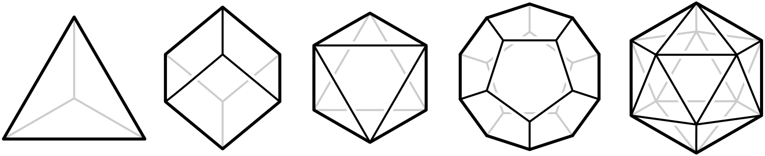




Keywords: vertices, edges, faces, facets, edge-graph, ...

Polytopes with symmetries and regularities



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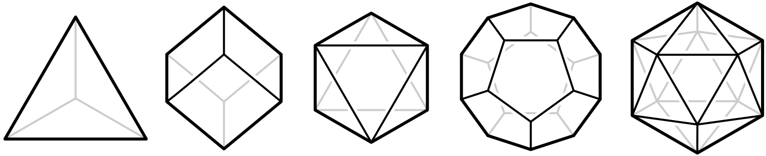

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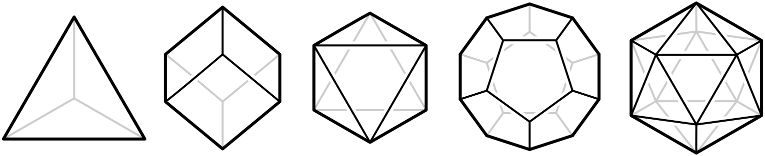

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"We are lacking examples." – Günter Ziegler

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Symmetry: an affine (or Euclidean) transformation of the ambient space that fixes the polytope set-wise.

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- inscribed (all vertices on a sphere)
- all edges of the same length
- all facets have the same shape
- simple (the edge-graph is regular of minimal degree)
- having a bipartite edge-graph
- ...

I. Symmetries of Orbit Polytopes

Orbit polytopes

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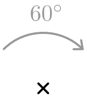

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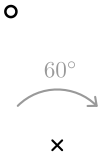

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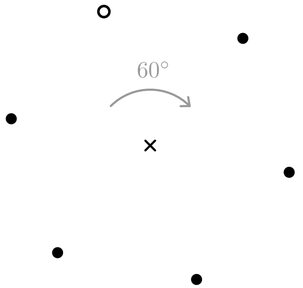

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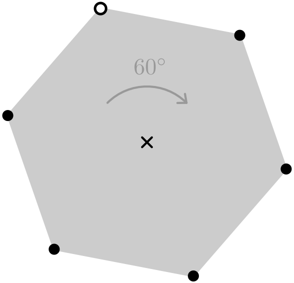

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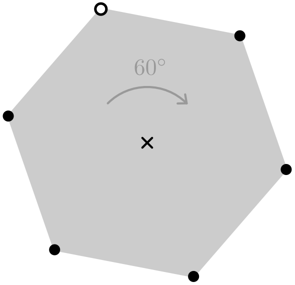

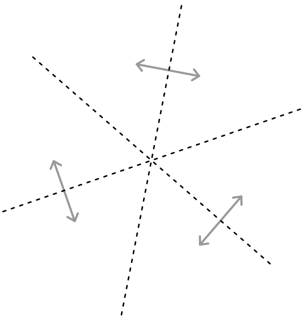

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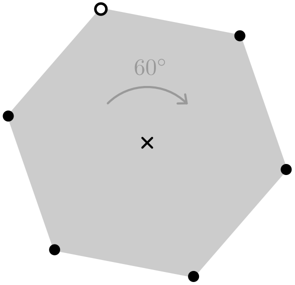

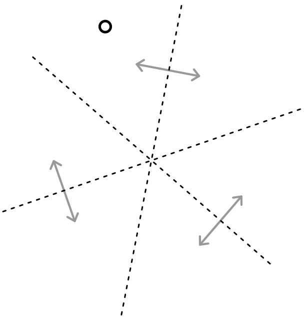

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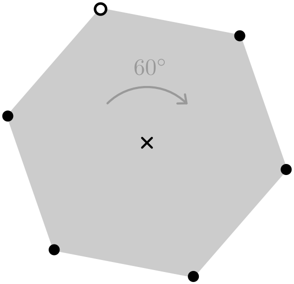

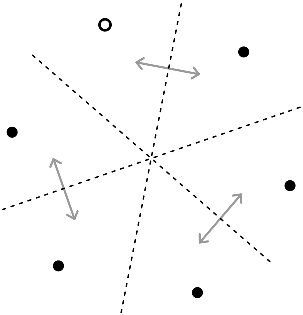

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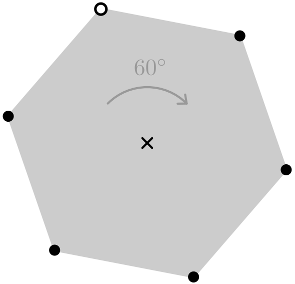

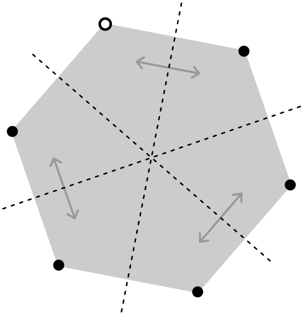

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(= vertex-transitive polytopes = isogonal polytopes)

