Spectral Graph Theory for Polytopes

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A Motivating Example
A curious observation ...

\[ P \subset \mathbb{R}^3 \]
A curious observation ...

\[ P \subset \mathbb{R}^3 \rightarrow G_P = (V, E) \]
A curious observation...

\[ P \subseteq \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A \in \mathbb{R}^{8 \times 8} \]
A curious observation ... 

\[ P \subset \mathbb{R}^3 \rightarrow G_P = (V, E) \rightarrow A \in \mathbb{R}^{8 \times 8} \]
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\[ P \subset \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A \in \mathbb{R}^{8 \times 8} \]
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\[ P \subset \mathbb{R}^3 \quad \rightarrow \quad G_P = (V, E) \quad \rightarrow \quad A \in \mathbb{R}^{8 \times 8} \]
A curious observation ...

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & 1 \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
1 & 1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & -1 & 1 \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1 \\
1 & 1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & -1 & 1 \\
\end{bmatrix} \in \mathbb{R}^8
\]
## A Motivating Example

A curious observation ...

\[ u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \in \mathbb{R}^8 \quad \rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^{8 \times 3} \]
A Motivating Example

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\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & -1 \\
3 & 1 & 1 \\
4 & 1 & 1 \\
5 & -1 & 1 \\
6 & -1 & 1 \\
7 & -1 & 1 \\
8 & -1 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & -1 \\
3 & 1 & -1 \\
4 & 1 & 1 \\
5 & -1 & 1 \\
6 & -1 & 1 \\
7 & -1 & -1 \\
8 & -1 & -1 \\
\end{bmatrix} \in \mathbb{R}^8 \quad \rightarrow \quad \begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & -1 \\
3 & 1 & 1 \\
4 & 1 & -1 \\
5 & -1 & 1 \\
6 & -1 & 1 \\
7 & -1 & -1 \\
8 & -1 & -1 \\
\end{bmatrix} \in \mathbb{R}^{8 \times 3}
\]
A curious observation ...

\[ u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \in \mathbb{R}^8 \quad \rightarrow \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^{8 \times 3} \]

\[ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \]
A Motivating Example

A curious observation ...

\[
\begin{align*}
    u_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
    u_2 &= \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \\
    u_3 &= \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \\
    \in \mathbb{R}^8 &\rightarrow \\
    \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} &\in \mathbb{R}^{8 \times 3}
\end{align*}
\]

\[
\begin{align*}
    v_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
    v_2 &= \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}
\end{align*}
\]
A curious observation ...

\[ u_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad u_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad u_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^8 \rightarrow \end{align*}

\[ u_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad u_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^{8 \times 3} \]

\[ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \ldots \]
A Motivating Example

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\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
\end{bmatrix}
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
\end{bmatrix},
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
\end{bmatrix},
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
\end{bmatrix}
\in \mathbb{R}^8 \rightarrow
\begin{bmatrix}
  u_1 & u_2 & u_3 \\
\end{bmatrix}
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & -1 \\
  1 & -1 & 1 \\
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & -1 \\
  1 & -1 & 1 \\
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & -1 \\
  1 & -1 & 1 \\
\end{bmatrix}
\in \mathbb{R}^{8 \times 3}
\]

\[
\begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
\end{bmatrix}
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
\end{bmatrix},
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
\end{bmatrix},
\begin{bmatrix}
  1 \\
  -1 \\
  -1 \\
\end{bmatrix}
\rightarrow
V \ni i \mapsto v_i \in \mathbb{R}^3
\]

A curious observation ...

Note: Works for many other polytopes too!
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Note: Works for many other polytopes too! But for which?
What I want to say ...

There is a construction ...

\[ G \mapsto P \]

There is a phenomenon ...

\[ P \mapsto G \mapsto P \]
A Motivating Example

What I want to say ...

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**EIGENPOLYTOPE**

There is a phenomenon ...

\[ P \leftrightarrow G \leftrightarrow P \]
What I want to say ...

