Capturing Polytopal Symmetries in the Edge-Graph

Martin Winter

Working group for Algorithmic and Discrete Mathematics

24. March, 2021

TECHNISCHE UNIVERSITÄT CHEMNITZ

Polytopes and geometric symmetries

Polytope: $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex* polytope),
- full-dimensional,
- vertices $v_1, ..., v_n \in \mathbb{R}^d$.

Polytopes and geometric symmetries

Polytope: $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex* polytope),
- full-dimensional,
- vertices $v_1, ..., v_n \in \mathbb{R}^d$.

Symmetries:

Polytopes and geometric symmetries

Polytope: $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex* polytope),
- full-dimensional,
- vertices $v_1, ..., v_n \in \mathbb{R}^d$.

Symmetries:

- $\mathrm{Aut}_{\mathrm{GL}}(P)$
- $\mathrm{Aut}_{\mathrm{O}}(P)$
- $\mathrm{Aut}_{\mathrm{aff}}(P)$
- $\mathrm{Aut}_{\mathrm{Euclid}}(P)$

Polytopes and geometric symmetries

Polytope: $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex* polytope),
- full-dimensional,
- vertices $v_1, ..., v_n \in \mathbb{R}^d$.

Symmetries:

- $\mathrm{Aut}_{\mathrm{GL}}(P)$
- $\mathrm{Aut}_{\mathrm{O}}(P)$
- $\mathrm{Aut}_{\mathrm{aff}}(P)$
- $\mathrm{Aut}_{\mathrm{Euclid}}(P)$

Polytopes and geometric symmetries

Polytope: $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex* polytope),
- full-dimensional,
- vertices $v_1, ..., v_n \in \mathbb{R}^d$.

Symmetries:

- $\mathrm{Aut}_{\mathrm{GL}}(P) := \{T \in \mathrm{GL}(\mathbb{R}^d) \mid TP = P\}$
- $\mathrm{Aut}_{\mathrm{O}}(P)$
- $\mathrm{Aut}_{\mathrm{aff}}(P)$
- $\mathrm{Aut}_{\mathrm{Euclid}}(P)$

Polytopes and geometric symmetries

Polytope: $P \subset \mathbb{R}^d, d \geq 2$

- convex hull of finitely many points (*convex* polytope),
- full-dimensional,
- vertices $v_1, ..., v_n \in \mathbb{R}^d$.

Symmetries:

- $\mathrm{Aut}_{\mathrm{GL}}(P) := \{T \in \mathrm{GL}(\mathbb{R}^d) \mid TP = P\} \;:= \mathrm{Aut}(P)$
- $\mathrm{Aut}_{\mathrm{O}}(P)$
- $\mathrm{Aut}_{\mathrm{aff}}(P)$
- $\mathrm{Aut}_{\mathrm{Euclid}}(P)$

Edge-graphs and combinatorial symmetries

Edge-graph: $G_P = (V, E)$

- finite simple graph with $V = \{1, ..., n\}$,
- $i \in v$ corresponds to vertex v_i,
- $ij \in E \Leftrightarrow \text{conv}\{v_i, v_j\}$ is an edge of P.

Edge-graphs and combinatorial symmetries

Edge-graph: $G_P = (V, E)$

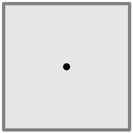
- finite simple graph with $V = \{1, ..., n\}$,
- $i \in v$ corresponds to vertex v_i,
- $ij \in E \;\Leftrightarrow\; \mathrm{conv}\{v_i, v_j\}$ is an edge of P.

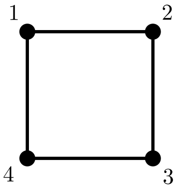
Symmetries:

permutations of the vertex set

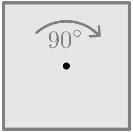
$$\mathrm{Aut}(G) := \{\, \sigma \in \underset{\uparrow}{\mathrm{Sym}(V)} \mid ij \in E \leftrightarrow \sigma(i)\sigma(j) \in E \,\}.$$

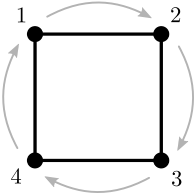
Polytope symmetries yield graph symmetries ...





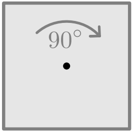
Polytope symmetries yield graph symmetries ...

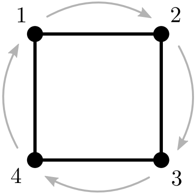




Polytope symmetries yield graph symmetries ...

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

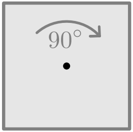


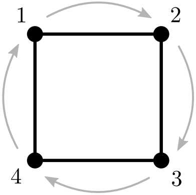


$$\sigma = (1234)$$

Polytope symmetries yield graph symmetries ...

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



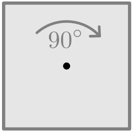


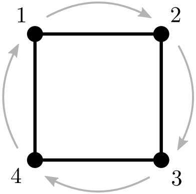
$$\sigma = (1234)$$

$$T \in \mathrm{Aut}(P) \quad \Longrightarrow \quad \exists\, \sigma_T \in \mathrm{Aut}(G_P).$$

Polytope symmetries yield graph symmetries ...

$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$





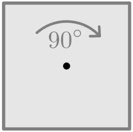
$\sigma = (1234)$

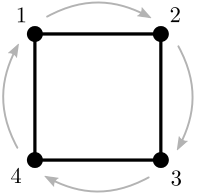
$$T \in \mathrm{Aut}(P) \quad \Longrightarrow \quad \exists\, \sigma_T \in \mathrm{Aut}(G_P).$$

s.t. $Tv_i = v_{\sigma_T(i)}$

Polytope symmetries yield graph symmetries ...