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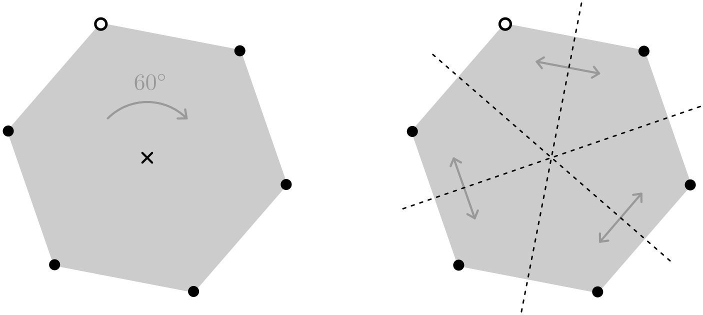

Orbit polytopes (= vertex-transitive polytopes = isogonal polytopes)

Definition.

Let $G \subset \mathrm{GL}(\mathbb{R}^d)$ be a matrix group and $x \in \mathbb{R}^d$ a point. The polytope

$$P(G, x) := \mathrm{conv}(Gx) = \mathrm{conv}\{gx \mid g \in G\}$$

is called the **orbit polytope** of G with **generator** x.

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$$G \;\mapsto\; \{x \mapsto P(G, x)\} \qquad \leftarrow \text{ orbit stratum of } G$$

Symmetries of orbit polytopes

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Related questions.

- What are the possible symmetry groups of vertex-transitive polytopes? (Babai)

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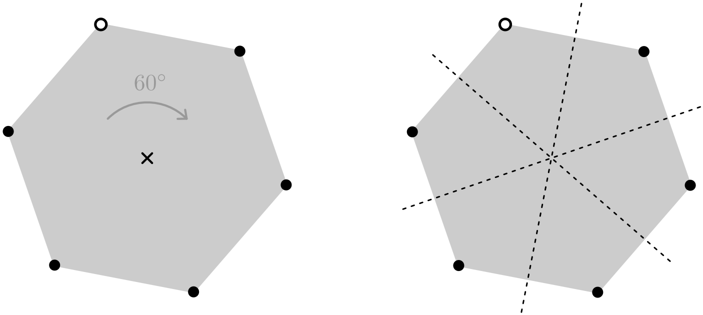
Related questions.

- What are the possible symmetry groups of vertex-transitive polytopes? (Babai)
- What are the possible symmetry groups of vertex-transitive graphs? (Frucht; Hetzel and Godsil)

Generically closed groups



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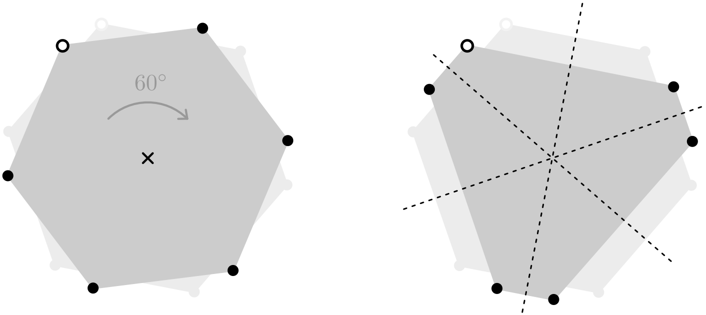

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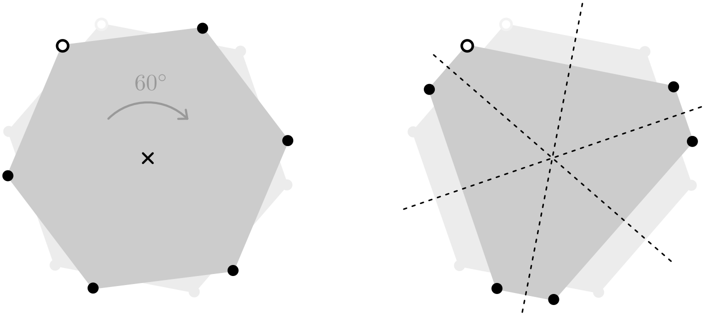

Definition.

A group $G \subset \mathrm{GL}(\mathbb{R}^d)$ is **generically closed** if there is a point $x \in \mathbb{R}^d$ with

$$\mathrm{Aut}_{\mathrm{GL}}(P(G,x)) = G.$$

When is a group generically closed?

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ideal part $\chi_I(G)$ ↓; non-ideal part $\chi_N(G)$ ↓

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Theorem. (Friese, Ladisch; 2017)

If the non-ideal part $\chi_N(G)$ of G is faithful, then G is generically closed.