There is a construction ...

\[ G \mapsto P \]

\[ \text{EIGENPOLYTOPE} \]

There is a phenomenon ...

\[ P \mapsto G \mapsto P \]

\[ \text{SPECTRAL POLYTOPE} \]
What I want to say ...

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There is a phenomenon ...

\[ P \mapsto G \mapsto P \mapsto G \mapsto \cdots \]

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What I want to say ...

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**Eigenpolytope**

There is a phenomenon ...

\[ P \mapsto G \mapsto P \mapsto G \mapsto \cdots \]

**Spectral Polytope**

Eigenpolytopes
Definition. (Eigenpolytope)

- Start with a graph $G = (V, E)$ with $V = \{1, ..., n\}$. 

- Choose one of them, say $\theta_i$ of multiplicity $d$.

- Choose an orthonormal basis $u_1, ..., u_d \in \mathbb{R}^n$ of the $\theta_i$-eigenspace and define the matrix $\Phi = \begin{bmatrix} | & | \\ u_1 & \cdots & u_d \end{bmatrix}$.

- Let $v_i \in \mathbb{R}^d$ be the $i$-th row of $\Phi$.

- The $\theta_i$-eigenpolytope of $G$ is $P_G(\theta_i) := \text{conv} \{v_i \mid i \in V\} \subset \mathbb{R}^d$. 
Definition. (Eigenpolytope)

- Start with a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, whose eigenvalues are $\theta_1 > \theta_2 > \cdots > \theta_m$. 
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$$\Phi := \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix} \in \mathbb{R}^{n \times d}$$
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$$\Phi := \begin{bmatrix} u_1 & \cdots & u_d \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}. $$

- Let $v_i \in \mathbb{R}^d$ be the $i$-th row of $\Phi$. 

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Example: cube graph

\[ \text{Spec("cube graph")} = \{ 3^1, 1^3, (-1)^3, (-3)^1 \} \]
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\[ \text{Spec}("\text{cube graph}") = \{ 3^1, 1^3, (-1)^3, (-3)^1 \} \]
Example: dodecahedron graph

\[ \text{Spec(”dodecahedron graph”) } = \{ 3^1, \sqrt{5}^3, 1^5, 0^4, (-2)^4, (-\sqrt{5})^3 \} \]
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Example: dodecahedron graph

$$\text{Spec("dodecahedron graph")} = \{3^1, \sqrt{5}^3, 1^5, 0^4, (-2)^4, (-\sqrt{5})^3\}$$
Other (non-)examples

Not spectral:

- Most prisms, e.g. edge-graph of triangle prism has spectrum 
  \{3, 1, 1, 0, 2, (−2)^2\}
- Most neighborly polytopes.
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Eigenpolytopes:
Other (non-)examples

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Eigenpolytopes:

- permutahedra (spectral),
Other (non-)examples

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Eigenpolytopes:
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- the Birkhoff polytope,
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Eigenpolytopes:
- permutahedra (spectral),
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- ...

Eigenpolytopes
Balanced and Spectral Polytopes
Balanced polytopes

A balanced polytope $P \subset \mathbb{R}^d$ with edge-graph $G_P = (V, E)$ and vertices $v_i \in \mathbb{R}^d$ for $i \in V$.

**Definition. (balanced polytope)**

$P$ is called **balanced** for some $\theta \in \mathbb{R}$ if

$$\sum_{j \in N(i)} v_j = \theta v_i, \quad \text{for all } i \in V.$$
Balanced and Spectral Polytopes

Balanced polytopes

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\[
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\]

\[
A \Psi = \theta \Psi, \quad \Psi := \begin{bmatrix}
  v_1 \\
  \vdots \\
  v_n
\end{bmatrix}
\]

**Consequences:**

\( \theta \) is an eigenvalue of \( G \), and

\( \text{the columns of } \Psi \text{ are } \theta \text{-eigenvectors, or} \)

\( \text{span } \Psi \subseteq \text{Eig } G(\theta) \).
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\[
\begin{bmatrix}
\vdots \\
v_1 \\
\vdots \\
v_n
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\[
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\vdots \\
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Balanced and Spectral Polytopes

