Taylor based nonequispaced fast Fourier transform

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Road map

Introduction

- $\mathbb{T}^d \simeq [0,1)^d$, $f\colon \mathbb{T}^d \to \mathbb{C}$ multivariate continuous function
- approximate f using a Fourier partial sum

$$\tilde{S}_{\mathrm{I}} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \mathrm{I} \subset \mathbb{Z}^d,\ |\mathrm{I}| < \infty$$

of approximated Fourier coefficients $\hat{\tilde{f}}_{\mathbf{k}}$ computed from L sampling values $f(\mathbf{y}_\ell)$, $L \geq |\mathrm{I}|$,

$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathrm{I}} := \operatorname*{arg\,min}_{\hat{\mathbf{g}}\in\mathbb{C}^{|\mathrm{I}|}} \|\mathbf{A}\,\hat{\mathbf{g}} - \mathbf{f}\|_2,$$

where $\mathbf{A} := (\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_\ell})_{\ell=0,\dots,L-1;\ \mathbf{k}\in\mathrm{I}}$ and $\mathbf{f} := f(\mathbf{y}_\ell)_{\ell=0}^{L-1}$

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- if $\mathbf{A}$ has full column rank $|\mathrm{I}|$:
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- fast (and stable) method for computing $\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathrm{I}}$?

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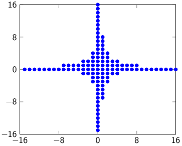

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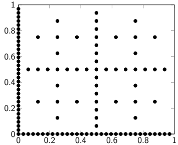

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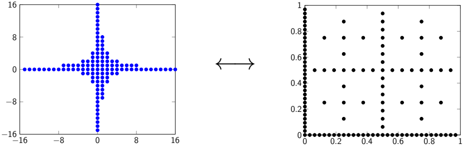

  - compute $(\hat{\tilde{f}}_{\mathbf{k}})_{\mathbf{k}\in\tilde{H}_N^d}$ in $\mathcal{O}\left(N \log^d N\right)$ arithmetic operations

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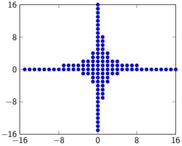

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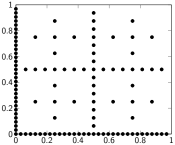

  - compute $(\hat{\hat{f}}_\mathbf{k})_{\mathbf{k}\in\tilde{H}_N^d}$ in $\mathcal{O}\left(N \log^d N\right)$ arithmetic operations
  - but numerically unstable, $\kappa(\mathbf{A})$ scales approx. like $\sqrt{|\tilde{H}_N^d|}$
    (Kämmerer, Kunis 2011)

Approx. reconstruction - index sets and function spaces

- weighted frequency index set $\mathcal{I}_N^{d,T}$, $N \geq 1$, $T \in [-\infty, 1)$,
  (similar index set in Griebel, Hamaekers 2013)

$$\mathcal{I}_N^{d,T} := \begin{cases} \left\{ \mathbf{k} \in \mathbb{Z}^d : \frac{\prod_{s=1}^{d} \max(1,|k_s|)}{\max(1,\|\mathbf{k}\|_1)^T} \leq N^{1-T} \right\}, & T > -\infty, \\ \left\{ \mathbf{k} \in \mathbb{Z}^d : \max(1, \|\mathbf{k}\|_1) \leq N \right\}, & T = -\infty \end{cases}$$

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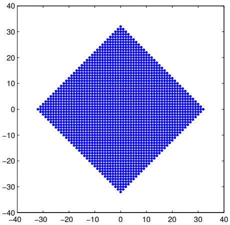

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ℓ_1 ball

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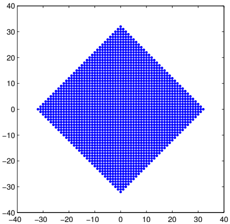

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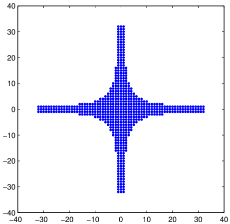