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$





$$\sigma = (1234)$$

$$T \in \mathrm{Aut}(P) \quad \Longrightarrow \quad \exists\, \sigma_T \in \mathrm{Aut}(G_P).$$

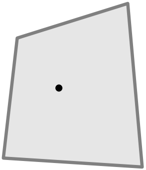
s.t. $Tv_i = v_{\sigma_T(i)}$

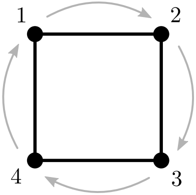
$$\exists\, T_\sigma \in \mathrm{Aut}(P) \quad \overset{?}{\Longleftarrow} \quad \sigma \in \mathrm{Aut}(G_P).$$

s.t. $T_\sigma v_i = v_{\sigma(i)}$

Polytope symmetries yield graph symmetries ...

$T = ?$





$\sigma = (1234)$

$$T \in \mathrm{Aut}(P) \quad \Longrightarrow \quad \exists\, \sigma_T \in \mathrm{Aut}(G_P).$$

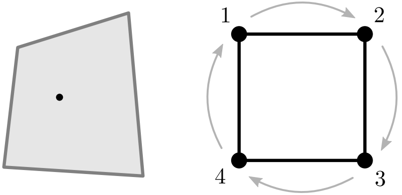
s.t. $Tv_i = v_{\sigma_T(i)}$

$$\exists\, T_\sigma \in \mathrm{Aut}(P) \quad \overset{?}{\Longleftarrow} \quad \sigma \in \mathrm{Aut}(G_P).$$

s.t. $T_\sigma v_i = v_{\sigma(i)}$

Polytope symmetries yield graph symmetries ...

$T = ?$



$\sigma = (1234)$

$$T \in \mathrm{Aut}(P) \quad \Longrightarrow \quad \exists\, \sigma_T \in \mathrm{Aut}(G_P).$$

s.t. $Tv_i = v_{\sigma_T(i)}$

$$\exists\, T_\sigma \in \mathrm{Aut}(P) \quad \overset{?}{\Longleftarrow} \quad \sigma \in \mathrm{Aut}(G_P).$$

s.t. $T_\sigma v_i = v_{\sigma(i)}$

Combinatorial symmetries vs. geometric symmetries

- Edge-graph can can be more symmetric than every realization of the polytope.

Combinatorial symmetries vs. geometric symmetries

- Edge-graph can can be more symmetric than every realization of the polytope.

Example: a neighborly polytope that is not a simplex (i.e. $G_P = K_n$).

$$\mathrm{Aut}(K_n) = \mathrm{Sym}(V).$$

Combinatorial symmetries vs. geometric symmetries

- Edge-graph can can be more symmetric than every realization of the polytope.

Example: a neighborly polytope that is not a simplex (i.e. $G_P = K_n$).

$$\mathrm{Aut}(K_n) = \mathrm{Sym}(V).$$

$\rightarrow$ the edge-graph carries very little information about the polytopes

Combinatorial symmetries vs. geometric symmetries

- Face lattice can be more symmetric than every realization of the polytope

  (face lattice = set of faces ordered by inclusion)

Combinatorial symmetries vs. geometric symmetries

- Face lattice can be more symmetric than every realization of the polytope
  (face lattice = set of faces ordered by inclusion)

Example: BEK polytope (Bokowski, Ewald, Kleinschmidt).

Combinatorial symmetries vs. geometric symmetries

- Face lattice can be more symmetric than every realization of the polytope

  (face lattice = set of faces ordered by inclusion)

Example: BEK polytope (Bokowski, Ewald, Kleinschmidt).

1	2	3	7	1	2	4	8	9	2	3	8	9	2	10	7
1	3	4	7	2	3	4	8	9	3	10	8	2	3	10	7
1	4	6	7	1	5	6	8	9	10	6	8	9	5	6	7
4	5	6	7	1	4	5	8	10	5	6	8	9	10	5	7
1	2	6	7	1	2	6	8	9	2	6	8	9	2	6	7
3	4	5	7	3	4	5	8	3	10	5	8	3	10	5	7
1	4	5	6	1	2	3	4	9	10	5	6	9	2	3	10

Combinatorial symmetries vs. geometric symmetries

- Face lattice can be more symmetric than every realization of the polytope
  (face lattice = set of faces ordered by inclusion)

Example: BEK polytope (Bokowski, Ewald, Kleinschmidt).

1	2	3	7	1	2	4	8	9	2	3	8	9	2	10	7
1	3	4	7	2	3	4	8	9	3	10	8	2	3	10	7
1	4	6	7	1	5	6	8	9	10	6	8	9	5	6	7
4	5	6	7	1	4	5	8	10	5	6	8	9	10	5	7
1	2	6	7	1	2	6	8	9	2	6	8	9	2	6	7
3	4	5	7	3	4	5	8	3	10	5	8	3	10	5	7
1	4	5	6	1	2	3	4	9	10	5	6	9	2	3	10

The face lattice has a "central symmetry" that cannot be realized geometrically.

$$1 \leftrightarrow 10, \quad 2 \leftrightarrow 5, \quad 3 \leftrightarrow 6, \quad 4 \leftrightarrow 7, \quad 7 \leftrightarrow 8.$$

Combinatorial symmetries vs. geometric symmetries

- A graph can be more symmetric than any polytope with this edge-graph.

Combinatorial symmetries vs. geometric symmetries

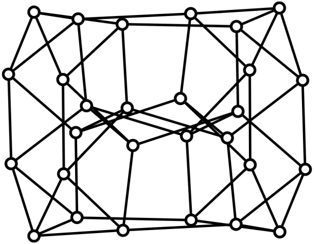
- A graph can be more symmetric than any polytope with this edge-graph.

Example: polar dual of BEK polytope.

Combinatorial symmetries vs. geometric symmetries

- A graph can be more symmetric than any polytope with this edge-graph.

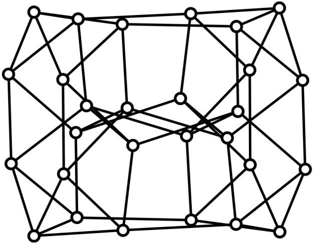
Example: polar dual of BEK polytope.