When is a group generically closed?

ideal part — non-ideal part

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$$\uparrow$$

$$\chi_N(g) = \chi_N(\mathrm{Id}) \implies g = \mathrm{Id}$$

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Absolutely irreducible groups: $d \geq 2$

$$\chi(G) = 1\chi_1 = 0 + \chi_N(G).$$

Abstract symmetry groups of orbit polytopes

Theorem. (FRIEDER, LADISCH, 2017)

A finite group G is not isomorphic to the affine symmetry group of an orbit polytope, if and only if it is one of the following holds:

(i) *G is abelian of exponent greater than 2.*

(ii) *G is generalized dicyclic.*

(iii) *G is isomorphic to $\mathbb{Z}_2^r$ for $r \in \{2, 3, 4\}$.*

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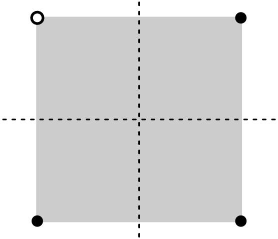

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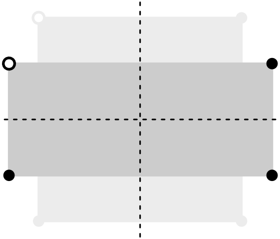
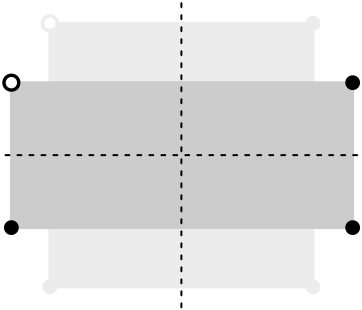


What about Euclidean symmetries?

$G \subset \mathrm{O}(\mathbb{R}^d)$, when is $\mathrm{Aut}_{\mathrm{O}}\big(P(G,x)\big) = G$?



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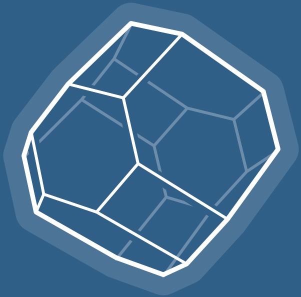
Open problem: what are the possible Euclidean symmetry groups of orbit polytopes?

II. Inscribed Zonotopes



Zonotopes

Definition.

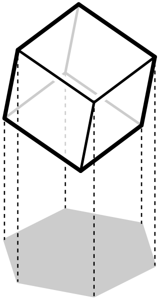
A **zonotope** $Z \subset \mathbb{R}^d$ is a polytope that satisfies any of the following equivalent conditions:

(i) Z is the projection of a cube.

(ii) Z is the Minkowski sum of line segments.

(iii) all faces of Z are centrally symmetric.

(iv) all 2-dimensional faces of Z are centrally symmetric.



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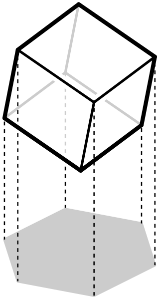

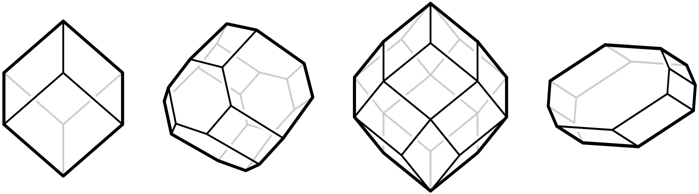

Zonotopes that are orbit polytopes

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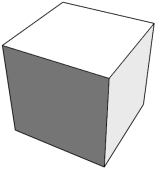

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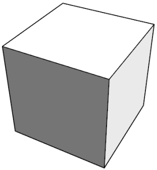

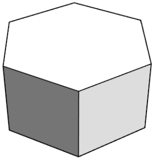

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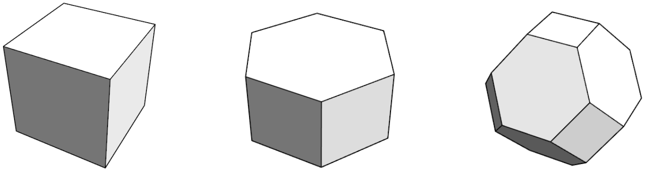

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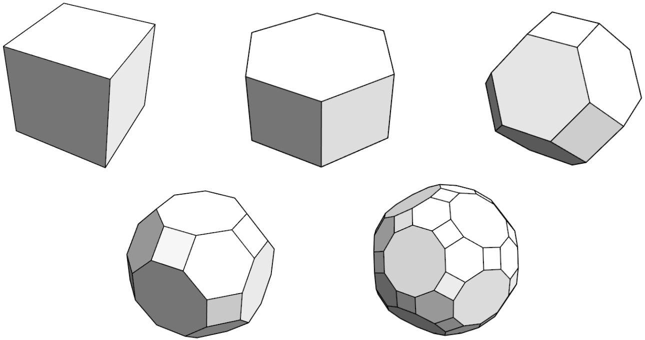

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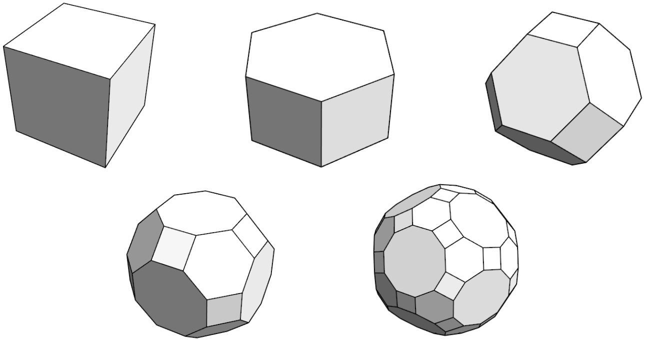

Definition.