$T = 0$

symmetric hyperbolic cross

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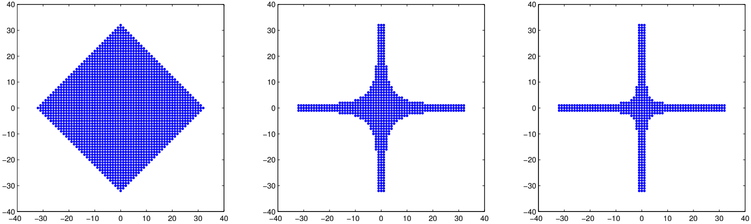

$T = -\infty$ | $T = 0$ | $T = 1/2$
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- Sobolev spaces of isotropic and mixed smoothness
  $\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\| < \infty \right\}$, $\beta \geq 0$, $\alpha > -\beta$,

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Aim:

- fast method and error estimates
  for approximating $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$ from L sampling values $f(\mathbf{y}_\ell)$
  at (perturbed) rank-1 lattice nodes $\mathbf{y}_\ell$

Approx. reconstruction - rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$
  recall: $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$ reconstructing rank-1 lattice on $\mathcal{I}_N^{d,T}$
  (Kämmerer 2012)
  - $\Lambda(\mathbf{z}, M) := \{\mathbf{x}_j\}_{j=0}^{M-1}$, $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$, $\mathbf{x}_j := \frac{j\mathbf{z}}{M} \bmod \mathbf{1}$

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  - allows exact and perfectly stable reconstruction of Fourier coefficients $\hat{p}_{\mathbf{k}}$ of trigonometric polynomial

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

from sampling values $p(\mathbf{x}_j)$, $x_j \in \Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$, i.e.,

$$\mathbf{A}^*\mathbf{A} = M\mathbf{E} \Longrightarrow \underbrace{\mathbf{A}^*\mathbf{A}}_{M\mathbf{E}} (\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathcal{I}_N^{d,T}} = \mathbf{A}^*\mathbf{f},$$

where $\mathbf{A} := \left(\mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}_j}\right)_{j=0,\ldots,M-1;\; \mathbf{k} \in \mathcal{I}_N^{d,T}}$

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  - allows exact and perfectly stable reconstruction of Fourier coefficients $\hat{p}_{\mathbf{k}}$ of trigonometric polynomial

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

    from sampling values $p(\mathbf{x}_j)$, $x_j \in \Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$, i.e.,

$$\mathbf{A}^*\mathbf{A} = M\mathbf{E} \Longrightarrow \underbrace{\mathbf{A}^*\mathbf{A}}_{M\mathbf{E}} (\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathcal{I}_N^{d,T}} = \mathbf{A}^*\mathbf{f},$$

    where $\mathbf{A} := \left(\mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}_j}\right)_{j=0,\ldots,M-1;\ \mathbf{k} \in \mathcal{I}_N^{d,T}}$
  - construction of $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$ with component-by-component search, $M \leq |\mathcal{I}_N^{d,T}|^2$

Approx. reconstruction - rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$
- approximate f using a Fourier partial sum

$$\tilde{S}_{\mathcal{I}_N^{d,T}} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}},$$

$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{I}_N^{d,T}} := \operatorname*{arg\,min}_{\hat{\mathbf{g}} \in \mathbb{C}^{\left|\mathcal{I}_N^{d,T}\right|}} \|\mathbf{A}\,\hat{\mathbf{g}} - \mathbf{f}\|_2 = \frac{1}{M} \mathbf{A}^* \mathbf{f},$$

where $\mathbf{f} := f(\mathbf{x}_j)_{j=0}^{M-1}$, $\mathbf{x}_j := (j\mathbf{z}/M) \bmod 1$

Approx. reconstruction - rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$
- approximate f using a Fourier partial sum

$$\tilde{S}_{\mathcal{I}_N^{d,T}} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}},$$

