Geometric expansion properties of polytope skeleta

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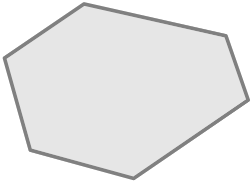
An intuitive problem



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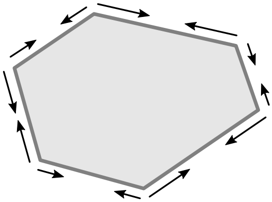

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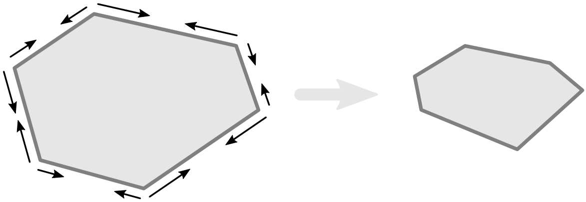

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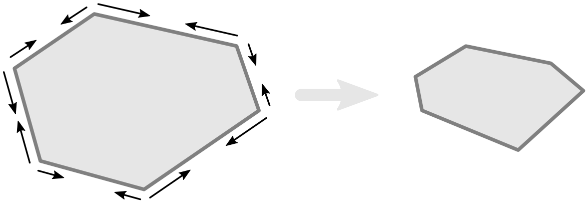

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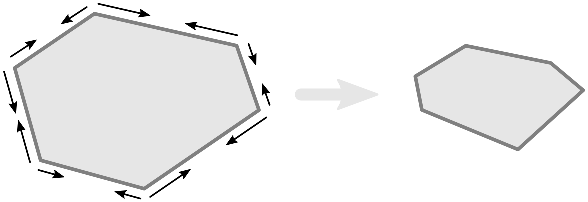

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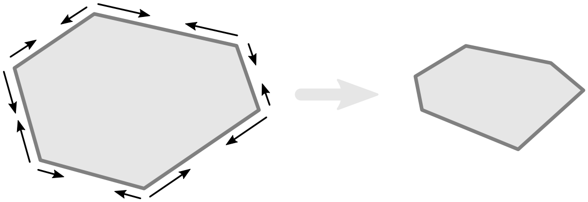

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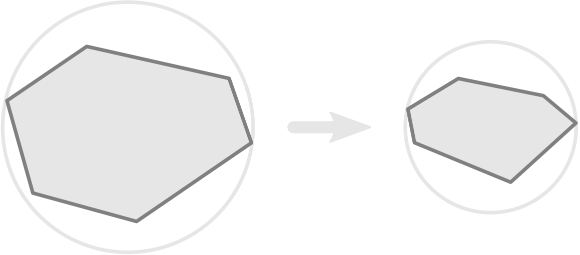

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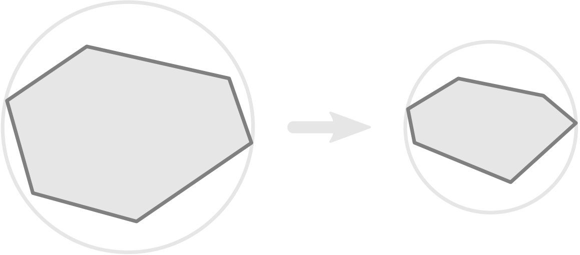

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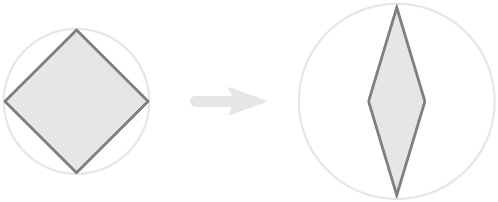

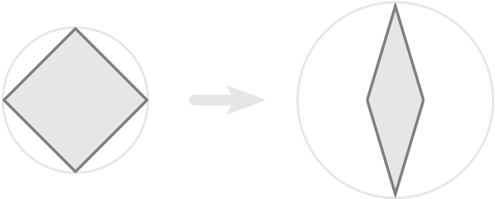

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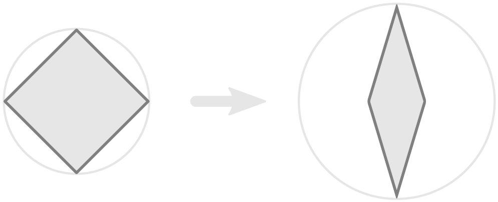

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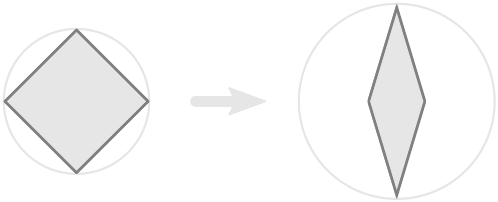

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Setting:

- convex polytope $P = \mathrm{conv}\{v_1, ..., v_n\} \subset \mathbb{R}^d$.