A G-**permutahedron** is a generic orbit polytope of a reflection group $G \subset \mathrm{O}(\mathbb{R}^d)$.

$A_d, B_d, D_d; I_2(p), H_3, H_4, F_4, E_6, E_7, E_8$

Zonotopes that are orbit polytopes

Theorem. (W., 2021)

If $Z \subset \mathbb{R}^d$ is a zonotope that is also an orbit polytope, then Z is a G-permutahedron.

Zonotopes that are inscribed

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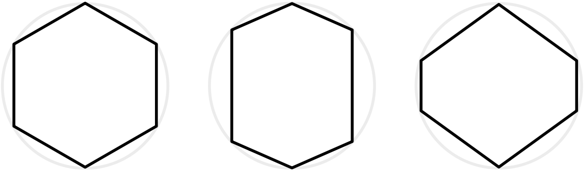

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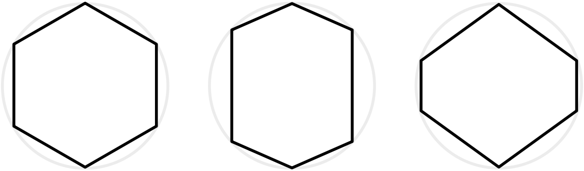
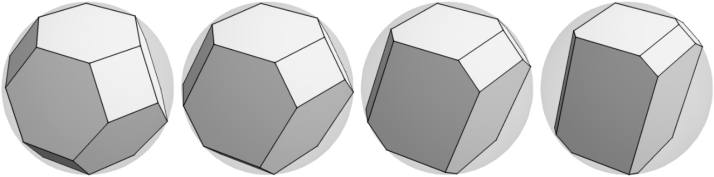


Theorem. (W., 2021)

If Z is inscribed and has all edges of the same length, then Z is a G-permutahedron.

Deforming the A_d-permutahedron



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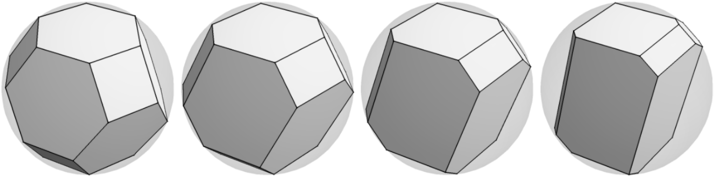

The A_d-permutahedron is generated by line-segments with directions

$$e_i - e_j \in \mathbf{1}^{\perp}, \quad \text{for distinct } i, j \in \{1, ..., n+1\}.$$

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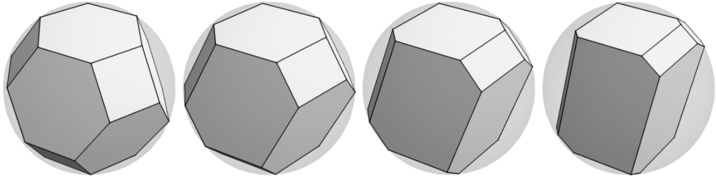


The A_d-permutahedron is generated by line-segments with directions

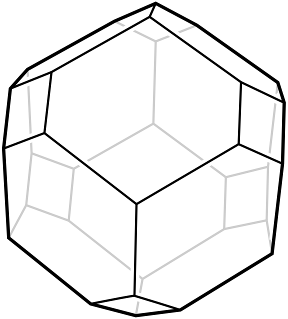
$$e_i - e_j \in \mathbf{1}^\perp, \quad \text{for distinct } i, j \in \{1, ..., n+1\}.$$

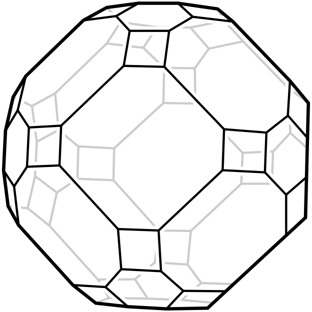
Choose parameters $\alpha_1, ..., \alpha_{n+1} \in \mathbb{R}_+$:

$$\alpha_i \alpha_j (\alpha_j e_i - \alpha_i e_j) \in (\alpha_1, ..., \alpha_{n+1})^\perp, \quad \text{for distinct } i, j \in \{1, ..., n+1\}.$$

Projections of permutahedra

Observation: the projection of an inscribed zonotope along an edge direction is again an inscribed zonotope.