$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{I}_N^{d,T}} := \operatorname*{arg\,min}_{\hat{\mathbf{g}} \in \mathbb{C}^{\left|\mathcal{I}_N^{d,T}\right|}} \|\mathbf{A}\,\hat{\mathbf{g}} - \mathbf{f}\|_2 = \frac{1}{M}\mathbf{A}^*\mathbf{f},$$

where $\mathbf{f} := f(\mathbf{x}_j)_{j=0}^{M-1}$, $\mathbf{x}_j := (j\mathbf{z}/M) \bmod 1$

- compute $\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{I}_N^{d,T}}$ with 1dim FFT(M),
  arithmetic complexity $\mathcal{O}\left(|\mathcal{I}_N^{d,T}|^2 \log |\mathcal{I}_N^{d,T}|\right)$

Approx. reconstruction - rank-1 lattice nodes

Theorem (Kämmerer, Potts, V. 2013)

Let a function $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, a weighted frequency index set $\mathcal{I}_N^{d,T}$ and a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$ be given, where $\beta \geq 0$, $\alpha > -\beta$, $N \geq 1$, $T < 1$.
Then, the approximation error is bounded by

$$\begin{aligned} &\|f - \tilde{S}_{\mathcal{I}_N^{d,T}} f | L^\infty(\mathbb{T}^d)\| \\ \leq \quad & 2\, N^{-(\alpha+\beta)}\, \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\| \\ & \cdot \begin{cases} N^{\frac{d-1}{d-T}(T\beta+\alpha)}, & T > -\frac{\alpha}{\beta}, \\ 1, & T = -\frac{\alpha}{\beta}, \\ d^{-\frac{T\beta+\alpha}{1-T}}, & T < -\frac{\alpha}{\beta}. \end{cases} \end{aligned}$$

Taylor and rank-1 lattice based NFFT - Method

- Let $\mathcal{I}_N^{d,T}$, $m \in \mathbb{N}$, $\Lambda(\mathbf{z}, M)$ be given. Approximate

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

  by Taylor expansion (idea based on Anderson, Dahleh 1996; Kunis 2008)

$$s_m(\mathbf{x}) \quad = \quad p(\mathbf{x}_{j'}) + \sum_{0 < |\boldsymbol{\nu}| < m} \frac{(\mathbf{x} - \mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \, (D^{\boldsymbol{\nu}} p)(\mathbf{x}_{j'})$$

  at $\mathbf{x}_{j'} = \underset{\mathbf{x}_j \in \Lambda(\mathbf{z}, M)}{\arg\min} \; \min_{\mathbf{h} \in \mathbb{Z}^d} \|\mathbf{x} - \mathbf{x}_j + \mathbf{h}\|_\infty$

- $\mathbf{x} := (x_1, \ldots, x_d)^\top$, $\boldsymbol{\nu} := (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$, $|\boldsymbol{\nu}| := \nu_1 + \ldots + \nu_d$, $D^{\boldsymbol{\nu}} p := \frac{\partial^{\nu_1}}{\partial {x_1}^{\nu_1}} \ldots \frac{\partial^{\nu_d}}{\partial {x_d}^{s\nu_d}}$, $\boldsymbol{\nu}! := \nu_1! \cdot \ldots \cdot \nu_d!$, $\mathbf{x}^{\boldsymbol{\nu}} := {x_1}^{\nu_1} \cdot \ldots \cdot {x_d}^{\nu_d}$

Taylor and rank-1 lattice based NFFT - Method

- Let $\mathcal{I}_N^{d,T}$, $m \in \mathbb{N}$, $\Lambda(\mathbf{z}, M)$ be given. Approximate

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

by Taylor expansion (idea based on Anderson, Dahleh 1996; Kunis 2008)

$$\begin{aligned} s_m(\mathbf{x}) &= p(\mathbf{x}_{j'}) + \sum_{0<|\boldsymbol{\nu}|<m} \frac{(\mathbf{x} - \mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} (D^{\boldsymbol{\nu}} p)(\mathbf{x}_{j'}) \\ &= \sum_{0\le|\boldsymbol{\nu}|<m} \frac{(\mathbf{x} - \mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} (2\pi \mathrm{i} \mathbf{k})^{\boldsymbol{\nu}} \, \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}_{j'}} \end{aligned}$$

at $\mathbf{x}_{j'} = \underset{\mathbf{x}_j \in \Lambda(\mathbf{z}, M)}{\arg\min} \; \min_{\mathbf{h} \in \mathbb{Z}^d} \|\mathbf{x} - \mathbf{x}_j + \mathbf{h}\|_\infty$

