Some modern open problems on convex polytopes with symmetries and regularities

Martin Winter

01. December, 2021
Convex polytopes

\[ P := \text{conv}\{v_1, \ldots, v_n\} \subset \mathbb{R}^d \]
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**Keywords:** vertices, edges, faces, facets, edge-graph, ...
Polytopes with symmetries and regularities
“[...] wayside shrines at which one should worship on the way to higher things.” – Peter McMullen
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“We are lacking examples.” – Günter Ziegler
Symmetries and regularities

**Symmetry:** an affine (or Euclidean) transformation of the ambient space that fixes the polytope set-wise.
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\text{Aut}_{\text{GL}}(P) := \{ X \in \text{GL}(\mathbb{R}^d) \mid XP = P \} \quad \text{(linear } \cong \text{ affine)}
\]
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\text{Aut}_O(P) := \{ X \in O(\mathbb{R}^d) \mid XP = P \} \quad \text{(orthogonal } \cong \text{ Euclidean)}
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**Regularity:**

- inscribed (all vertices on a sphere)
- all edges of the same length
- all facets have the same shape
- simple (the edge-graph is regular of minimal degree)
- having a bipartite edge-graph
- ...
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I. Symmetries of Orbit Polytopes
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Orbit polytopes

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I. Symmetries of Orbit Polytopes

**Orbit polytopes** ( = vertex-transitive polytopes = isogonal polytopes )

**Definition.**

Let $G \subset \text{GL}(\mathbb{R}^d)$ be a matrix group and $x \in \mathbb{R}^d$ a point. The polytope

$$P(G, x) := \text{conv}(Gx) = \text{conv}\{gx \mid g \in G\}$$

is called the **orbit polytope** of $G$ with **generator** $x$. 
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**Claim:** orbit polytopes should be a tool and an object of study in the field of representation theory.
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**Claim:** *orbit polytopes should be a tool and an object of study in the field of representation theory.*

\[
G \mapsto \{x \mapsto P(G, x)\} \quad \leftarrow \text{orbit stratum of } G
\]
Symmetries of orbit polytopes

**Observation:** all $g \in G$ are symmetries of $P(G, x)$, that is

$$G \subset \text{Aut}_{GL}(P(G, x)).$$
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When do we have equality?
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Related questions.

- What are the possible symmetry groups of vertex-transitive polytopes? (Babai)
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**Related questions.**

- What are the possible symmetry groups of vertex-transitive polytopes? *(Babai)*
- What are the possible symmetry groups of vertex-transitive graphs? *(Frucht; Hetzel and Godsil)*
I. Symmetries of Orbit Polytopes

Definition.

A group $G \subset GL(\mathbb{R}^d)$ is generically closed if there is a point $x \in \mathbb{R}^d$ with $Aut GL(P(G,x)) = G$.
Generically closed groups

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[Diagram showing symmetries of orbit polytopes with 60° rotations and 8-fold symmetry]
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When is a group generically closed?

\[ \chi(G) \]
When is a group generically closed?

\[ g \mapsto \text{tr}(g) \]

\[ \chi(G) \]

**Theorem.** (Friese, Ladisch; 2017) If the non-ideal part \( \chi_N(G) \) of \( G \) is faithful, then \( G \) is generically closed. 

\[ \chi_N(g) = \chi_N(\text{Id}) \Rightarrow g = \text{Id} \]
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\[ \chi(G) = m_1 \chi_1 + \cdots + m_r \chi_r \]
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\[ \chi(G) = m_1 \chi_1 + \cdots + m_r \chi_r = \sum_{i=m_i=\dim \chi_i} m_i \chi_i + \sum_{i=m_i<\dim \chi_i} m_i \chi_i \]

\[ \leq \dim \chi_1 \leq \dim \chi_r \]
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\[
\begin{align*}
g \mapsto \text{tr}(g) \\
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\[ \chi_I(G) \downarrow \]

\[ \chi_N(G) \downarrow \]

\[ \leq \text{dim } \chi_1 \]