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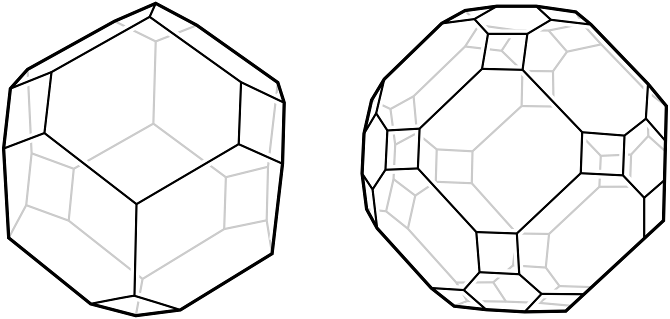
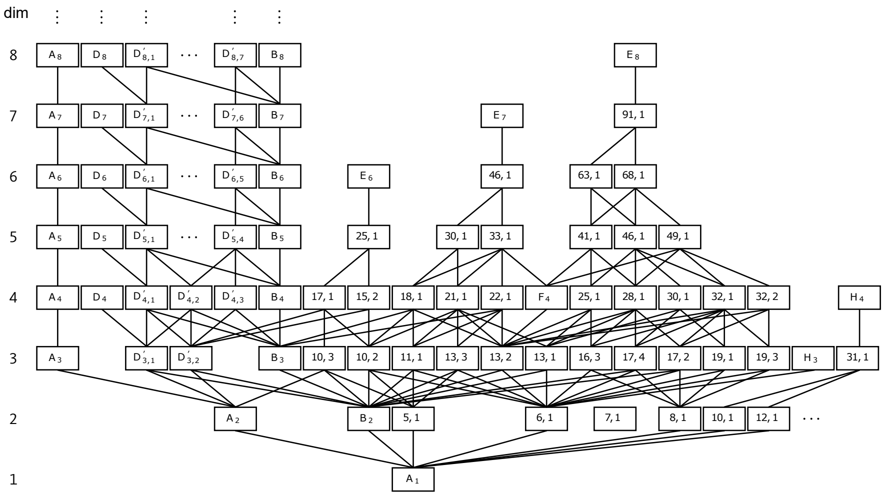


Open problem: are there any others?

Inscribed zonotopes

(by Michael Cuntz)



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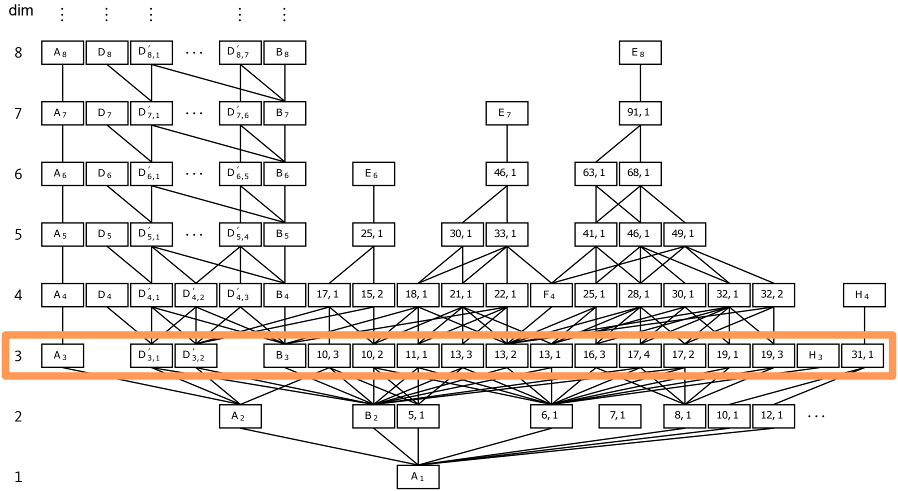
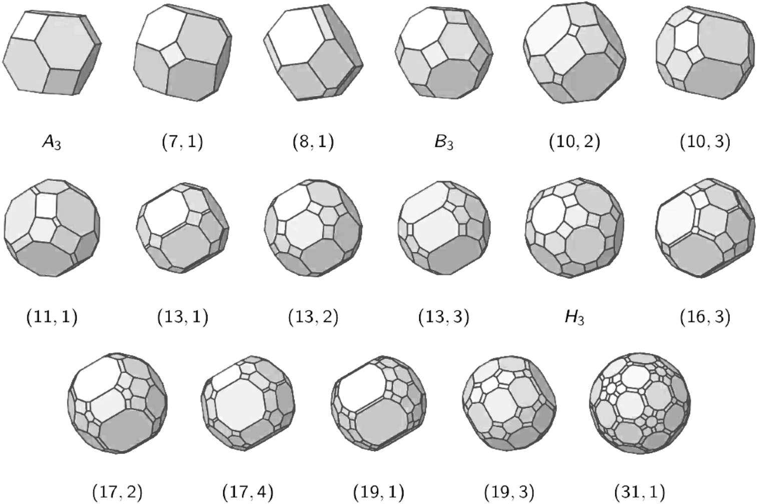


Inscribed zonohedra

(by Sebastian Manecke and Raman Sanyal)



Why is this interesting?

Why is this interesting? – Hyperplane arrangements

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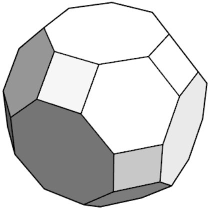

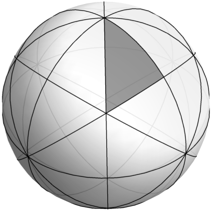

zonotopes $\Longleftrightarrow$	**hyperplane arrangements**

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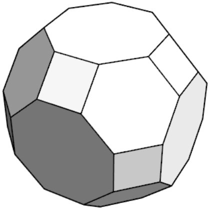
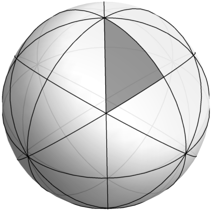
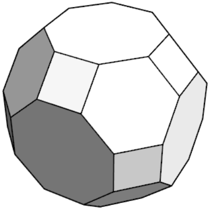
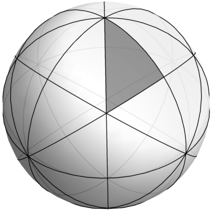




zonotopes	$\Longleftrightarrow$ **hyperplane arrangements**
vertex-transitive	

Why is this interesting? – Hyperplane arrangements





zonotopes $\Longleftrightarrow$	**hyperplane arrangements**
vertex-transitive $\cap$ inscribed	