- $\mathbf{x} := (x_1, \ldots, x_d)^\top$, $\boldsymbol{\nu} := (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$, $|\boldsymbol{\nu}| := \nu_1 + \ldots + \nu_d$, $D^{\boldsymbol{\nu}} p := \frac{\partial^{\nu_1}}{\partial {x_1}^{\nu_1}} \ldots \frac{\partial^{\nu_d}}{\partial {x_d}^{s\nu_d}}$, $\boldsymbol{\nu}! := \nu_1! \cdot \ldots \cdot \nu_d!$, $\mathbf{x}^{\boldsymbol{\nu}} := {x_1}^{\nu_1} \cdot \ldots \cdot {x_d}^{\nu_d}$

Taylor and rank-1 lattice based NFFT - Method

- Let $\mathcal{I}_N^{d,T}$, $m \in \mathbb{N}$, $\Lambda(\mathbf{z}, M)$ be given. Approximate

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

  by Taylor expansion (idea based on Anderson, Dahleh 1996; Kunis 2008)

$$\begin{aligned} s_m(\mathbf{x}) &= p(\mathbf{x}_{j'}) + \sum_{0<|\boldsymbol{\nu}|<m} \frac{(\mathbf{x}-\mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} (D^{\boldsymbol{\nu}} p)(\mathbf{x}_{j'}) \\ &= \sum_{0\le|\boldsymbol{\nu}|<m} \frac{(\mathbf{x}-\mathbf{x}_{j'})^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} (2\pi \mathrm{i} \mathbf{k})^{\boldsymbol{\nu}} \, \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}_{j'}} \end{aligned}$$

  at $\mathbf{x}_{j'} = \operatorname*{arg\,min}_{\mathbf{x}_j \in \Lambda(\mathbf{z},M)} \min_{\mathbf{h} \in \mathbb{Z}^d} \|\mathbf{x} - \mathbf{x}_j + \mathbf{h}\|_\infty$

- For fixed $\boldsymbol{\nu} \in \mathbb{N}_0^d$, compute $(D^{\boldsymbol{\nu}} p)(\mathbf{x}_j) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} (2\pi \mathrm{i} \mathbf{k})^{\boldsymbol{\nu}} \, \hat{p}_{\mathbf{k}} \, \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}_j}$

  for all $\mathbf{x}_j \in \Lambda(\mathbf{z}, M)$ with 1dim FFT(M) in $\mathcal{O}(d\,|\mathcal{I}_N^{d,T}| + M \log M)$

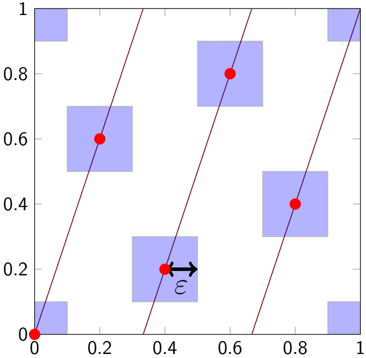
- $(s_m(\mathbf{y}_\ell))_{\ell=0}^{L-1}$: arithmetic complexity $\mathcal{O}\left(m^d (d\,|\mathcal{I}_N^{d,T}| + M \log M + L)\right)$