Combinatorial symmetries vs. geometric symmetries

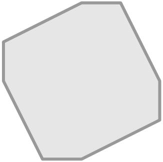
- A graph can be more symmetric than any polytope with this edge-graph.

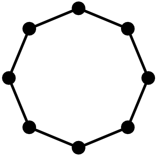
Example: polar dual of BEK polytope.



4-regular $+$ non-planar $\rightarrow$ unique realization (in $\mathbb{R}^4$)

Capturing symmetries via colors





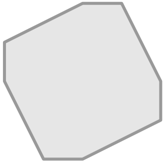
Capturing symmetries via colors

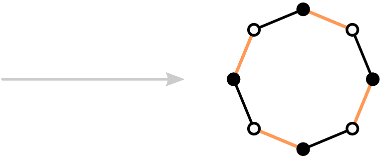
Capturing symmetries via colors

$$\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$$

Capturing symmetries via colors

$(G_P, \mathfrak{c}) =: G_P^{\mathfrak{c}}$

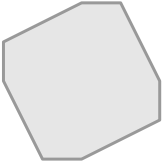


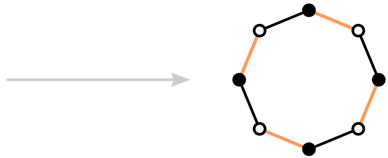


$\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$

Capturing symmetries via colors

$(G_P, \mathfrak{c}) =: G_P^{\mathfrak{c}}$





$\mathfrak{c}\colon V \,\dot\cup\, E \to \mathfrak{C}$

$$\mathrm{Aut}(G_P^{\mathfrak{c}}) := \left\{ \sigma \in \mathrm{Aut}(G_P) \;\middle|\; \begin{array}{ll} \mathfrak{c}(i) = \mathfrak{c}(\sigma(i)) & \text{for all } i \in V \\ \mathfrak{c}(ij) = \mathfrak{c}(\sigma(i)\sigma(j)) & \text{for all } ij \in E \end{array} \right\}.$$

Main result

Theorem. (W., 2021+)

There is a coloring $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ *of the edge-graph so that*

$$\operatorname{Aut}(G_P^{\mathfrak{c}}) \cong \operatorname{Aut}(P).$$

Main result

Theorem. (W., 2021+)

There is a coloring $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ *of the edge-graph so that*

$$\operatorname{Aut}(G_P^{\mathfrak{c}}) \;\cong\; \underset{\mathrm{GL}(\mathbb{R}^d)}{\underset{\cap}{\operatorname{Aut}(P)}}\,.$$

Main result

Theorem. (W., 2021+)

There is a coloring $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ *of the edge-graph so that*

$$\underset{\underset{\mathrm{Sym}(V)}{\cap|}}{\mathrm{Aut}(G_P^{\mathfrak{c}})} \;\cong\; \underset{\underset{\mathrm{GL}(\mathbb{R}^d)}{\cap}}{\mathrm{Aut}(P)}\,.$$

Main result

Theorem. (W., 2021+)

There is a coloring $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ *of the edge-graph so that*

$$\underset{\mathrm{Sym}(V)}{\underset{\cap|}{\mathrm{Aut}(G_P^{\mathfrak{c}})}} \;\underset{\uparrow\,\textit{natural}}{\cong}\; \underset{\mathrm{GL}(\mathbb{R}^d)}{\underset{\cap}{\mathrm{Aut}(P)}}\,.$$

Main result

Theorem. (W., 2021+)

There is a coloring $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ *of the edge-graph so that*

$$\operatorname{Aut}(G_P^{\mathfrak{c}}) \cong \operatorname{Aut}(P).$$

$$\operatorname{Aut}(G_P^{\mathfrak{c}}) \subseteq \operatorname{Sym}(V), \qquad \operatorname{Aut}(P) \subset \operatorname{GL}(\mathbb{R}^d)$$

The isomorphism is *natural*:

$$\sigma \mapsto T_\sigma$$

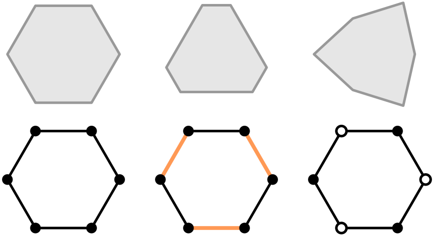
natural: T_σ permutes the vertices of P as σ permutes the vertices of G_P, i.e.

$$T_\sigma v_i = v_{\sigma(i)}, \qquad \text{for all } i \in V.$$

The orbit coloring

Definition.

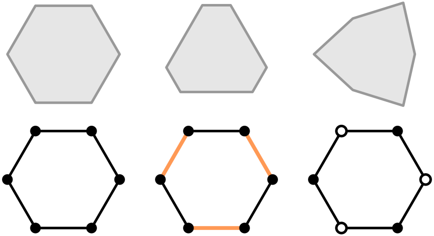
The **orbit coloring** assigns the same color to two vertices resp. edges of G_P if and only if they are in the same orbit *w.r.t.* the action of $\mathrm{Aut}(P)$.



The orbit coloring

Definition.

The **orbit coloring** assigns the same color to two vertices resp. edges of G_P if and only if they are in the same orbit *w.r.t.* the action of $\mathrm{Aut}(P)$.



Theorem.

The orbit coloring is the *finest* *coloring with* $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}(P)$.