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then the circumradius of Q is at most as large as the circumcenter of P.

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then $r(Q) \leq r(P)$.

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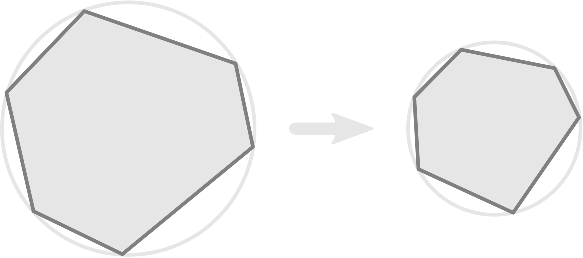

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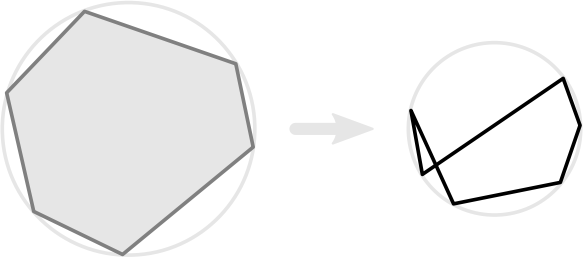

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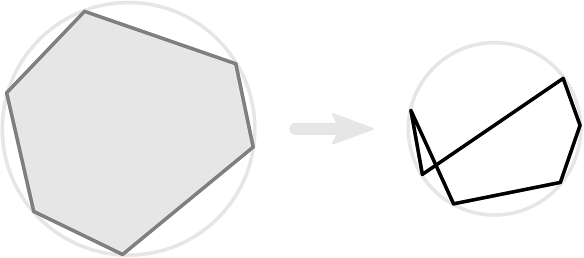

graph embedding $w\colon V \to \mathbb{R}^d$.

Case $d = 2$: polygon skeletons

Proof sketch.

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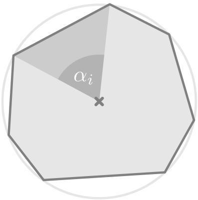

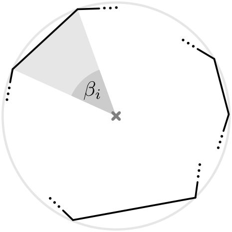

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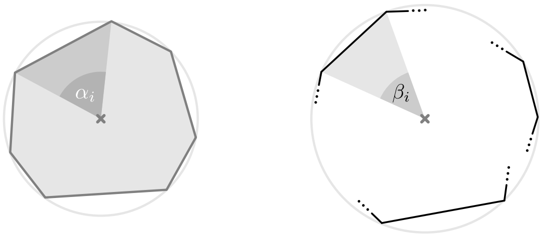

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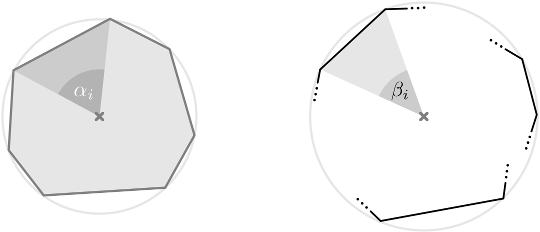

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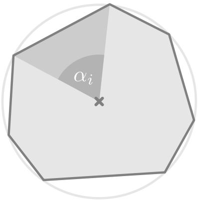

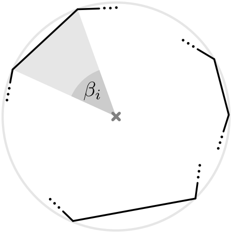

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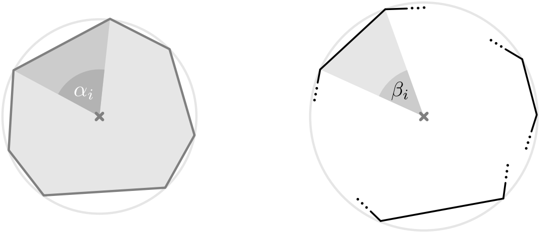