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Abelian groups: $d \geq 2$

$$\chi(G) = 1\chi_1 + \cdots + 1\chi_r = \chi_I(G) + 0.$$
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**Absolutely irreducible groups:** $d \geq 2$

$$\chi(G) = 1\chi_1 = 0 + \chi_N(G).$$
I. Symmetries of Orbit Polytopes

Abstract symmetry groups of orbit polytopes

Theorem. (FRIEDER, LADISCH, 2017)

A finite group $G$ is not isomorphic to the affine symmetry group of an orbit polytope, if and only if it is one of the following holds:

(i) $G$ is abelian of exponent greater than 2.
(ii) $G$ is generalized dicyclic.
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What about Euclidean symmetries?

\[ G \subset O(\mathbb{R}^d), \quad \text{when is } \text{Aut}_O(P(G, x)) = G? \]
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What about Euclidean symmetries?

**Theorem.** (Babai, 1977)

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Open problem: what are the possible Euclidean symmetry groups of orbit polytopes?
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II. Inscribed Zonotopes
A zonotope $Z \subset \mathbb{R}^d$ is a polytope that satisfies any of the following equivalent conditions:

(i) $Z$ is the projection of a cube.

(ii) $Z$ is the Minkowski sum of line segments.

(iii) all faces of $Z$ are centrally symmetric.

(iv) all 2-dimensional faces of $Z$ are centrally symmetric.
Zonotopes

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II. Inscribed Zonotopes

Definition.

A $G$-permutahedron is a generic orbit polytope of a reflection group $G \subset O(R^d)$.

$A_d, B_d, D_d; I_2(p), H_3, H_4, F_4, E_6, E_7, E_8

Martin Winter 14 / 27
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Zonotopes that are orbit polytopes

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II. Inscribed Zonotopes

Zonotopes that are orbit polytopes

**Theorem.** (W., 2021)

If $Z \subset \mathbb{R}^d$ is a zonotope that is also an orbit polytope, then $Z$ is a $G$-permutahedron.

Let’s relax the symmetry restrictions — inscribed zonotopes.
II. Inscribed Zonotopes

Zonotopes that are inscribed

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Let’s relax the symmetry restrictions → **inscribed zonotopes**.

**Theorem.** (W., 2021)

If $Z$ is inscribed and has all edges of the same length, then $Z$ is a $G$-permutahedron.
Deforming the $A_d$-permutahedron
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The $A_d$-permutahedron is generated by line-segments with directions

$$e_i - e_j \in \mathbf{1}^\perp, \quad \text{for distinct } i, j \in \{1, \ldots, n + 1\}.$$
Deforming the $A_d$-permutahedron

The $A_d$-permutahedron is generated by line-segments with directions

$$e_i - e_j \in 1^\perp, \quad \text{for distinct } i, j \in \{1, \ldots, n + 1\}.$$

Choose parameters $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{R}_+$:

$$\alpha_i \alpha_j (\alpha_j e_i - \alpha_i e_j) \in (\alpha_1, \ldots, \alpha_{n+1})^\perp, \quad \text{for distinct } i, j \in \{1, \ldots, n + 1\}.$$
Observation: the projection of an inscribed zonotope along an edge direction is again an inscribed zonotope.
Projections of permutahedra

**Observation:** the projection of an inscribed zonotope along an edge direction is again an inscribed zonotope.

**Open problem:** are there any others?
II. Inscribed Zonotopes

Inscribed zonotopes

(by Michael Cuntz)
II. Inscribed Zonotopes

Inscribed zonotopes

(by Michael Cuntz)
II. Inscribed Zonotopes

Inscribed zonohedra (by Sebastian Manecke and Raman Sanyal)

$A_3$ (7, 1) (8, 1) $B_3$ (10, 2) (10, 3)

(11, 1) (13, 1) (13, 2) (13, 3) $H_3$ (16, 3)