A similar result

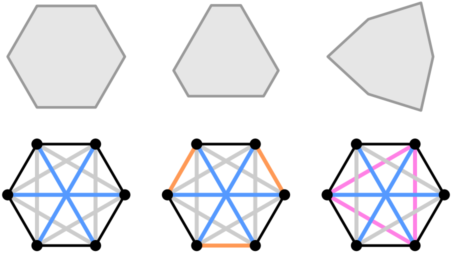
Theorem. (Bremner, Sikirić, Pasechnik, Rehn, Schürman)

Consider complete graph $K_n^{\mathfrak{c}}$ *with edge-coloring*

$$\mathfrak{c}\colon E \to \mathbb{R}, \qquad \mathfrak{c}(ij) := \|v_i - v_j\|, \qquad \textit{whenever } i \neq j.$$

Then $\operatorname{Aut}(K_n^{\mathfrak{c}}) \cong \operatorname{Aut}_{\text{Euclid}}(P)$.

→ BSPRS coloring *or* complete coloring



A similar result

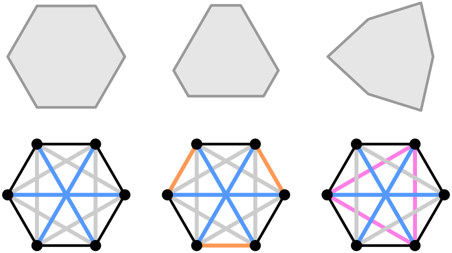
Theorem. (Bremner, Sikirić, Pasechnik, Rehn, Schürman)

Consider complete graph $K_n^{\mathfrak{c}}$ *with edge-coloring*

$$\mathfrak{c}\colon E \to \mathbb{R}, \qquad \mathfrak{c}(ij) := \|Q(v_i - v_j)\|, \quad \textit{whenever } i \neq j.$$

Then $\mathrm{Aut}(K_n^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{aff}}(P)$.

→ BSPRS coloring *or* complete coloring



Why isn't this obvious?

- What if I consider a subgroup of $\mathrm{Aut}(P)$?

Why isn't this obvious?

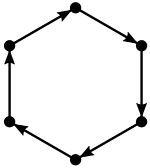
- What if I consider a subgroup of $\mathrm{Aut}(P)$? For example,

$$\mathrm{Aut}(P)_+ := \{T \in \mathrm{Aut}(P) \mid \det(T) > 0\}.$$

Why isn't this obvious?

- What if I consider a subgroup of $\mathrm{Aut}(P)$? For example,

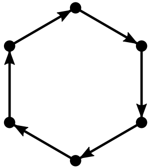
$$\mathrm{Aut}(P)_+ := \{T \in \mathrm{Aut}(P) \mid \det(T) > 0\}.$$



Why isn't this obvious?

- What if I consider a subgroup of $\mathrm{Aut}(P)$? For example,

$$\mathrm{Aut}(P)_+ := \{T \in \mathrm{Aut}(P) \mid \det(T) > 0\}.$$

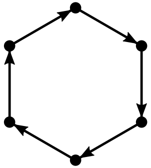


- Why don't we need to color higher dimensional face?

Why isn't this obvious?

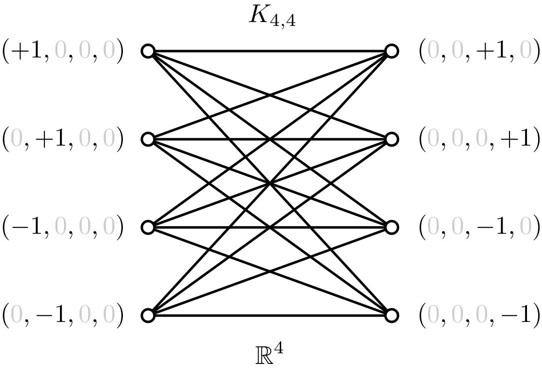
- What if I consider a subgroup of $\mathrm{Aut}(P)$? For example,

$$\mathrm{Aut}(P)_+ := \{T \in \mathrm{Aut}(P) \mid \det(T) > 0\}.$$

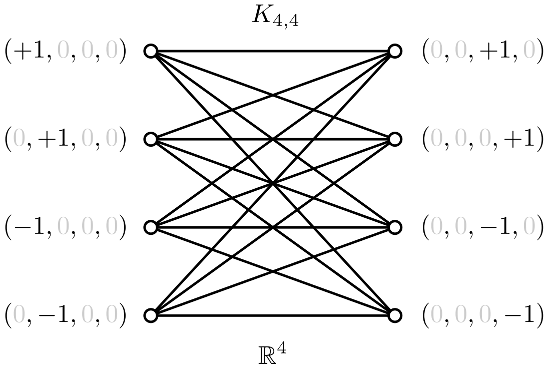


- Why don't we need to color higher dimensional face?
- It means we cannot change the symmetry of the polytope without chaning the orbits.

Capturing symmetries of graph embeddings



Capturing symmetries of graph embeddings



vertex-transitive $+$ edge-transitive $+$ less symmetric than $K_{4,4}$

Proof: a linear algebra criterion

Proof: a linear algebra criterion

Suppose we have a symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we need a $T_\sigma \in \mathrm{Aut}(P)$:

Proof: a linear algebra criterion

Suppose we have a symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we need a $T_\sigma \in \mathrm{Aut}(P)$:

$$T_\sigma v_i = v_{\sigma(i)}, \quad \text{for all } i \in V.$$

Proof: a linear algebra criterion

Suppose we have a symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we need a $T_\sigma \in \mathrm{Aut}(P)$:

$$T_\sigma v_i = v_{\sigma(i)}, \quad \text{for all } i \in V.$$

$$\begin{aligned} T_\sigma(v_1, ..., v_n) &= (v_{\sigma(1)}, ..., v_{\sigma(n)}) \\ &= (v_1, ..., v_n)\Pi_\sigma. \end{aligned}$$

Proof: a linear algebra criterion

Suppose we have a symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we need a $T_\sigma \in \mathrm{Aut}(P)$:

$$T_\sigma v_i = v_{\sigma(i)}, \quad \text{for all } i \in V.$$

$$\begin{aligned} T_\sigma \overbrace{(v_1, ..., v_n)}^{\Phi} &= (v_{\sigma(1)}, ..., v_{\sigma(n)}) \\ &= \underbrace{(v_1, ..., v_n)}_{\Phi} \Pi_\sigma. \end{aligned}$$