Taylor and rank-1 lattice based NFFT - Error estimates

- Let $\Lambda(\mathbf{z}, M)$ be given.
- set of admissible evaluation nodes

$$\mathcal{Y}_\varepsilon := \{\mathbf{x} \in \mathbb{T}^d \colon \exists \mathbf{x}_{j'} \in \Lambda(\mathbf{z}, M) \colon \min_{\mathbf{k} \in \mathbb{Z}^d} \|\mathbf{x} - \mathbf{x}_{j'} + \mathbf{k}\|_\infty \leq \varepsilon\},$$

for a parameter $\varepsilon \geq 0$



$\mathbf{z} = (1, 3)^\top$, $M = 5$, $\varepsilon = 0.1$

Taylor and rank-1 lattice based NFFT - Error estimates

Theorem (V. 2013)

Let $\mathcal{I}_N^{d,T}$, $p(\mathbf{x}) := \sum_{\mathbf{k}\in\mathcal{I}_N^{d,T}} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}}$, $\Lambda(\mathbf{z}, M)$,
$\mathcal{Y}_\varepsilon := \{\mathbf{x}\in\mathbb{T}^d \colon \exists \mathbf{x}_{j'}\in\Lambda(\mathbf{z},M)\colon\ \min_{\mathbf{h}\in\mathbb{Z}^d}\|\mathbf{x}-\mathbf{x}_{j'}+\mathbf{h}\|_\infty \le \varepsilon\}$
as well as $\beta\ge 0$ *and* α, $0<\alpha+\beta\le m$, *be given, where* $N\ge 1$, $\hat{p}_{\mathbf{k}}\in\mathbb{C}$, $T<1$, $\varepsilon\ge 0$, $m\in\mathbb{N}$. *Then, for the approximate evaluation of the trigonometric polynomial* p *by a truncated Taylor series* s_m *at nodes* $\mathbf{y}\in\mathcal{Y}_\varepsilon$, *the remainder is bounded by*

$$|(p-s_m)(\mathbf{y})| \le \frac{(2\pi)^m}{m!} d^{\frac{m}{1-T}} \varepsilon^m N^{m-\alpha-\beta} \sum_{\mathbf{k}\in\mathcal{I}_N^{d,T}} |\hat{p}_{\mathbf{k}}|\, \omega^{\alpha,\beta}(\mathbf{k}) \cdot \begin{cases} N^{\frac{d-1}{d-T}(T\beta+\alpha)}, & T>-\frac{\alpha}{\beta},\\ 1, & T=-\frac{\alpha}{\beta},\\ d^{-\frac{T\beta+\alpha}{1-T}}, & T<-\frac{\alpha}{\beta}. \end{cases}$$

Approx. reconstruction - perturbed rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$

Approx. reconstruction - perturbed rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$
- approximate f using a Fourier partial sum

$$\tilde{S}_{\mathcal{I}_N^{d,T}} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

of approximated Fourier coefficients $\hat{\tilde{f}}_{\mathbf{k}}$ computed from M sampling values $f(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$,

$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{I}_N^{d,T}} := \underset{\hat{\mathbf{g}} \in \mathbb{C}^{\left|\mathcal{I}_N^{d,T}\right|}}{\arg\min} \|\tilde{\mathbf{A}}\, \hat{\mathbf{g}} - \mathbf{f}\|_2,$$

where $\tilde{\mathbf{A}}$ is approximation of $\mathbf{A} := (\mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{y}_\ell})_{\ell=0,\dots,L-1;\, \mathbf{k} \in \mathrm{I}}$ using Taylor and rank-1 lattice based NFFT, $\mathbf{f} := f(\mathbf{y}_j)_{j=0}^{M-1}$

Approx. reconstruction - perturbed rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$
- approximate f using a Fourier partial sum
$$\tilde{S}_{\mathcal{I}_N^{d,T}} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{I}_N^{d,T}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$
of approximated Fourier coefficients $\hat{\tilde{f}}_{\mathbf{k}}$ computed from M sampling values $f(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$,
$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathcal{I}_N^{d,T}} := \underset{\hat{\mathbf{g}}\in\mathbb{C}^{\left|\mathcal{I}_N^{d,T}\right|}}{\arg\min} \|\tilde{\mathbf{A}}\,\hat{\mathbf{g}} - \mathbf{f}\|_2,$$
where $\tilde{\mathbf{A}}$ is approximation of $\mathbf{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_\ell}\right)_{\ell=0,\ldots,L-1;\;\mathbf{k}\in\mathrm{I}}$ using Taylor and rank-1 lattice based NFFT, $\mathbf{f} := f(\mathbf{y}_j)_{j=0}^{M-1}$
- normal equation $\tilde{\mathbf{A}}^*\tilde{\mathbf{A}}\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k}\in\mathcal{I}_N^{d,T}} = \tilde{\mathbf{A}}^*\mathbf{f}$