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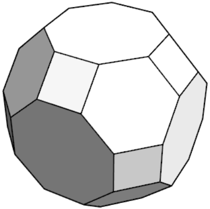

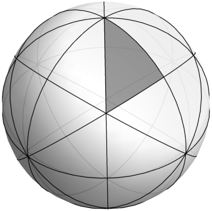

zonotopes $\Longleftrightarrow$	**hyperplane arrangements**
vertex-transitive $\cap$ inscribed $\cap$ simple	

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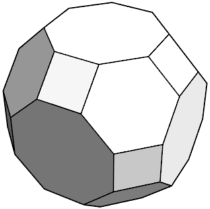

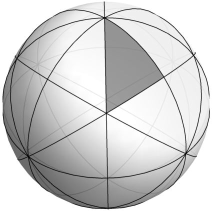

zonotopes $\Longleftrightarrow$	**hyperplane arrangements**
vertex-transitive	
$\cap$	
inscribed	
$\cap$	
simple	simplicial 3 families + 95 exceptions

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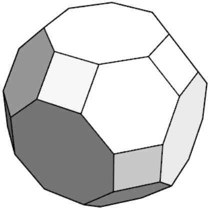

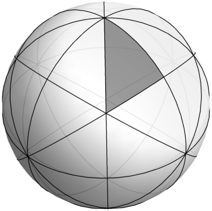

zonotopes	$\Longleftrightarrow$ **hyperplane arrangements**
vertex-transitive	
$\cap$	
inscribed	inscribable
$\cap$	$\cap$
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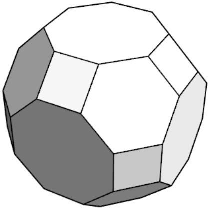
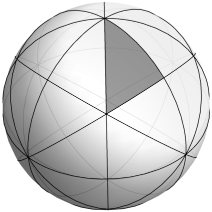


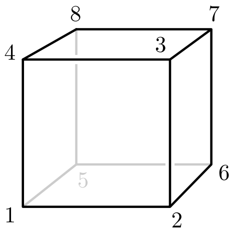


zonotopes $\Longleftrightarrow$	**hyperplane arrangements**
vertex-transitive	chamber-transitive
$\cap$	$\cap$
inscribed	inscribable
$\cap$	$\cap$
simple	simplicial
	3 families + 95 exceptions

III. Spectral Polytopes

A curious observation ...

$P \subset \mathbb{R}^3$



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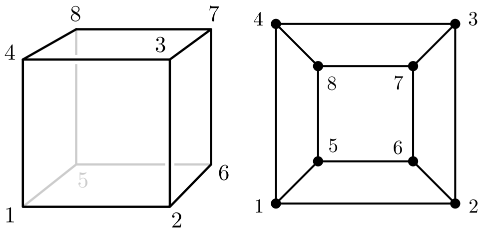
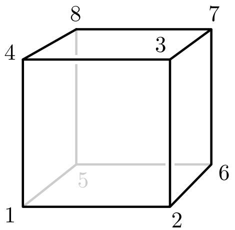
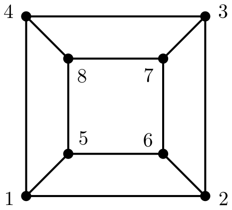
$$P \subset \mathbb{R}^3 \quad \longrightarrow \quad G_P = (V, E)$$



A curious observation ...

$$P \subset \mathbb{R}^3 \quad \longrightarrow \quad G_P = (V, E) \quad \longrightarrow \quad A(G_P) \in \mathbb{R}^{8\times 8}$$





	1	2	3	4	5	6	7	8
1	0	1	0	1	1	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	0	1	0
4	1	0	1	0	0	0	0	1
5	1	0	0	0	0	1	0	1
6	0	1	0	0	1	0	1	0
7	0	0	1	0	0	1	0	1
8	0	0	0	1	1	0	1	0

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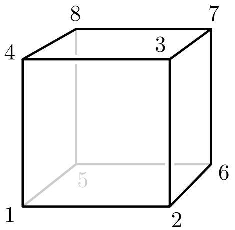

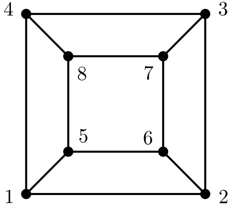

	1	2	3	4	5	6	7	8
1	0	1	0	1	1	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	0	1	0
4	1	0	1	0	0	0	0	1
5	1	0	0	0	0	1	0	1
6	0	1	0	0	1	0	1	0
7	0	0	1	0	0	1	0	1
8	0	0	0	1	1	0	1	0

$$\downarrow$$

$$\{\, 3^1, 1^3, (-1)^3, (-3)^1 \,\}$$

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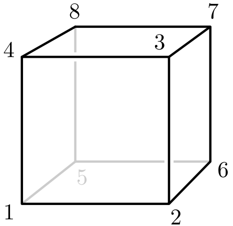