Proof: a linear algebra criterion

Suppose we have a symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we need a $T_\sigma \in \mathrm{Aut}(P)$:

$$T_\sigma v_i = v_{\sigma(i)}, \quad \text{for all } i \in V.$$

$$\begin{aligned} T_\sigma \overbrace{(v_1, \ldots, v_n)}^{\Phi} &= (v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \\ &= \underbrace{(v_1, \ldots, v_n)}_{\Phi} \Pi_\sigma. \end{aligned}$$

$$T_\sigma \Phi = \Phi \Pi_\sigma. \qquad (\Phi \in \mathbb{R}^{d\times n})$$

Proof: a linear algebra criterion

Suppose we have a symmetry $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$, we need a $T_\sigma \in \mathrm{Aut}(P)$:

$$T_\sigma v_i = v_{\sigma(i)}, \qquad \text{for all } i \in V.$$

$$\begin{aligned} T_\sigma \overbrace{(v_1, \ldots, v_n)}^{\Phi} &= (v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \\ &= \underbrace{(v_1, \ldots, v_n)}_{\Phi} \Pi_\sigma. \end{aligned}$$

$$T_\sigma \Phi = \Phi \Pi_\sigma. \qquad (\Phi \in \mathbb{R}^{d \times n})$$

Ansatz: $T_\sigma := \Phi \Pi_\sigma \Phi^\dagger. \qquad (\Phi\Phi^\dagger = \mathrm{Id})$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\qquad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

TECHNISCHE UNIVERSITÄT CHEMNITZ

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\quad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi =$$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi \Pi_\sigma \Phi^\dagger.$ $\qquad (T_\sigma \Phi \overset{?}{=} \Phi \Pi_\sigma)$

$$T_\sigma \Phi = \Phi \Pi_\sigma \Phi^\dagger \Phi$$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\qquad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\Phi^\dagger\Phi$$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi \Pi_\sigma \Phi^\dagger.$ $\qquad (T_\sigma \Phi \stackrel{?}{=} \Phi \Pi_\sigma)$

$$T_\sigma \Phi = \Phi \Pi_\sigma \overbrace{\Phi^\dagger \Phi}^{\text{orthogonal projector onto } U := \operatorname{span} \Phi^\dagger \subseteq \mathbb{R}^n}$$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\qquad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\overbrace{\Phi^\dagger\Phi}^{\text{orthogonal projector onto } U := \operatorname{span}\Phi^\dagger \subseteq \mathbb{R}^n} = \Phi\Pi_\sigma\pi_U.$$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\qquad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\overbrace{\Phi^\dagger\Phi}^{\text{orthogonal projector onto } U := \operatorname{span}\Phi^\dagger \subseteq \mathbb{R}^n} = \Phi\Pi_\sigma\pi_U.$$

But: if U is a Π_σ-invariant, then $\quad (\Pi_\sigma U = U)$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\qquad (T_\sigma\Phi \stackrel{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi = \Phi\Pi_\sigma \overbrace{\Phi^\dagger\Phi}^{\text{orthogonal projector onto } U := \operatorname{span}\Phi^\dagger \subseteq \mathbb{R}^n} = \Phi\Pi_\sigma\pi_U.$$

But: if U is a Π_σ-invariant, then $\quad (\Pi_\sigma U = U)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\pi_U = \Phi\pi_U\Pi_\sigma = \underbrace{\Phi\Phi^\dagger}_{\mathrm{Id}}\Phi\Pi_\sigma = \Phi\Pi_\sigma.$$

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger.$ $\quad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\overbrace{\Phi^\dagger\Phi}^{\text{orthogonal projector onto } U := \operatorname{span}\Phi^\dagger \subseteq \mathbb{R}^n} = \Phi\Pi_\sigma\pi_U.$$

But: if U is a Π_σ-invariant, then $(\Pi_\sigma U = U)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\pi_U = \Phi\pi_U\Pi_\sigma = \underbrace{\Phi\Phi^\dagger}_{\text{Id}}\Phi\Pi_\sigma = \Phi\Pi_\sigma.$$

→ We have to ensure that all $\Pi_\sigma, \sigma \in \operatorname{Aut}(G_P^{\mathfrak{c}})$ commute with π_U.

Proof: a linear algebra criterion

Ansatz: $T_\sigma := \Phi\Pi_\sigma\Phi^\dagger. \qquad (T_\sigma\Phi \overset{?}{=} \Phi\Pi_\sigma)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\overbrace{\Phi^\dagger\Phi}^{\text{orthogonal projector onto } U := \operatorname{span}\Phi^\dagger \subseteq \mathbb{R}^n} = \Phi\Pi_\sigma\pi_U.$$

But: if U is a Π_σ-invariant, then $\quad (\Pi_\sigma U = U)$

$$T_\sigma\Phi = \Phi\Pi_\sigma\pi_U = \Phi\pi_U\Pi_\sigma = \underbrace{\Phi\Phi^\dagger}_{\text{Id}}\Phi\Pi_\sigma = \Phi\Pi_\sigma.$$

$\rightarrow$ We have to ensure that all $\Pi_\sigma, \sigma \in \operatorname{Aut}(G_P^{\mathfrak{c}})$ commute with π_U. **But how?**

Proof: the colored adjacency matrix

A permutation $\sigma \in \mathrm{Sym}(V)$ is in $\mathrm{Aut}(G_P^{\mathfrak{c}})$ if and only if

$$A^{\mathfrak{c}}\Pi_\sigma = \Pi_\sigma A^{\mathfrak{c}}. \quad \text{with } A^{\mathfrak{c}}_{ij} = \begin{cases} \mathfrak{c}(i) & \text{if } i = j \\ \mathfrak{c}(ij) & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

Proof: the colored adjacency matrix

A permutation $\sigma \in \mathrm{Sym}(V)$ is in $\mathrm{Aut}(G_P^{\mathfrak{c}})$ if and only if

$$A^{\mathfrak{c}}\Pi_\sigma = \Pi_\sigma A^{\mathfrak{c}}. \quad \text{with } A_{ij}^{\mathfrak{c}} = \begin{cases} \mathfrak{c}(i) & \text{if } i = j \\ \mathfrak{c}(ij) & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

Observations:

→ the eigenspaces of $A^{\mathfrak{c}}$ are Π_σ-invariant for all $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$.