Approx. reconstruction - perturbed rank-1 lattice nodes

- Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$
- approximate f using a Fourier partial sum

$$\tilde{S}_{\mathcal{I}_N^{d,T}} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{I}_N^{d,T}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

  of approximated Fourier coefficients $\hat{\tilde{f}}_{\mathbf{k}}$ computed from M sampling values $f(\mathbf{y}_j)$ at perturbed rank-1 lattice nodes $\mathbf{y}_j$,

$$\left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{I}_N^{d,T}} := \underset{\hat{\mathbf{g}} \in \mathbb{C}^{\left|\mathcal{I}_N^{d,T}\right|}}{\arg\min} \|\tilde{\mathbf{A}}\,\hat{\mathbf{g}} - \mathbf{f}\|_2,$$

  where $\tilde{\mathbf{A}}$ is approximation of $\mathbf{A} := (\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{y}_\ell})_{\ell=0,\ldots,L-1;\,\mathbf{k}\in\mathrm{I}}$ using Taylor and rank-1 lattice based NFFT, $\mathbf{f} := f(\mathbf{y}_j)_{j=0}^{M-1}$
- normal equation $\tilde{\mathbf{A}}^* \tilde{\mathbf{A}} \left(\hat{\tilde{f}}_{\mathbf{k}}\right)_{\mathbf{k} \in \mathcal{I}_N^{d,T}} = \tilde{\mathbf{A}}^* \mathbf{f}$
- compute $(\hat{\tilde{f}}_{\mathbf{k}})_{\mathbf{k}\in\mathrm{I}}$ using CGNR or LSQR method in K iterations arithmetic complexity $\mathcal{O}\left(K\; m^d(|\mathcal{I}_N^{d,T}|^2 \log|\mathcal{I}_N^{d,T}|\right)$

Approx. reconstruction - perturbed rank-1 lattice nodes

Lemma (Kämmerer, Potts, V. 2013)

Let a weighted frequency index set $\mathcal{I}_N^{d,T}$, a reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$ and sampling nodes $\mathbf{y}_j$, $\min_{\mathbf{h}\in\mathbb{Z}^d} \|\mathbf{y}_j - \mathbf{x}_j + \mathbf{h}\|_\infty \leq \varepsilon$, be given, where $N \geq 1$, $T < 1$,
$0 \leq \varepsilon < \left(2\pi\left(d^{1+\left(\frac{T}{1-T}\right)_+}\right)N\right)^{-1} \ln 2$,
$(a)_+ := \max(0, a)$.

Then, the largest singular value $\sigma_{\max}(\tilde{\mathbf{A}}) < \sqrt{M} \mathrm{e}^{2\pi\left(d^{1+\left(\frac{T}{1-T}\right)_+}\right)N\varepsilon}$, and the smallest singular value

$$\sigma_{\min}(\tilde{\mathbf{A}}) > \sqrt{M}\left(2 - \mathrm{e}^{2\pi\left(d^{1+\left(\frac{T}{1-T}\right)_+}\right)N\varepsilon}\right).$$

For $\varepsilon = 0$, we have $\sigma_{\max}(\tilde{\mathbf{A}}) = \sigma_{\min}(\tilde{\mathbf{A}}) = \sqrt{M}$.