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- the columns of \( \Psi \) are \( \theta \)-eigenvectors, or

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\text{span } \Psi \subseteq \text{Eig}_G(\theta).
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Balanced polytopes

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Consequences:

- $\theta$ is an eigenvalue of $G$, and
- the columns of $\Psi$ are $\theta$-eigenvectors, or

$$\text{span } \Psi = \text{Eig}_G(\theta).$$
Definition. (spectral polytope)

$P$ is called *spectral* for some $\theta \in \mathbb{R}$ if

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Spectral polytopes

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$P$ is called *spectral* for some $\theta \in \mathbb{R}$ if

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$\iff$ $P$ is the eigenpolytope of its edge-graph (in the right way).
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**Theorem.** (W., 2020)

*If $P$ is spectral, then*

...
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**Theorem.** *(W., 2020)*

If $P$ is spectral, then

- $P$ is uniquely determined by its edge-graph (up to scale and orientation),
Spectral polytopes

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If $P$ is spectral, then

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- $P$ realizes all the symmetries of its edge-graph.
Spectral polytopes

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**Theorem.** *(W., 2020)*

If $P$ is spectral, then

- $P$ is uniquely determined by its edge-graph (up to scale and orientation),
- $P$ realizes all the symmetries of its edge-graph.

**Question:** Can we classify spectral polytopes?
Spectral graphs

Definition. (spectral graph)

$G$ is *spectral* if it is the edge-graph of a spectral polytope.
Spectral graphs

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**Spectral graphs**

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**Alternative:**

**Definition.** *(spectral graph, graph theory version)*

$G$ is *spectral* if

$\forall i \in V$ there is an eigenvector $u \in \mathbb{R}^n$ with unique largest component $u_i$,

$\forall i, j \in E \iff$ there is an eigenvector $u \in \mathbb{R}^n$ with only largest components $u_i$ and $u_j$. 
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Spectral graphs

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\[ \iff G \text{ is the edge-graph of its eigenpolytope (in the right way)}. \]

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2. \( ij \in E \iff \) there is an eigenvector \( u \in \mathbb{R}^n \) with only largest components \( u_i \) and \( u_j \).

**Question:** Can we classify spectral graphs?
Characterizing Spectral Polytopes
A geometric characterization

Let $P \subset \mathbb{R}^d$ with edge-graph $G_P = (V, E)$, and vertices $v_i, i \in V$.

$$P^\circ = \{ x \in \mathbb{R}^d \mid \langle v_i, x \rangle \leq 1 \text{ for } i = 1, \ldots, n \}.$$
A geometric characterization

Let $P \subset \mathbb{R}^d$ with edge-graph $G_P = (V, E)$, and vertices $v_i, i \in V$.

$$P^\circ(c) = \{ x \in \mathbb{R}^d \mid \langle v_i, x \rangle \leq c_i \text{ for } i = 1, \ldots, n \}.$$ 

with $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. 
A geometric characterization

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A geometric characterization

Let $P \subset \mathbb{R}^d$ with edge-graph $G_P = (V, E)$, and vertices $v_i, i \in V$.

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![Diagram showing geometric characterization of spectral polytopes](image)
A geometric characterization

Theorem. (W., 2020, based on IZMESTIEV, 2010)

Define the matrix $X \in \mathbb{R}^{n \times n}$ by

$$X_{ij} = \left. \frac{\partial \text{vol}(P^\circ(c))}{\partial c_i \partial c_j} \right|_{c=(1,\ldots,1)}.$$
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If it holds

(i) $X_{ii}$ is independent of $i \in V$, and

(ii) $X_{ij}$ is independent of $ij \in E$.

then $P$ is spectral.
A geometric characterization

**Theorem.** (W., 2020, based on IZMESTIEV, 2010)

Define the matrix $X \in \mathbb{R}^{n \times n}$ by

$$X_{ij} = \frac{\partial \text{vol}(P^c(c))}{\partial c_i \partial c_j} \bigg|_{c=(1,...,1)}.$$

If it holds

(i) $X_{ii}$ is independent of $i \in V$, and
(ii) $X_{ij}$ is independent of $i,j \in E$.

then $P$ is spectral to eigenvalue $\theta_2$. 