Proof: the colored adjacency matrix

A permutation $\sigma \in \mathrm{Sym}(V)$ is in $\mathrm{Aut}(G_P^{\mathfrak{c}})$ if and only if

$$A^{\mathfrak{c}}\Pi_\sigma = \Pi_\sigma A^{\mathfrak{c}}. \quad \text{with } A_{ij}^{\mathfrak{c}} = \begin{cases} \mathfrak{c}(i) & \text{if } i = j \\ \mathfrak{c}(ij) & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

Observations:

→ the eigenspaces of $A^{\mathfrak{c}}$ are Π_σ-invariant for all $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$.

→ we need to make $U := \operatorname{span}\Phi^\dagger$ an eigenspace of $A^{\mathfrak{c}}$.

Proof: the colored adjacency matrix

A permutation $\sigma \in \mathrm{Sym}(V)$ is in $\mathrm{Aut}(G_P^{\mathfrak{c}})$ if and only if

$$A^{\mathfrak{c}}\Pi_\sigma = \Pi_\sigma A^{\mathfrak{c}}. \quad \text{with } A^{\mathfrak{c}}_{ij} = \begin{cases} \mathfrak{c}(i) & \text{if } i = j \\ \mathfrak{c}(ij) & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

Observations:

→ the eigenspaces of $A^{\mathfrak{c}}$ are Π_σ-invariant for all $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$.

→ we need to make $U := \operatorname{span}\Phi^\dagger$ an eigenspace of $A^{\mathfrak{c}}$.

→ we need to choose a clever coloring!

A simple construction

$$A^{\mathfrak{c}} := \Phi^{\dagger}\Phi = \pi_U.$$

A simple construction

$$A^{\mathfrak{c}} := \Phi^\dagger \Phi = \pi_U.$$

→ $U := \operatorname{span} \Phi^\dagger$ is the eigenspace of $A^{\mathfrak{c}}$ to eigenvalue $\theta = 1$.

A simple construction

$$A^{\mathfrak{c}} := \Phi^{\dagger}\Phi = \pi_U.$$

$\rightarrow$ $U := \operatorname{span}\Phi^{\dagger}$ is the eigenspace of $A^{\mathfrak{c}}$ to eigenvalue $\theta = 1$.

$$\approx \quad A^{\mathfrak{c}}_{ij} = \langle v_i, v_j \rangle.$$

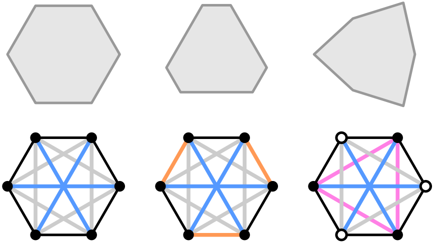
A simple construction

$$A^{\mathfrak{c}} := \Phi^\dagger \Phi = \pi_U.$$

→ $U := \operatorname{span} \Phi^\dagger$ is the eigenspace of $A^{\mathfrak{c}}$ to eigenvalue $\theta = 1$.

$$\approx \quad A^{\mathfrak{c}}_{ij} = \langle v_i, v_j \rangle.$$

→ similar to BSPRS coloring but with vertex-colors.



Proof: a better construction

$$P^\circ := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq 1 \text{ for all } i \in V\}.$$

Proof: a better construction

$$P^\circ(c) := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq c_i \text{ for all } i \in V\}, \qquad c \in \mathbb{R}^n$$

Proof: a better construction

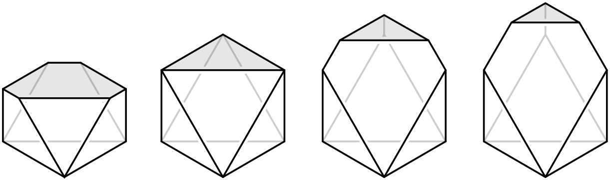
$$P^\circ(c) := \{x \in \mathbb{R}^d \mid \langle x, v_i\rangle \le c_i \text{ for all } i \in V\}, \qquad c \in \mathbb{R}^n$$

Then $P^\circ(1,...,1) = P^\circ$.

Proof: a better construction

$$P^\circ(c) := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \le c_i \text{ for all } i \in V\}, \qquad c \in \mathbb{R}^n$$

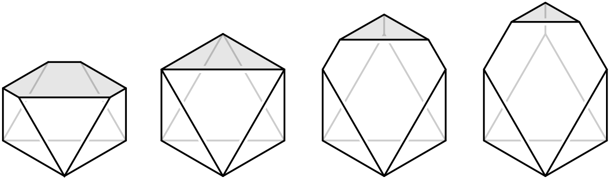
Then $P^\circ(1, ..., 1) = P^\circ$.



Proof: a better construction

$$P^\circ(c) := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \le c_i \text{ for all } i \in V\}, \qquad c \in \mathbb{R}^n$$

Then $P^\circ(1, ..., 1) = P^\circ$.

