Capturing Polytopal Symmetries in the Edge-Graph

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Polytopes and geometric symmetries

**Polytope:** $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex polytope*),
- full-dimensional,
- vertices $v_1, \ldots, v_n \in \mathbb{R}^d$. 

**Symmetries:**

$\text{Aut}_{\text{GL}}(P) := \{T \in \text{GL}(\mathbb{R}^d) | TP = P\}$

$\text{Aut}_{\text{O}}(P) := \{T \in \text{O}(\mathbb{R}^d) | TP = P\}$
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**Edge-graphs and combinatorial symmetries**

**Edge-graph:** \( G_P = (V, E) \)
- finite simple graph with \( V = \{1, \ldots, n\} \),
- \( i \in V \) corresponds to vertex \( v_i \),
- \( ij \in E \iff \text{conv}\{v_i, v_j\} \) is an edge of \( P \).
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- $i \in V$ corresponds to vertex $v_i$,
- $ij \in E \iff \text{conv}\{v_i, v_j\}$ is an edge of $P$.

**Symmetries:**

\[
\text{Aut}(G_P) := \{ \sigma \in \text{Sym}(V) \mid ij \in E \iff \sigma(i)\sigma(j) \in E \}.
\]
Polytope symmetries yield graph symmetries ...
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\[ T = ? \]

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Polytope symmetries yield graph symmetries ...
Capturing symmetries via colors

There is a coloring \( c: V \cup E \to C \) of the edge-graph so that \( \text{Aut}(G_cP_c) \sim = \text{Aut}(GL(P)) \).

\[ \text{Aut}(G_cP_c) = \{ \sigma \in \text{Aut}(G_P) \mid c(i) = c(\sigma(i)) \text{ for all } i \in V \land c(ij) = c(\sigma(i)\sigma(j)) \text{ for all } ij \in E \} \]
Capturing symmetries via colors

There is a coloring $c : V \cup E \to C$ of the edge-graph so that $\text{Aut}(G^cP) \cong \text{Aut}(GL(P))$.

$$\text{Aut}(G^cP) := \{ \sigma \in \text{Aut}(G^P) \mid c(i) = c(\sigma(i)) \text{ for all } i \in V^{cc}(ij) = c(\sigma(i)\sigma(j)) \text{ for all } ij \in E \}.$$
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Capturing symmetries via colors

\[ (G_P, c) =: G_P^c \]

\[ c : V \cup E \rightarrow \mathcal{C} \]

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\text{Aut}(G_P^c) = \{ \sigma \in \text{Aut}(G_P) \mid c(i) = c(\sigma(i)) \text{ for all } i \in V, c(ij) = c(\sigma(i)\sigma(j)) \text{ for all } ij \in E \}.\]
Capturing symmetries via colors

Mail result. (W., 2021+)

There is a coloring \( c: V \cup E \to C \) of the edge-graph so that

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\text{Aut}(G^c_P) := \left\{ \sigma \in \text{Aut}(G_P) \bigg| \begin{array}{ll}
  c(i) = c(\sigma(i)) & \text{for all } i \in V \\
  c(ij) = c(\sigma(i)\sigma(j)) & \text{for all } ij \in E
\end{array} \right\}.
\]
The metric coloring

Idea:

Let $G$ be a graph with coloring $c: V \cup E \to \mathbb{R}$ defined by $c(ij) := \|v_i - v_j\|$, for all $ij \in E$.

$c(i) := \|v_i\|$, for all $i \in V$.

Question:

Do we have $\text{Aut}(G_c) \sim = \text{Aut}(O(P))$?
The metric coloring

Idea:

\[ c : V \cdot \cup E \rightarrow \mathbb{R} \]

\[ c(\langle i, j \rangle) := \| v_i - v_j \|, \text{ for all } \langle i, j \rangle \in E \]

\[ c(i) := \| v_i \|, \text{ for all } i \in V \]

Question.

Do we have \( \text{Aut}(G_{cP}) \cong \text{Aut}(O(P)) \)?
The metric coloring

**Idea:** $G^c_P$ with coloring $c : V \cup E \to \mathbb{R}$ defined by

$$c(ij) := \|v_i - v_j\|, \quad \text{for all } ij \in E$$
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Question.