(17, 2) (17, 4) (19, 1) (19, 3) (31, 1)
II. Inscribed Zonotopes

Why is this interesting?
Why is this interesting? – Hyperplane arrangements
Why is this interesting? – Hyperplane arrangements

II. Inscribed Zonotopes

- Hyperplane arrangements
  ⇐⇒
- zonotopes
  ⇐⇒
- hyperplane arrangements

3 families + 95 exceptions
Why is this interesting? – Hyperplane arrangements

zonotopes \iff \text{hyperplane arrangements}

\text{vertex-transitive}
II. Inscribed Zonotopes

Why is this interesting? – Hyperplane arrangements

zonotopes \iff \text{hyperplane arrangements}

vertex-transitive \cap \text{inscribed}
Why is this interesting? – Hyperplane arrangements

**zonotopes**

- vertex-transitive
- inscribed
- simple

**hyperplane arrangements**

II. Inscribed Zonotopes
II. Inscribed Zonotopes

Why is this interesting? – Hyperplane arrangements

zonotopes $\iff$ hyperplane arrangements

- vertex-transitive
- inscribed
- simple $\subset$ inscribable $\subset$ simple

simplicial
3 families + 95 exceptions
Why is this interesting? – Hyperplane arrangements

- Inscribed zonotopes
- Vertex-transitive
- Inscribed
- Simple

↔

- Hyperplane arrangements
- Inscribleable
- Simplicial

3 families + 95 exceptions
Why is this interesting? – Hyperplane arrangements

<table>
<thead>
<tr>
<th>zonotopes</th>
<th>hyperplane arrangements</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertex-transitive</td>
<td>chamber-transitive</td>
</tr>
<tr>
<td>∩ inscribed</td>
<td>∩ inscribable</td>
</tr>
<tr>
<td>∩ simple</td>
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</tr>
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</table>

3 families + 95 exceptions
III. Spectral Polytopes
A curious observation ...

\[ P \subset \mathbb{R}^3 \]
A curious observation ...

\[ P \subset \mathbb{R}^3 \longrightarrow G_P = (V, E) \]
A curious observation ...

\[ P \subset \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A(G_P) \in \mathbb{R}^{8 \times 8} \]
A curious observation ...

\[ P \subset \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A(G_P) \in \mathbb{R}^{8 \times 8} \]

\[
\begin{matrix}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
6 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
7 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
8 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{matrix}
\]

\[
\{ 3^1, 1^3, (-1)^3, (-3)^1 \} \]
A curious observation ...

\[ P \subset \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A(G_P) \in \mathbb{R}^{8 \times 8} \]

\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[ \{3^1, 1^3, (-1)^3, (-3)^1\} \]

\[ \theta_1 > \theta_2 > \cdots > \theta_m \]
A curious observation ...

\[ P \subset \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A(G_P) \in \mathbb{R}^{8 \times 8} \]
A curious observation ...

\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
-1 \\
-1 \\
-1 \\
-1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
1 \\
-1 \\
-1 \\
1 \\
1 \\
1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
-1 \\
-1 \\
-1 \\
1 \\
\end{bmatrix} \in \mathbb{R}^8
\]
III. Spectral Polytopes

A curious observation ...

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^8 \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^{8 \times 3}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
A curious observation ...

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1
\end{bmatrix} \in \mathbb{R}^8 \quad \rightarrow \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1
\end{bmatrix} \in \mathbb{R}^{8 \times 3}
\]
A curious observation ...

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
& \in \mathbb{R}^8 \quad \rightarrow \\
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
& \in \mathbb{R}^{8 \times 3}
\end{align*}
\]

\[
v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3
\]
A curious observation ...

\[
\begin{bmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}
\in \mathbb{R}^8 \rightarrow
\begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & 1 \\
3 & 1 & 1 \\
4 & 1 & 1 \\
5 & 1 & 1 \\
6 & 1 & 1 \\
7 & 1 & 1 \\
8 & 1 & 1 \\
\end{bmatrix}
\in \mathbb{R}^{8\times3}
\]

\[
v_1 = \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
1 \\
1 \\
-1 \\
\end{bmatrix}
\]
A curious observation ...

