(Random) Trees of Intermediate Volume Growth

Martin Winter
(joint work with George Kontogeorgiou)

University of Warwick

12. July, 2022
Volume Growth in Graphs

$|B_v(r)|$
**Volume growth**

ball \( B_v(r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \} \)

\[ |B_v(0)| = 1 \]
Volume growth

ball \ldots \ B_v(r) := \{ x \in V(G) \mid \text{dist}(x,v) \leq r \}

|B_v(1)| = 5
Volume growth

$$B_v(r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \}$$

$$|B_v(2)| = 13$$
Volume growth

\[ B_v(r) := \{ x \in V(G) \mid \text{dist}(x, v) \leq r \} \]

\[ |B_v(3)| = 25 \]
Examples: polynomial and exponential

Application: geometric group theory → Cayley graphs
**Arbitrary growth at a vertex**

For each *strictly increasing* function $g: \mathbb{N}_0 \to \mathbb{N}$ it is easy to find a graph $G$ with

$$|B_v(r)| = g(r), \quad \text{for all } r \geq 0$$

at a fixed vertex $v \in V(G)$.
Uniform growth

Target growth: \( g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \)

**Definition.**

A graph \( G \) is of **uniform volume growth** \( g \) if there are \( r_0 \geq 1 \) and \( c_1, c_2, C_1, C_2 > 0 \) so that for all \( v \in V(G) \) and \( r \geq r_0 \)

\[
C_1 \cdot g(c_1 r) \leq |B_v(r)| \leq C_2 \cdot g(c_2 r).
\]

We write \( |B_v(r)| \sim g(r) \).

**Note:** this is different from \( |B_v(r)| = \theta(g) \).
**Uniform growth**

\[ |B_v(r)| \sim r^2 \]

but

\[ |B_v(r)| \not\sim r^2 \]

\[ |B_v(r)| = \theta(r^2) \]
Growth of planar graphs

Two interesting classes:
- planar triangulations
- trees
GROWTH OF PLANAR GRAPHS

Two interesting classes:

- planar triangulations
- trees
Trees
Uniform growth of trees

What kind of uniform growth can a tree have?

- linear ✓
- exponential ✓
Uniform growth of trees

What kind of uniform growth can a tree have?

- linear ✓
- exponential ✓
- polynomial ??
- intermediate ??
- oscillating ??

(Benjamini, Schramm; 2001)

$$|B_v(r)| \sim r^{\alpha}, \text{ where } \alpha = \log |E| \log d = \log 5 \log \frac{2}{3} \approx 1.464973.$$
Uniform growth of trees

What kind of uniform growth can a tree have?

- linear ✓
- exponential ✓
- polynomial ✓
- intermediate ??
- oscillating ??

\[ |B_v(r)| \sim r^\alpha, \quad \text{where} \quad \alpha = \frac{\log |E|}{\log d} = \frac{\log 5}{\log 3} \approx 1.464973. \]

(Benjamini, Schramm; 2001)
**The Question**

Q: “Are there random trees of uniform intermediate volume growth?”

– Itai Benjamini

super-polynomial: $e^{\omega(\log(r))}$

sub-exponential: $e^{o(r)}$

\[ \downarrow \]
The Question

Q: “Are there random trees of uniform intermediate volume growth?”

– Itai Benjamini

super-polynomial: \( e^{\omega(\log(r))} \)
sub-exponential: \( e^{o(r)} \)
The Question

Q: “Are there random trees of uniform intermediate volume growth?”

– Itai Benjamini

Why could there be doubt?

- intermediate growth is known to be a delicate issue in other settings
- e.g. Cayley graphs → Grigorchuk group
- no intermediate growth Cayley graph is a tree
They exist!
They exist!
The Construction

$T_0 \subset T_1 \subset T_2 \subset T_3 \subset \cdots$
CONSTRUCTION — A SEQUENCE OF TREES

Given: sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$T_0$
Construction – A Sequence of Trees

Given: sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$

$T_0$
CONSTRUCTION – A SEQUENCE OF TREES

Given: sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$
**Construction – A Sequence of Trees**

**Given:** sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2$ 

$T_1$
CONSTRUCTION — A SEQUENCE OF TREES

Given: sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$
CONSTRUCTION – A SEQUENCE OF TREES

Given: sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$
**Construction — a sequence of trees**

**Given:** sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2$ 3 4 5
**Construction – a sequence of trees**

**Given:** sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$
**CONSTRUCTION — A SEQUENCE OF TREES**

**Given:** sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$
**Construction — A Sequence of Trees**

**Given:** sequence $\delta_1, \delta_2, \delta_3, \ldots \in \mathbb{N}$, $\delta_n \geq 1$

$\delta_n := n + 2 \quad 3 \quad 4 \quad 5$

$$T := \bigcup_{n} T_n$$
**Heuristics Argument**

Properties of $T_n$:

- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
- distance from root to apocentric vertex: $2^n - 1$
Heuristics argument

Properties of $T_n$:
- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
- distance from root to apocentric vertex: $2^n - 1$

$$|B_v(2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1)$$
HEURISTICS ARGUMENT

Properties of $T_n$:

- number of vertices: $(\delta_1 + 1) \cdots (\delta_n + 1)$
- distance from root to apocentric vertex: $2^n - 1$

$$|B_v(2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1) \quad \implies \quad \delta_n \approx \frac{g(2^n - 1)}{g(2^{n-1} - 1)} - 1$$
Example: polynomial growth

\[ |B_v(r)| = (r + 1)^2 \]

\[ \delta_n := 3 \]
**Example: Polynomial Growth**

\[ |B_v(r)| = (r + 1)^2 \implies |B_v(2^n - 1)| = (2^n)^2 = 4^n = (3 + 1) \cdots (3 + 1) \]

\[ \delta_n := 3 \]
**Example: exponential growth**

\[ \delta_n := 2^{2^n} \]

\[ |B_v(2^n - 1)| = (\delta_1 + 1) \cdots (\delta_n + 1) = \prod_{k=1}^{n} \left( 2^{2^{k-1}} + 1 \right) = \sum_{i=0}^{2^n-1} 2^i = 2 \cdot 2^{2^n-1} - 1 \]
Example: intermediate growth?

\[ \delta_n := n + 2 \]

\[ |B_v(r)| \sim r^{\log \log r} \]
Main Result

For every function $g: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ there is a tree of uniform growth $g$. 
For every function? Almost ...

\( g \) increasing

“\( g \) at least linear”

“\( g \) at most exponential”
For every function? Almost ...

$g$ increasing

“$g$ at least linear”

“$g$ at most exponential”
Main Result

**For every function? Almost ...**

\[ g \text{ increasing} \]

\[ "g \text{ at least linear"} \iff \delta_n \geq 1 \]

\[ "g \text{ at most exponential"} \]
For every function? Almost ... 

\( g \) increasing

"\( g \) at least linear" \( \iff \delta_n \geq 1 \iff g \) super-additive

"\( g \) at most exponential"
For every function? Almost ...

$g$ increasing

“$g$ at least linear” $\iff$ $\delta_n \geq 1$ $\iff$ $g$ super-additive

“$g$ at most exponential” $\iff$ $T$ of bounded degree
For every function? Almost ...

\( g \) increasing

“\( g \) at least linear” \( \iff \delta_n \geq 1 \iff \) \( g \) super-additive

“\( g \) at most exponential” \( \iff T \) of bounded degree

\[
\Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}
\]

\( \bar{\Delta} := \sup_n [\Delta(n)] < \infty \iff \delta_{n+1} \leq \bar{\Delta} \cdot \delta_1 \cdots \delta_n \)
Main Result: \( T \) has uniform growth

\[
\Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}, \quad \bar{\Delta} := \sup_n \Delta(n), \quad \Gamma := \sup_{m \geq n} \left[ \frac{\Delta(m)}{\Delta(n)} \right].
\]

**Theorem.** (Kontogeorgiou, W.; 2022+)

For super-additive \( g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) exists a tree \( T \) so that for all \( v \in V(T) \) and \( r \geq r_0 \)

\[
|B_v(r)| \geq C_1 \cdot g(r/4)
\]

if \( \bar{\Delta} < \infty \) then

\[
|B_v(r)| \leq C_2 \cdot g(2r)^2
\]

if \( \Gamma < \infty \) then

\[
|B_v(r)| \leq C_3 \cdot g(4r)
\]

In particular, if \( \Gamma < \infty \), then \( T \) is of uniform growth \( g \).
Main Result

**Main result:** $T$ has uniform growth

$$\Delta(n) := \frac{\delta_n}{\delta_1 \cdots \delta_{n-1}}, \quad \tilde{\Delta} := \sup_n [\Delta(n)], \quad \Gamma := \sup_{m \geq n} \left[ \frac{\Delta(m)}{\Delta(n)} \right].$$

**Theorem.** (Kontogeorgiou, W.; 2022+)

For super-additive $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ exists a tree $T$ so that for all $v \in V(T)$ and $r \geq r_0$

$$|B_v(r)| \geq C_1 \cdot g(r/4)$$

if $\tilde{\Delta} < \infty$ then

$$|B_v(r)| \leq C_2 \cdot g(2r)^2$$

if $\Gamma < \infty$ then

$$|B_v(r)| \leq C_3 \cdot g(4r)$$

In particular, if $\Gamma < \infty$, then $T$ is of uniform growth $g$.

**Theorem.**

If $g$ is super-additive and log-concave, then there is a tree of uniform growth $g$. 
Random Trees
**The original question**

**Q:** “Are there random trees of uniform intermediate volume growth?”

“unimodular random rooted trees”

– Itai Benjamini
**The original question**

Q: “Are there random trees of uniform intermediate volume growth?”

“unimodular random rooted trees”

\[ T_0, T_1, T_2, T_3, \ldots \xrightarrow{\text{BS}} \mathcal{T} \]

**Benjamini-Schramm limit** (BS)... graph limit for graphs of bounded degree
**Two interesting facts**

**A threshold phenomenon:**

**Theorem.**

*If* \( \delta_n \geq n^\alpha \) *for some* \( \alpha > 1 \), *then* \( T \) *is a.s. 1-ended.*

\[
|B_v(r)| \sim g(r) = \Omega(r^{\alpha \log \log r})
\]

**A deterministic tree:**

**Theorem.**

*If* \( \delta_{n+1} = \delta_n \delta_{n-1} \) *then* \( T \) *is a deterministic unimodular tree.*

\[
|B_v(r)| \sim \exp(r^{1/\sqrt{5}})
\]
Open Questions
Just one question

Question.

If $G$ has uniform volume growth $g$, does $G$ have a spanning tree of the same uniform volume growth $g$?

Note: true for lattice graphs
Thank you.