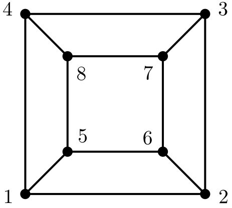

	1	2	3	4	5	6	7	8
1	0	1	0	1	1	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	0	1	0
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$$\downarrow$$

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$$\theta_1 > \theta_2 > \cdots > \theta_m$$

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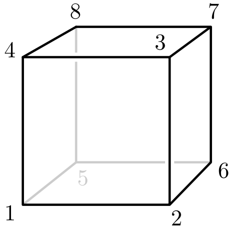
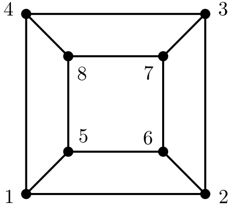




	1	2	3	4	5	6	7	8
1	0	1	0	1	1	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	0	1	0
4	1	0	1	0	0	0	0	1
5	1	0	0	0	0	1	0	1
6	0	1	0	0	1	0	1	0
7	0	0	1	0	0	1	0	1
8	0	0	0	1	1	0	1	0

$$\downarrow$$

$$\{\, 3^1, 1^3, (-1)^3, (-3)^1 \,\}$$

$$\theta_1 > \theta_2 > \cdots > \theta_m$$

A curious observation ...

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} u_1 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} u_2 \\ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} u_3 \\ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \end{array} \in \mathbb{R}^8$$

A curious observation ...

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \overset{u_1}{\begin{bmatrix} 1\\1\\1\\1\\-1\\-1\\-1\\-1 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1\\1\\-1\\-1\\1\\1\\-1\\-1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1\\-1\\-1\\1\\1\\-1\\-1\\1 \end{bmatrix}} \in \mathbb{R}^8 \quad \longrightarrow \quad \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\\ -1 & 1 & -1\\ -1 & -1 & -1\\ -1 & -1 & 1 \end{bmatrix} \end{array} \in \mathbb{R}^{8\times 3}$$

A curious observation ...

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \overset{u_1}{\begin{bmatrix} 1\\1\\1\\1\\-1\\-1\\-1\\-1 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1\\1\\-1\\-1\\1\\1\\-1\\-1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1\\-1\\-1\\1\\1\\-1\\-1\\1 \end{bmatrix}} \in \mathbb{R}^8 \quad \longrightarrow \quad \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \in \mathbb{R}^{8\times 3}$$

A curious observation ...

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{array}{c} u_1 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \end{array},
\begin{array}{c} u_2 \\ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \end{array},
\begin{array}{c} u_3 \\ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \end{array}
\in \mathbb{R}^8 \quad \longrightarrow \quad
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1
\end{bmatrix} \end{array}
\in \mathbb{R}^{8\times 3}
$$

$$
v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3
$$

A curious observation ...

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \overset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \in \mathbb{R}^8 \longrightarrow \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \overset{\begin{matrix} u_1 & u_2 & u_3 \end{matrix}}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}} \in \mathbb{R}^{8\times 3}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

A curious observation ...

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\begin{array}{c} u_1 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} u_2 \\ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} u_3 \\ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \end{array} \in \mathbb{R}^8 \quad \longrightarrow \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \in \mathbb{R}^{8\times 3}$$

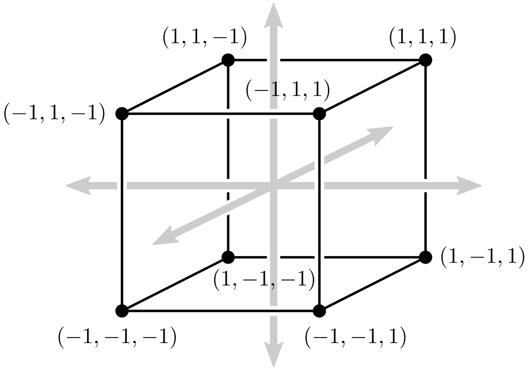
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \ldots$$

A curious observation ...

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \overset{u_1}{\begin{bmatrix} 1\\1\\1\\1\\-1\\-1\\-1\\-1 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1\\1\\-1\\-1\\1\\1\\-1\\-1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 1\\-1\\-1\\1\\1\\-1\\-1\\1 \end{bmatrix}} \in \mathbb{R}^8 \longrightarrow \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \overset{\quad u_1 \quad u_2 \quad u_3}{\begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\\ -1 & 1 & -1\\ -1 & -1 & -1\\ -1 & -1 & 1 \end{bmatrix}} \in \mathbb{R}^{8\times 3}$$

$$v_1 = \begin{bmatrix} 1\\1\\1 \end{bmatrix}, v_2 = \begin{bmatrix} 1\\1\\-1 \end{bmatrix}, v_3 = \begin{bmatrix} 1\\-1\\-1 \end{bmatrix}, \ldots \implies \boxed{V \ni i \;\mapsto\; v_i \in \mathbb{R}^3}$$

A curious observation ...