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1
\end{bmatrix}
\in \mathbb{R}^8 \rightarrow
\begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & -1 \\
3 & -1 & -1 \\
4 & 1 & 1 \\
5 & -1 & 1 \\
6 & -1 & -1 \\
7 & -1 & -1 \\
8 & -1 & -1
\end{bmatrix}
\in \mathbb{R}^{8\times3}
\]

\[
v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
\quad v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},
\quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, ...
\]
A curious observation ...

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1 \\
\end{bmatrix} \in \mathbb{R}^8 \rightarrow \begin{bmatrix}
v_1 \\
v_2 \\
v_3 \
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1 \\
\end{bmatrix} \in \mathbb{R}^{8 \times 3}
\]

\[
v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \ldots \quad \Rightarrow \quad V \ni i \mapsto v_i \in \mathbb{R}^3
\]
A curious observation ...
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For which polytopes does it work?
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For which polytopes does it work?  🔄 spectral polytopes/graphs
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- Why have we used $\theta_2$?
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- For which polytopes does it work? ← spectral polytopes/graphs
- Why have we used $\theta_2$? ← probably nodal domains
III. Spectral Polytopes

A curious observation ...

- For which polytopes does it work?  \( \text{spectral polytopes/graphs} \)
- Why have we used \( \theta_2 \)?  \( \text{probably nodal domains} \)
- Why is this interesting?
III. Spectral Polytopes

Spectral polytopes/graphs

**Observation:** If $P \subset \mathbb{R}^d$ is spectral, then ...

- $P$ can be reconstructed from its edge-graph.
- $P$ is exactly as symmetric as its edge-graph.

Open questions:

- how else can we characterize spectral polytopes/graphs? Can they be classified?
- are there spectral polytopes/graphs with few symmetries?

We want large eigenspaces without much symmetry $\Rightarrow$ distance-regular graphs
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We want large eigenspaces without much symmetry $\implies$ distance-regular graphs
### Theorem. (Godsil, 1997)

If $G$ is distance-regular and spectral, then it is one of the following:

1. a cycle graph,
2. the edge-graph of the icosahedron,
3. the edge-graph of the dodecahedron,
4. a complete graph minus a matching,
5. a Jonson graph $J(n, k)$,
6. a halved cube graph $\frac{1}{2}Q_n$,
7. a Hamming graph $H(d, q)$,
8. the Schlaffli graph,
9. the Gosset graph.
III. Spectral Polytopes

Distance-regular spectral graphs

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(i) a cycle graph,
(ii) the edge-graph of the icosahedron,
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(v) a Jonson graph $J(n, k)$, \hspace{1cm} \text{distance-transitive}
(vi) a halved cube graph $\frac{1}{2}Q_n$,
(vii) a Hamming graph $H(d, q)$,
(viii) the Schlӓfli graph,
(ix) the Gosset graph.
### Theorem. (W., 2020)

If $P \subset \mathbb{R}^d$ is distance-transitive, then $P$ is one of the following:

1. a regular polygon,
2. the icosahedron,
3. the dodecahedron,
4. a crosspolytope,
5. a hyper-simplex (this includes regular simplices),
6. a demi-cube,
7. a cartesian power of a simplex (this includes hypercubes),
8. the 6-dimensional $2_{21}$-polytope,
9. the 7-dimensional $3_{21}$-polytope.
### Theorem. (W., 2020)

If $P \subset \mathbb{R}^d$ is edge-transitive with $d \geq 4$, then

(i) $P$ is spectral.

(ii) $P$ is uniquely determined by its edge-graph up to scale and orientation.

(iii) $P$ is as symmetric as its edge-graph.

Claim: spectral graph theory should be a tool in polytope theory.

Open questions: can we classify edge-transitive polytopes via spectral graph theory?
Edge-transitive polytopes

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Thank you.