Expansion and rigidity properties of polytope skeleta

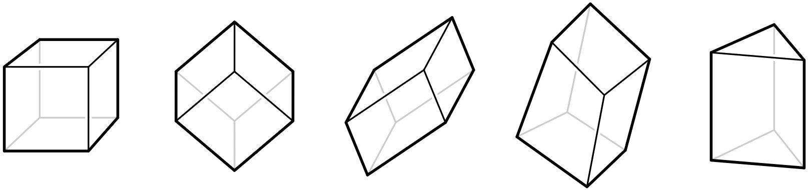
Martin Winter

University of Warwick

14. September, 2022

The setting

convex polytope ... $P := \mathrm{conv}\{p_1, ..., p_n\} \subset \mathbb{R}^d$.



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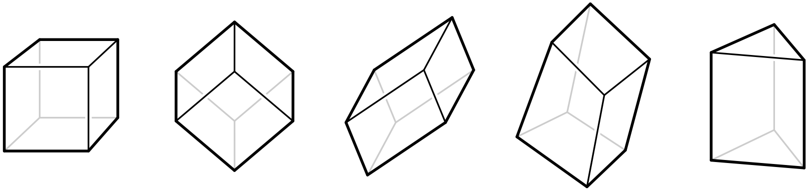

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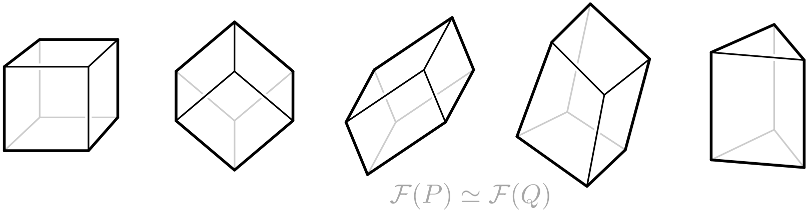

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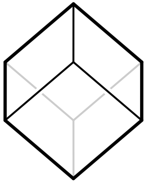

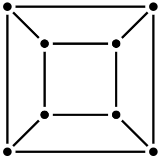

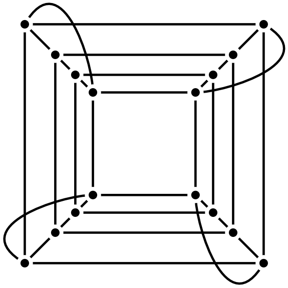

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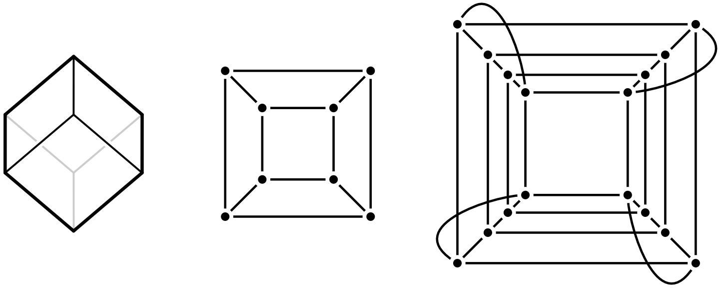

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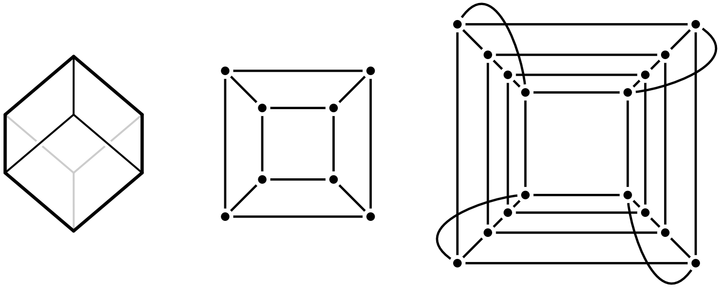

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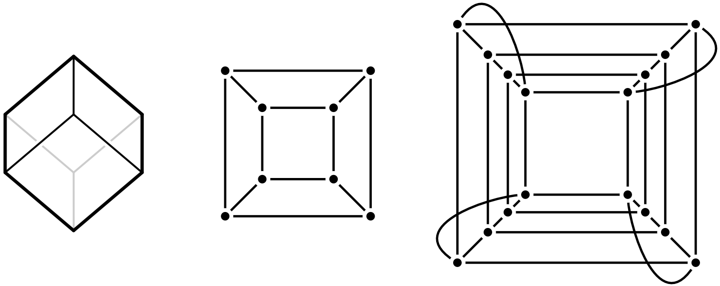
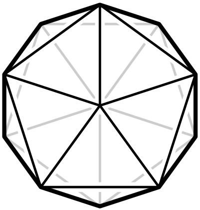


Question: Given the edge-graph, can I reconstruct the polytope up to combinatorial equivalence? **No!** → complete graphs, cube graphs, ...
→ not even up to dimension!