Do we have $\text{Aut}(G^c_P) \cong \text{Aut}_O(P)$?
The metric coloring

Idea: $K_n^c$ with coloring $c: V \cup E \to \mathbb{R}$ defined by

\[
\begin{align*}
  c(ij) &:= \|v_i - v_j\|, & \text{for all } ij \in E \\
  c(i) &:= \|v_i\|, & \text{for all } i \in V
\end{align*}
\]

Theorem. (Bremner, Sikirić, Pasechnik, Rehn, Schürman)

We have $\text{Aut}(K_n^c) \cong \text{Aut}_O(P)$. 
The orbit coloring

**Definition.**

The **orbit coloring** assigns the same color to two vertices resp. edges of $G_P$ if and only if they are in the same orbit w.r.t. the action of $\text{Aut}_{GL}(P)$.
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The **orbit coloring** assigns the same color to two vertices resp. edges of $G_P$ if and only if they are in the same orbit w.r.t. the action of $\text{Aut}_{\text{GL}}(P)$.

Theorem.
*The orbit coloring is the finest coloring with* $\text{Aut}(G^c_P) \cong \text{Aut}_{\text{GL}}(P)$. 
The Izmestiev construction

$$P^\circ := \{ x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq 1 \text{ for all } i \in V \}.$$
The Izmestiev construction

\[ P^o(c) := \{ x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq c_i \ \text{for all} \ i \in V \}, \quad c \in \mathbb{R}^n \]
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Then \( P^\circ(1, \ldots, 1) = P^\circ \).
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Then \( P^\circ(1, \ldots, 1) = P^\circ. \)

\[ M_{ij} := \left. \frac{\partial^2 \text{vol}(P^\circ(c))}{\partial c_i \partial c_j} \right|_{c=(1, \ldots, 1)} \]
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if \( ij \in E \)

\[ = - \frac{\text{vol}(f^\circ_{ij})}{\|v_i\|\|v_j\| \sin \angle(v_i, v_j)}. \]

\[ \text{dual face to edge } ij \]
The Theorem of Izmestiev

Izmestiev matrix $\rightarrow M \in \mathbb{R}^{n \times n}$ with $M_{ij} := \frac{\partial^2 \text{vol}(P^o(c))}{\partial c_i \partial c_j} \bigg|_{c=(1,\ldots,1)}$.

**Theorem. (Izmestiev)**

(i) $M_{ij} < 0$ if $ij \in E$.

(ii) $M_{ij} = 0$ if $ij \notin E$ and $i \neq j$.

(iii) $\dim \ker M = d$.

(iv) $M\Phi^T = 0$, where $\Phi := (v_1, \ldots, v_n) \in \mathbb{R}^{d \times n}$.

(v) $M$ has a unique negative eigenvalue of multiplicity one.
The Izmestiev coloring

**Definition.**

The Izmestiev coloring is \( c: V \cup E \rightarrow \mathbb{R} \) with

\[
c(i) := M_{ii}, \quad c(ij) := M_{ij},
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where \( M \) is the Izmestiev matrix.
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**Theorem.**

The Izmestiev coloring satisfies $\text{Aut}(G^c_P) \cong \text{Aut}_{GL}(P)$. 
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**Theorem.**

*The Izmestiev coloring satisfies* \( \text{Aut}(G_P^c) \cong \text{Aut}_{GL}(P) \).

**Corollary.**

*The orbit coloring satisfies* \( \text{Aut}(G_P^c) \cong \text{Aut}_{GL}(P) \).
Theorem.

There is a coloring with $\text{Aut}(G_P^c) \cong \text{Aut}_O(P)$. 

Idea:

Use $c: V \cdot \cup E \to \mathbb{R} \times \mathbb{R}$ with $c(i) := (M_{ii}, \|v_i\|)$, $c(ij) := (M_{ij}, \|v_i - v_j\|)$.

Open problem.

Can we also use $c(i) := \|v_i\|$, $c(ij) := \|v_i - v_j\|$?
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Is it sufficient to color edges (but not the vertices) of $d \geq 3$?
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Is it sufficient to color edges (but not the vertices) of $d \geq 3$?

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Is there a coloring $c: V \cup E \to \mathcal{C}$ with

$$\text{Aut}(G^c_P) \cong \text{Aut}_{\text{PGL}}(P)$$?
Thank you.