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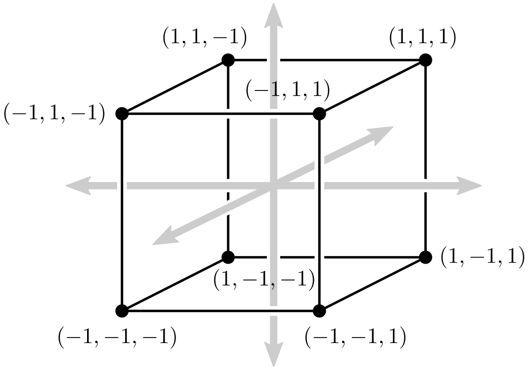

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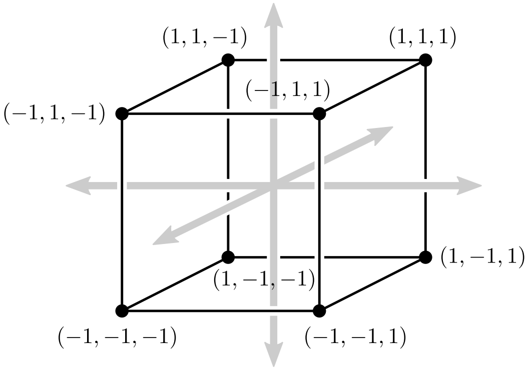

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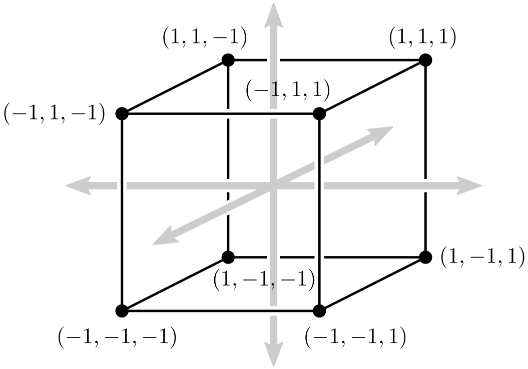

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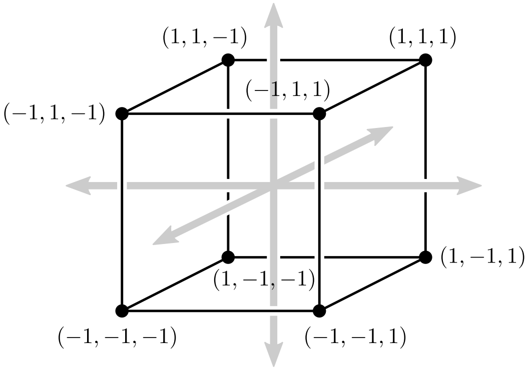

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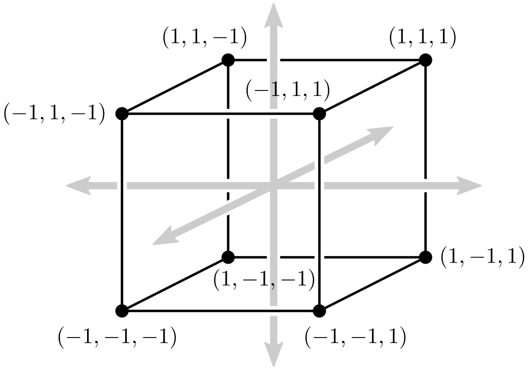

- For which polytopes does it work? ← **spectral polytopes/graphs**
- Why have we used θ_2? ← probably **nodal domains**
- Why is this interesting?

Spectral polytopes/graphs

Observation: If $P \subset \mathbb{R}^d$ is spectral, then ...

- P can be reconstructed from its edge-graph.
- P is exactly as symmetric as its edge-graph.

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We want large eigenspaces without much symmetry $\implies$ *distance-regular graphs*

Distance-regular spectral graphs

Theorem. (Godsil, 1997)

If G is distance-regular and spectral, then it is one of the following:

(i) *a cycle graph,*

(ii) *the edge-graph of the icosahedron,*

(iii) *the edge-graph of the dodecahedron,*

(iv) *a complete graph minus a matching,*

(v) *a Jonson graph $J(n,k)$,*

(vi) *a halved cube graph $^1/_2 Q_n$,*

(vii) *a Hamming graph $H(d,q)$,*

(viii) *the Schläfli graph,*

(ix) *the Gosset graph.*

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} *distance-transitive*

Distance-transitive polytopes

Theorem. (W., 2020)

If $P \subset \mathbb{R}^d$ is distance-transitive, then P is one of the following:

(i) *a regular polygon,*
(ii) *the icosahedron,*
(iii) *the dodecahedron,*
(iv) *a crosspolytope,*
(v) *a hyper-simplex (this includes regular simplices),*
(vi) *a demi-cube,*
(vii) *a cartesian power of a simplex (this includes hypercubes),*
(viii) *the 6-dimensional 2_{21}-polytope,*
(ix) *the 7-dimensional 3_{21}-polytope.*

Edge-transitive polytopes

Theorem. (W., 2020)

If $P \subset \mathbb{R}^d$ is edge-transitive with $d \geq 4$, then

(i) P *is spectral.*

(ii) P *is uniquely determined by its edge-graph up to scale and orientation.*

(iii) P *is as symmetric as its edge-graph.*

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Claim: *spectral graph theory should be a tool in polytope theory.*

Open questions: can we classify edge-transitive polytopes via spectral graph theory?

Thank you.