Approx. reconstruction - perturbed rank-1 lattice nodes

Theorem (Kämmerer, Potts, V. 2013)

Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$, $\mathcal{Y} = \{\mathbf{y}_j\}_{j=0}^{M-1}$ *be given, where* $\beta \geq 0$, $0 < \alpha + \beta \leq m$, $N \geq 1$, $T < 1$,
$\min_{\mathbf{h}\in\mathbb{Z}^d} \|\mathbf{y}_j - \mathbf{x}_j + \mathbf{h}\|_\infty \leq \varepsilon$ *for all* $\mathbf{x}_j \in \Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$,
$0 \leq \varepsilon < \left(2\pi\left(d^{1+\left(\frac{T}{1-T}\right)_+}\right)N\right)^{-1} \ln 2$, $(a)_+ := \max(0, a)$.
Then, the approximation error

$$\|f - \tilde{S}_{\mathcal{I}_N^{d,T}} f | L^2(\mathbb{T}^d)\|$$

$$\leq \left(1 + \frac{\sqrt{M}}{\sigma_{\min}(\tilde{\mathbf{A}})}\left(1 + \frac{\left(2\pi d^{\frac{1}{1-T}}\varepsilon N\right)^m}{m!}\right)\right)$$

$$N^{-(\alpha+\beta)}\|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\| \begin{cases} N^{\frac{d-1}{d-T}(T\beta+\alpha)}, & T > -\frac{\alpha}{\beta}, \\ 1, & T = -\frac{\alpha}{\beta}, \\ d^{-\frac{T\beta+\alpha}{1-T}}, & T < -\frac{\alpha}{\beta}. \end{cases}$$

Approx. reconstruction - perturbed rank-1 lattice nodes

Theorem (Kämmerer, Potts, V. 2013)

Let $f \in \mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$, $\mathcal{I}_N^{d,T}$, $\Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$, $\mathcal{Y} = \{\mathbf{y}_j\}_{j=0}^{M-1}$ *be given, where* $\beta \geq 0$, $0 < \alpha + \beta \leq m$, $N \geq 1$, $T < 1$, $\min_{\mathbf{h}\in\mathbb{Z}^d} \|\mathbf{y}_j - \mathbf{x}_j + \mathbf{h}\|_\infty \leq \varepsilon$ *for all* $\mathbf{x}_j \in \Lambda(\mathbf{z}, M, \mathcal{I}_N^{d,T})$, $0 \leq \varepsilon < \left(2\pi\left(d^{1+\left(\frac{T}{1-T}\right)_+}\right)N\right)^{-1} \ln 2$, $(a)_+ := \max(0, a)$.
Then, the approximation error

$$\|f - \tilde{S}_{\mathcal{I}_N^{d,T}} f | L^2(\mathbb{T}^d)\|$$

$$\leq \left(1 + \frac{1}{2 - \mathrm{e}^{2\pi\left(d^{1+\left(\frac{T}{1-T}\right)_+}\right)N\varepsilon}} \left(1 + \frac{\left(d^{\frac{T}{1-T}} \ln 2\right)^m}{m!}\right)\right)$$

$$N^{-(\alpha+\beta)} \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\| \begin{cases} N^{\frac{d-1}{d-T}(T\beta+\alpha)}, & T > -\frac{\alpha}{\beta}, \\ 1, & T = -\frac{\alpha}{\beta}, \\ d^{-\frac{T\beta+\alpha}{1-T}}, & T < -\frac{\alpha}{\beta}. \end{cases}$$

Summary

- fast and perfectly stable approximate reconstruction of functions from Sobolev spaces $\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$
  by sampling at rank-1 lattice nodes and
  using weighted frequency index sets $\mathcal{I}_N^{d,T}$
  + error estimates,
  best error for $T = -\alpha/\beta$
- fast algorithm for approximate evaluation of trigonometric polynomials supported on weighted frequency index sets $\mathcal{I}_N^{d,T}$
  presented + error estimates,
  best error for $T = -\alpha/\beta$
- fast and stable approximate reconstruction of functions from Sobolev spaces $\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)$
  by sampling at perturbed rank-1 lattice nodes and
  using weighted frequency index sets $\mathcal{I}_N^{d,T}$
  + error estimates (incl. all constants),
  best error for $T = -\alpha/\beta$