Geometric interpretation: $X_{ij} = \text{vol}(\sigma_i \cap \sigma_j) \parallel v_i \parallel \parallel v_j \parallel \sin \angle(v_i, v_j)$, for $ij \in E$.
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$$X_{ij} = \frac{\text{vol}(\sigma_i \cap \sigma_j)}{\|v_i\| \|v_j\| \sin \angle(v_i, v_j)}, \quad \text{for } ij \in E$$

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**Question:** Is this also necessary?
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Define the matrix $X \in \mathbb{R}^{n \times n}$ by

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If it holds

(i) $X_{ii}$ is independent of $i \in V$, and
(ii) $X_{ij}$ is independent of $ij \in E$. \quad \{\text{true if } P \text{ is vertex & edge-transitive}\}

then $P$ is spectral to eigenvalue $\theta_2$.

**Geometric interpretation:** $X_{ij} = \frac{\text{vol}(\sigma_i \cap \sigma_j)}{\|v_i\|\|v_j\| \sin \angle(v_i, v_j)}$, for $ij \in E$

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Question 1

*Is there a spectral graph/polytope to an eigenvalue other than $\theta_2$?*
Question I

Is there a spectral graph/polytope to an eigenvalue other than $\theta_2$?

Theorem. (W., 2020)

Let $P/G$ be spectral. If $G/P$ is edge-transitive, then

- it is spectral to eigenvalue $\theta_2$
Question 1

Is there a spectral graph/polytope to an eigenvalue other than $\theta_2$?

**Theorem.** (W., 2020)

Let $P/G$ be spectral. If $G/P$ is edge-transitive, then

- it is spectral to eigenvalue $\theta_2$, and
- if $P/G$ is not vertex-transitive, then it is the following:

![Diagram of spectral polytopes](image-url)
Question II

Are spectral graphs/polytopes necessarily of high symmetry?
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Are spectral graphs/polytopes necessarily of high symmetry?

**Theorem.** *(Godsil, 1998)*

Let $G$ be spectral. If $G$ is distance-regular, then $G$ is one of the following:

1. a cycle graph,
2. the dodecahedron graph,
3. the icosahedron graph,
4. a crown graph,
5. a Johnson graph,
6. a Hamming graph,
7. a halved cube graph,
8. the Gosset graph,
9. the Schl"afli graph.
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Theorem. (Godsil, 1998)

Let $G$ be spectral. If $G$ is distance-regular, then $G$ is one of the following:

- a cycle graph,
- the dodecahedron graph,
- the icosahedron graph,
- a crown graph,
- a Johnson graph, (v)  a Hamming graph,
- a halved cube graph,
- the Gosset graph, or
- the Schl"afli graph.

all of these are distance-transitive
The End

Questions?
Corollary.

If $P \subset \mathbb{R}^d$ is vertex- and edge-transitive, then

- $P$ is spectral to the eigenvalue $\theta_2$,
- $P$ is uniquely determined by its edge-graph (up to scale and orientation),
- $P$ realizes all the symmetries of its edge-graph,
- $\text{Aut}(P)$ is irreducible as matrix group,
- if $P$ has circumradius $r$ and edge-length $\ell$, then
  \[ \frac{\ell}{r} = \sqrt{2 \left(1 - \frac{\theta_2}{\deg(G_P)}\right)} , \]
- if $P^\circ$ has dihedral angle $\alpha$, then
  \[ \cos(\alpha) = -\frac{\theta_2}{\deg(G_P)} . \]
More general ...

*The symmetry group of $P$ is as large as possible, given the orbits on its 1-skeleton.*

**Theorem.** *(W., 2020+)*

Let $G$ be the orbit-colored edge-graph of $P$. Then $\text{Aut}(G) \cong \text{Aut}(P)$. 