Additional data helps

Reconstruction is possible

- if dimension ≤ 3 (Steinitz)
- if simple (Blind & Mani; later Kalai)
- for zonotope
- for matroid base polytope
- if very symmetric



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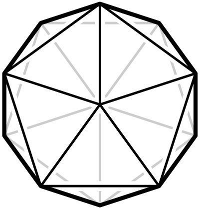

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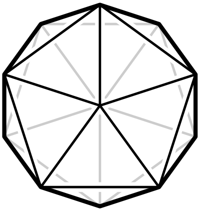

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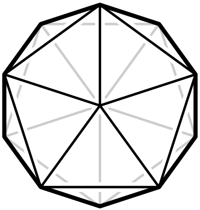

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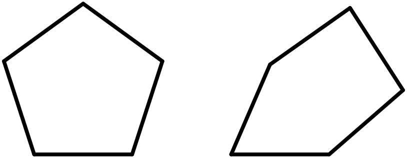

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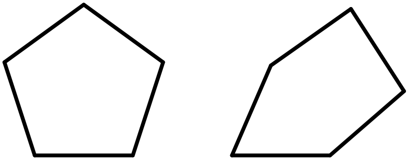

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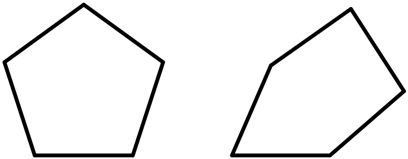

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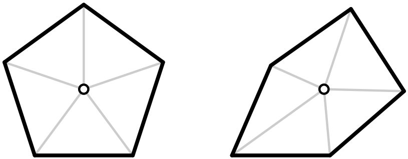

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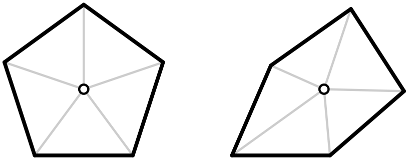

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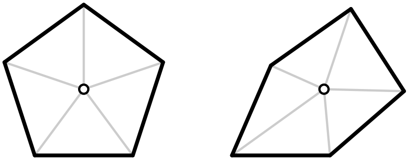

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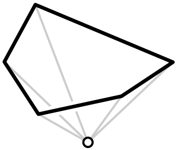
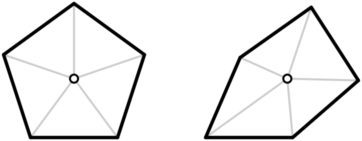
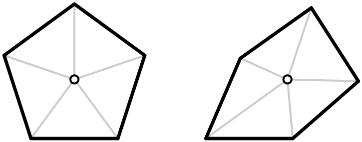




Conjecture.

A polytope can be reconstructed from its edge-graph, its edge lengths and the distances of each vertex from some point in the interior.



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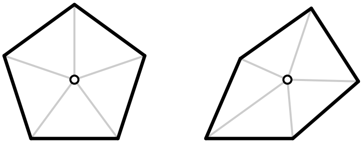

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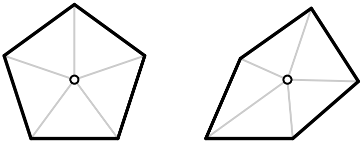
- ... if we fix the combinatorial type (not only the edge-graph).