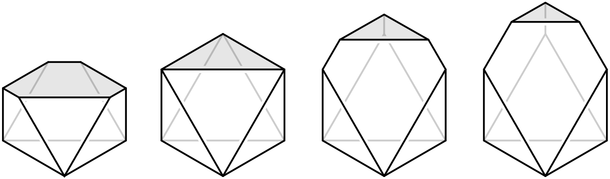


$$A^{\mathfrak{c}}_{ij} := \frac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i \, \partial c_j}\Big|_{c=(1,...,1)}$$

Proof: a better construction

$$P^\circ(c) := \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \le c_i \text{ for all } i \in V\}, \qquad c \in \mathbb{R}^n$$

Then $P^\circ(1, ..., 1) = P^\circ$.



$$A^{\mathfrak{c}}_{ij} := \frac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i \, \partial c_j}\bigg|_{c=(1,...,1)} = -\frac{\mathrm{vol}(f^\circ_{ij})}{\|v_i\| \|v_j\| \sin \sphericalangle(v_i, v_j)}.$$

Proof: a better construction

Izmestiev matrix $\to$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

Proof: a better construction

Izmestiev matrix $\rightarrow\; A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

Proof: a better construction

Izmestiev matrix $\to\ A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}.$

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

Proof: a better construction

Izmestiev matrix $\to$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\ldots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

Proof: a better construction

Izmestiev matrix $\rightarrow$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\ldots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

only "true edges" of P *are colored.*

(iii) $\dim\ker A^{\mathfrak{c}} = d$.

Proof: a better construction

Izmestiev matrix $\to$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of* P *are colored.*

(iii) $\dim \ker A^{\mathfrak{c}} = d$.

(iv) $A^{\mathfrak{c}} \Phi^T = 0$.

Proof: a better construction

Izmestiev matrix $\rightarrow$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\ldots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.
(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

(iii) $\dim\ker A^{\mathfrak{c}} = d$.
(iv) $A^{\mathfrak{c}}\Phi^T = 0$.
(v) $A^{\mathfrak{c}}$ *has a unique negative eigenvalue of multiplicity one.*

Proof: a better construction

$$\textbf{Izmestiev matrix} \;\to\; A^{\mathfrak{c}} \in \mathbb{R}^{n\times n} \text{ with } A^{\mathfrak{c}}_{ij} := \frac{\partial^2 \operatorname{vol}(P^\circ(c))}{\partial c_i\,\partial c_j}\Big|_{c=(1,\dots,1)}.$$

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.
(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

(iii) $\dim\ker A^{\mathfrak{c}} = d$.
(iv) $A^{\mathfrak{c}}\Phi^T = 0$.
(v) $A^{\mathfrak{c}}$ *has a unique negative eigenvalue of multiplicity one.*

$$\ker A^{\mathfrak{c}} \supseteq \operatorname{span}\Phi^T$$

Proof: a better construction

Izmestiev matrix $\rightarrow$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

(iii) $\dim\ker A^{\mathfrak{c}} = d$.

(iv) $A^{\mathfrak{c}}\Phi^T = 0$.

(v) $A^{\mathfrak{c}}$ *has a unique negative eigenvalue of multiplicity one.*

$$\ker A^{\mathfrak{c}} = \mathrm{span}\,\Phi^T$$

Proof: a better construction

Izmestiev matrix $\to$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\,\partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

(iii) $\dim\ker A^{\mathfrak{c}} = d$.

(iv) $A^{\mathfrak{c}}\Phi^T = 0$.

(v) $A^{\mathfrak{c}}$ *has a unique negative eigenvalue of multiplicity one.*

$$\ker A^{\mathfrak{c}} = \mathrm{span}\,\Phi^{\dagger}$$

Proof: a better construction

Izmestiev matrix $\to$ $A^{\mathfrak{c}} \in \mathbb{R}^{n\times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i\, \partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

(iii) $\dim\ker A^{\mathfrak{c}} = d$.

(iv) $A^{\mathfrak{c}}\Phi^T = 0$.

(v) $A^{\mathfrak{c}}$ *has a unique negative eigenvalue of multiplicity one.*

$$\ker A^{\mathfrak{c}} = \operatorname{span}\Phi^{\dagger} \;=: U$$

Proof: a better construction

Izmestiev matrix $\rightarrow$ $A^{\mathfrak{c}} \in \mathbb{R}^{n \times n}$ with $A^{\mathfrak{c}}_{ij} := \dfrac{\partial^2 \mathrm{vol}(P^\circ(c))}{\partial c_i \, \partial c_j}\Big|_{c=(1,\dots,1)}$.

Theorem. (Izmestiev)

(i) $A^{\mathfrak{c}}_{ij} < 0$ *if* $ij \in E$.

(ii) $A^{\mathfrak{c}}_{ij} = 0$ *if* $ij \notin E$ *and* $i \neq j$.

} *only "true edges" of P are colored.*

(iii) $\dim \ker A^{\mathfrak{c}} = d$.

(iv) $A^{\mathfrak{c}} \Phi^T = 0$.

(v) $A^{\mathfrak{c}}$ *has a unique negative eigenvalue of multiplicity one.*

$$\ker A^{\mathfrak{c}} = \mathrm{span}\, \Phi^\dagger \;=: U$$

$\rightarrow$ U is an eigenspace of $A^{\mathfrak{c}}$, hence Π_σ-invariant for all $\sigma \in \mathrm{Aut}(G_P^{\mathfrak{c}})$.

The Izmestiev coloring

Definition.

The **Izemstiev coloring** is $\mathfrak{c}\colon V \cup E \to \mathbb{R}$ with

$$\mathfrak{c}(i) := A^{\mathfrak{c}}_{ii}, \qquad \mathfrak{c}(ij) := A^{\mathfrak{c}}_{ij},$$

where $A^{\mathfrak{c}}$ is the Izmestiev matrix.

The Izmestiev coloring

Definition.

The **Izemstiev coloring** is $\mathfrak{c}\colon V \cup E \to \mathbb{R}$ with

$$\mathfrak{c}(i) := A^{\mathfrak{c}}_{ii}, \qquad \mathfrak{c}(ij) := A^{\mathfrak{c}}_{ij},$$

where $A^{\mathfrak{c}}$ is the Izmestiev matrix.

Theorem.

The Izmestiev coloring satisfies $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}(P)$.

The Izmestiev coloring

Definition.