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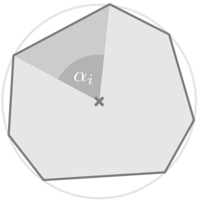

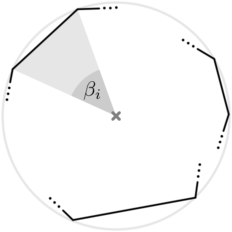

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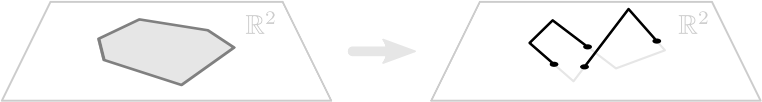

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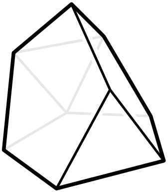

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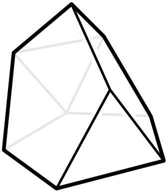

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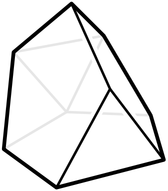

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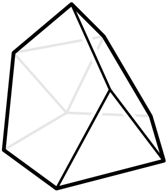

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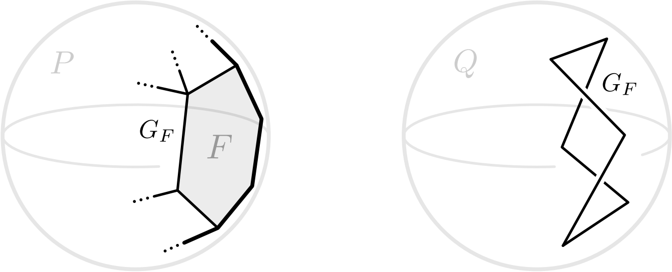

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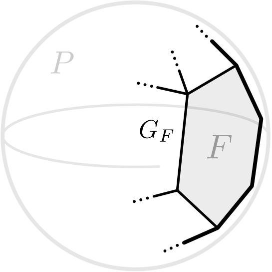

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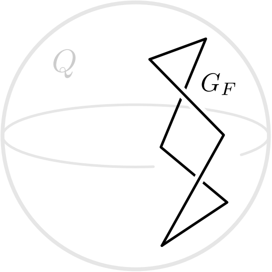

G_F "weirdly" embedded in Q

The embedding of G_F in Q must be "less expanded" than F.

Higher dimensions: the Kirszbraun Theorem

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We want to show: polytope edges are enough!

Another measure for expansion

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Definition.

An m-centered $v\colon V \to \mathbb{R}^d$ is m-**expanded** if for every m-centered $w\colon V \to \mathbb{R}^f$ with

$$\|w_i - w_j\| \le \|v_i - v_j\|, \quad \text{for all } ij \in E$$

holds $\|w\|_m \le \|v\|_m$.

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(iii) M has single positive eigenvalue $\theta > 0$.

Constructing expanded embeddings

Definition.

A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is a **promising matrix** for G if

(i) $M_{ij} = 0$ whenever $i \neq j$ and $ij \notin E$,

(ii) $M_{ij} \geq 0$ whenever $ij \in E$,

(iii) M has single positive eigenvalue $\theta > 0$.

Example: (negated) Colin de Verdière matrices.

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If some $w\colon V \to \mathbb{R}^d$ has the same m-expansion as v, then

- *w has the same edge-lengths as v, and*
- *w is a linear transformation of v.*

So, what about inscribed polytopes?

If P is inscribed with center $0 \in P$, then

$$\|v\|_m^2 = \sum_i m_i \|v_i\|^2 = r(v)^2 \cdot \underbrace{\Big(\sum_i m_i\Big)}_{\mu}.$$

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If $w\colon V \to \mathbb{R}^f$ is inscribed with shorter edges, then

$$\mu r(w) = \|w\|_m^2 \leq \|v\|_m^2 = \mu r(v) \quad \Longrightarrow \quad r(w) \leq r(v).$$

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Theorem. (W., 2022+)

Given an inscribed polytope $P \in \mathbb{R}^d$ and a centrally symmetric inscribed embedding $w\colon V \to \mathbb{R}^f$ so that

- *each edge in w is at most as long as its counterpart in P,*

then $r(w) \le r(P)$.

Thank you.