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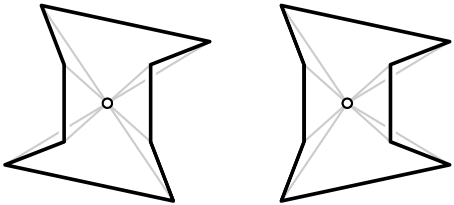
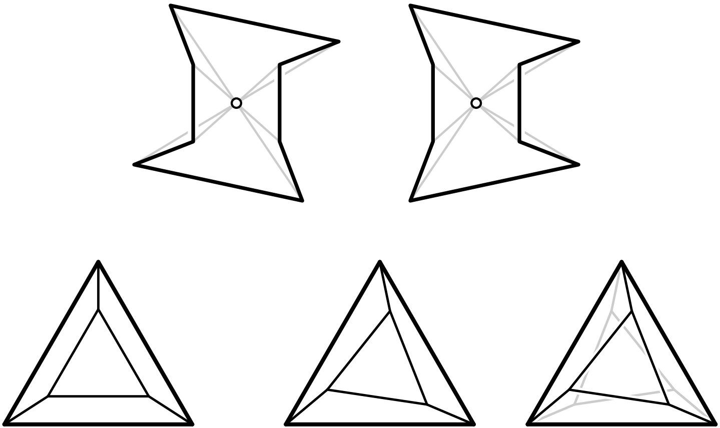
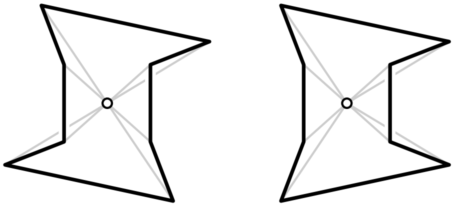
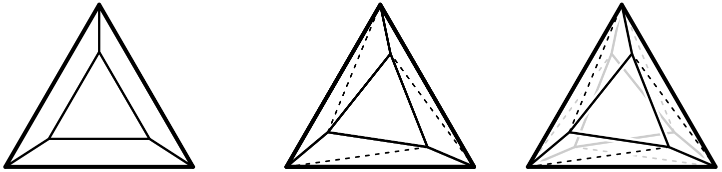
Theorem. (W., 2022+)

Let $P, Q \subset \mathbb{R}^d$ be combinatorially equivalent polytopes with the same edge lengths and vertex-origin distances. If $0 \in \mathrm{int}(Q)$, then $P \simeq Q$.









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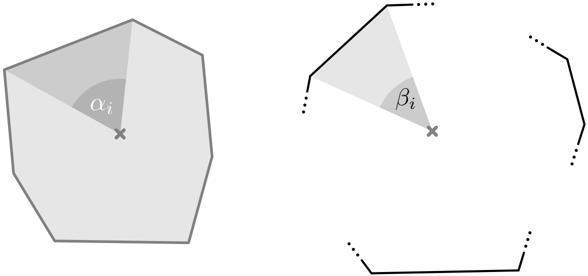
"A polytopes cannot get larger (say, more expanded) if its edges are only getting shorter."

Special case: polygons



$$\alpha_1 + \cdots + \alpha_n = 2\pi, \qquad \beta_1 + \cdots + \beta_n < 2\pi.$$

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Another measure of polytope size

Fix $\alpha \in \Delta_n := \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\}$

$$\alpha\text{-}\textbf{expansion:} \qquad \|P\|_\alpha^2 := \tfrac{1}{2} \sum_{i,j} \alpha_i \alpha_j \|p_i - p_j\|^2$$

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"Polytopes are maximally expanded for their edge-lengths."

The Theorem of Izmestiev

"Polytope skeleta are spectral embeddings."

Theorem. (Izmestiev, 2007)

Given a polytope $P \subset \mathbb{R}^d$ with edge-graph $G = (V, E)$ and $0 \in \mathrm{int}(P)$, then there exists a matrix $M \in \mathbb{R}^{n \times n}$ so that

(i) $M_{ij} > 0$ *whenever* $ij \in E$,

(ii) $M_{ij} = 0$ *whenever* $i \neq j$ *and* $ij \notin E$,

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(iv) $M X_P = 0$, *where* $X_P^\top = (p_1, ..., p_n) \in \mathbb{R}^{d \times n}$,

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$$\begin{aligned}\|P\|_\alpha^2 &= \tfrac{1}{2}\sum_{ij\in E} M_{ij}\|p_i - p_j\|^2 + \mathrm{tr}(MX_PX_P^\top) - \Big\|\sum_i \alpha_i p_i\Big\|^2 \\ &\geq \tfrac{1}{2}\sum_{ij\in E} M_{ij}\|q_i - q_j\|^2 + \mathrm{tr}(MX_QX_Q^\top) - \Big\|\sum_i \alpha_i q_i\Big\|^2 = \|Q\|_\alpha^2\end{aligned}$$

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□

What else ... ?

Corollary.

The dimension of the realization space of P is $\leq f_0 + f_1 - d - 1$.

Conjecture.

A polytope is <u>not</u> determined by its edge-graph and edge-lengths if and only if it has a line segment or n-gon ($n \geq 4$) as Minkowski summand:

$$P = S + P'.$$

Thank you.