The **Izemstiev coloring** is $\mathfrak{c}\colon V \cup E \to \mathbb{R}$ with

$$\mathfrak{c}(i) := A^{\mathfrak{c}}_{ii}, \qquad \mathfrak{c}(ij) := A^{\mathfrak{c}}_{ij},$$

where $A^{\mathfrak{c}}$ is the Izmestiev matrix.

Theorem.

The Izmestiev coloring satisfies $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}(P)$.

Corollary.

The orbit coloring satisfies $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}(P)$.

Theorem.

There is a coloring with $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.

Theorem.

There is a coloring with $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.

Idea: use $\mathfrak{c}\colon V \cup E \to \mathbb{R} \times \mathbb{R}$ with

$$\mathfrak{c}(i) := (A_{ii}^{\mathfrak{c}}, \|v_i\|), \quad \mathfrak{c}(ij) := (A_{ij}^{\mathfrak{c}}, \|v_i - v_j\|).$$

Theorem.

There is a coloring with $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.

Idea: use $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathbb{R} \times \mathbb{R}$ with

$$\mathfrak{c}(i) := (A_{ii}^{\mathfrak{c}}, \|v_i\|), \quad \mathfrak{c}(ij) := (A_{ij}^{\mathfrak{c}}, \|v_i - v_j\|).$$

Question

Can we also use

$$\mathfrak{c}(i) := \|v_i\|, \quad \mathfrak{c}(ij) := \|v_i - v_j\| \;?$$

Theorem.

There is a coloring with $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.

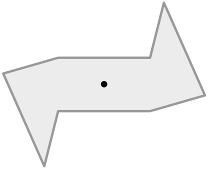
Idea: use $\mathfrak{c}\colon V \cup E \to \mathbb{R} \times \mathbb{R}$ with

$$\mathfrak{c}(i) := (A_{ii}^{\mathfrak{c}}, \|v_i\|), \quad \mathfrak{c}(ij) := (A_{ij}^{\mathfrak{c}}, \|v_i - v_j\|).$$

Question

Can we also use

$$\mathfrak{c}(i) := \|v_i\|, \quad \mathfrak{c}(ij) := \|v_i - v_j\| \ ?$$



Theorem.

There is a coloring with $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.

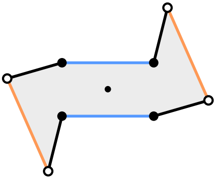
Idea: use $\mathfrak{c}\colon V \cup E \to \mathbb{R} \times \mathbb{R}$ with

$$\mathfrak{c}(i) := (A_{ii}^{\mathfrak{c}}, \|v_i\|), \quad \mathfrak{c}(ij) := (A_{ij}^{\mathfrak{c}}, \|v_i - v_j\|).$$

Question

Can we also use

$$\mathfrak{c}(i) := \|v_i\|, \quad \mathfrak{c}(ij) := \|v_i - v_j\| \; ?$$



Theorem.

There is a coloring with $\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{O}}(P)$.

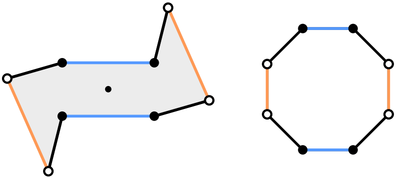
Idea: use $\mathfrak{c}\colon V \cup E \to \mathbb{R} \times \mathbb{R}$ with

$$\mathfrak{c}(i) := (A_{ii}^{\mathfrak{c}}, \|v_i\|), \quad \mathfrak{c}(ij) := (A_{ij}^{\mathfrak{c}}, \|v_i - v_j\|).$$

Question

Can we also use

$$\mathfrak{c}(i) := \|v_i\|, \quad \mathfrak{c}(ij) := \|v_i - v_j\| \,?$$



Are vertex colors necessary?

Conjecture.

In dimension $d \geq 3$ it suffices to color only the edges (and not the vertices).

Are vertex colors necessary?

Conjecture.

In dimension $d \geq 3$ it suffices to color only the edges (and not the vertices).

A counterexample would be a **strange** polytope with

Are vertex colors necessary?

Conjecture.

In dimension $d \geq 3$ it suffices to color only the edges (and not the vertices).

A counterexample would be a **strange** polytope with

- an edge graph that is **bipartite** and **regular**,

Are vertex colors necessary?

Conjecture.

In dimension $d \geq 3$ it suffices to color only the edges (and not the vertices).

A counterexample would be a **strange** polytope with

- an edge graph that is **bipartite** and **regular**,
- exactly **two vertex orbits** (corresponding to the bipartition classes),

Are vertex colors necessary?

Conjecture.

In dimension $d \geq 3$ it suffices to color only the edges (and not the vertices).

A counterexample would be a **strange** polytope with

- an edge graph that is **bipartite** and **regular**,
- exactly **two vertex orbits** (corresponding to the bipartition classes),
- a combinatorial symmetry of G_P that exchanges the bipartition classes,

Are vertex colors necessary?

Conjecture.

In dimension $d \geq 3$ it suffices to color only the edges (and not the vertices).

A counterexample would be a **strange** polytope with

- an edge graph that is **bipartite** and **regular**,
- exactly **two vertex orbits** (corresponding to the bipartition classes),
- a combinatorial symmetry of G_P that exchanges the bipartition classes,
- no geometry symmetry of P that exchanges the bipartition classes.

Other questions

Question

Is there a coloring $\mathfrak{c}\colon V \mathbin{\dot\cup} E \to \mathfrak{C}$ with

$$\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{PGL}}(P)\ ?$$

Other questions

Question

Is there a coloring $\mathfrak{c}\colon V \cup E \to \mathfrak{C}$ with

$$\mathrm{Aut}(G_P^{\mathfrak{c}}) \cong \mathrm{Aut}_{\mathrm{PGL}}(P)\ ?$$

Question

Can this be generalized to certain types of graph embeddings?

Thank you.