TECHNISCHE UNIVERSITÄT CHEMNITZ

Faculty of Mathematics
Applied Functional Analysis
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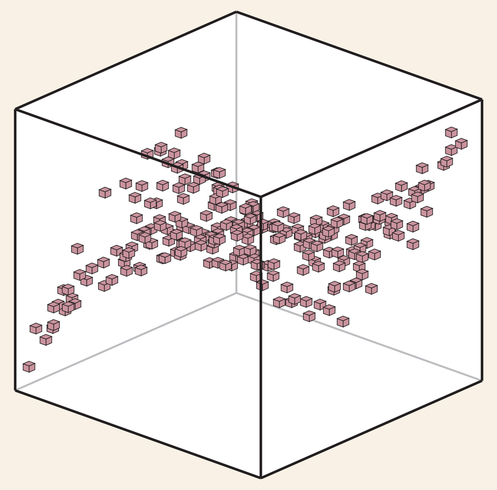
High-dimensional approximation and sparse FFT using multiple rank-1 lattices

Frequency index set I



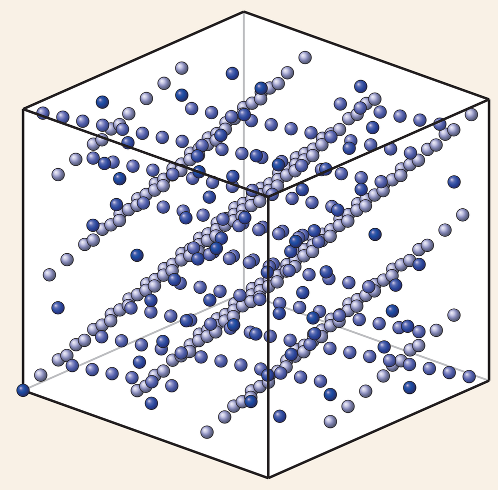
Task

We want to **reconstruct** a **high-dimensional** (e.g. $d = 10$) periodic signal using a trigonometric polynomial $p\colon \mathbb{T}^d \simeq [0,1)^d \to \mathbb{C}$,

$$p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \hat{p}_{\boldsymbol{k}} \in \mathbb{C},$$

from samples, where $I \subset \mathbb{Z}^d$ is a suitable and **possibly unknown** frequency index set.

Sampling nodes



Known frequency index set I

Multiple rank-1 lattices as sampling sets

A **multiple rank-1 lattice** Λ is the union of $L \in \mathbb{N}$ many rank-1 lattices,

$$\Lambda = \Lambda(\boldsymbol{z}_1, M_1, \ldots, \boldsymbol{z}_L, M_L) := \bigcup_{\ell=1}^{L} \Lambda(\boldsymbol{z}_\ell, M_\ell),$$

$$\Lambda(\boldsymbol{z}, M) := \{ j\boldsymbol{z}/M \bmod \boldsymbol{1} \colon j = 0, \ldots, M-1 \} \subset \mathbb{T}^d,$$

and consists of $|\Lambda| \leq 1 - L + \sum_{\ell=1}^{L} M_\ell$ many nodes.

Reconstructing multiple rank-1 lattice Λ for I, sufficient condition:

$$\boldsymbol{k}\cdot\boldsymbol{z}_\ell \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z}_\ell \pmod{M_\ell} \text{ for all } \boldsymbol{k} \in I_\ell,\ \boldsymbol{k}' \in I, \boldsymbol{k} \neq \boldsymbol{k}', \qquad \bigcup_{\ell=1}^{L} I_\ell = I.$$

Fast construction and high-dimensional FFT

Fast probabilistic construction algorithm for reconstructing Λ is available. Under mild assumptions with high probability, we have $|\Lambda| \lesssim |I| \log |I|$ and the construction requires $\mathcal{O}(|I|(d + \log|I|)\log|I|)$ arithmetic operations.

Oversampling factor $|\Lambda|/|I| \lesssim L$ does not depend on the dimension d.

FFT requires only $\mathcal{O}\big(|I|\,(d + \log|I|)\log|I|\big)$ arithmetic operations.

Approximation results

$$f \in \mathcal{A}^{\beta}_{\mathrm{mix}} := \Big\{ g \in L_1(\mathbb{T}^d) \colon \|g|\mathcal{A}^{\beta}_{\mathrm{mix}}\| := \textstyle\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{g}_{\boldsymbol{k}}|\, \Pi_{s=1}^{d} \max(1, |k_s|)^{\beta} < \infty \Big\}.$$

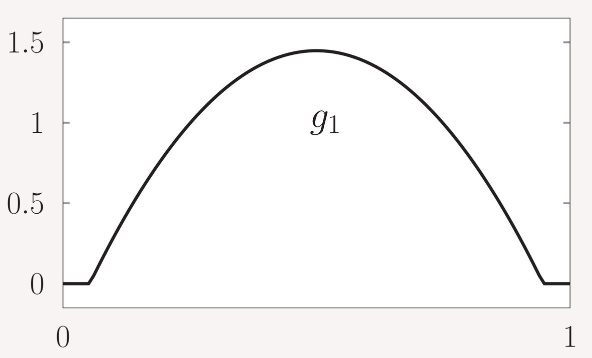
For hyperbolic cross frequency index sets I, one can show

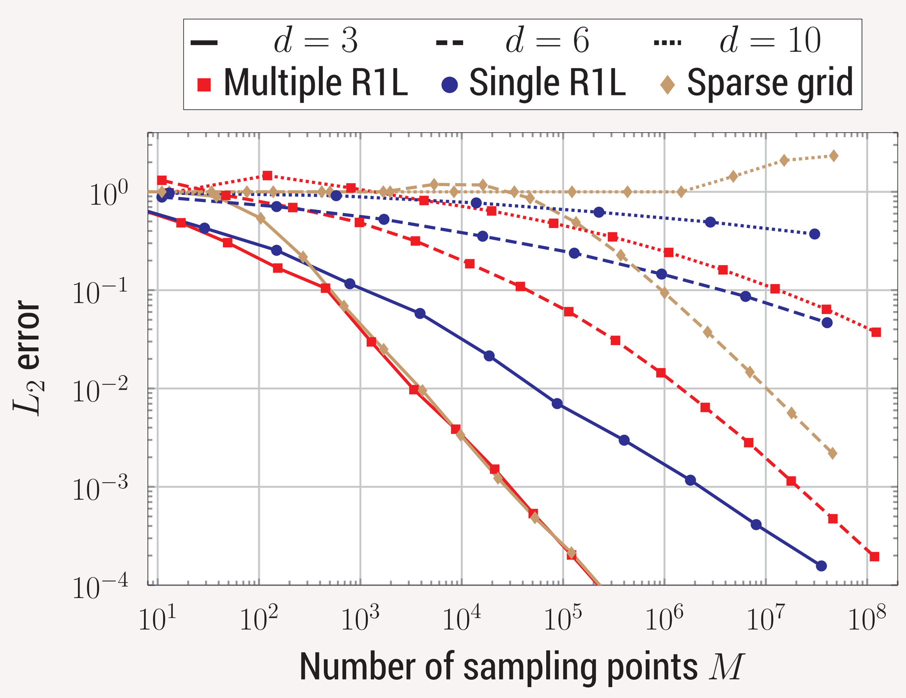
$$\|f - S_I^{\Lambda} f | L_2(\mathbb{T}^d)\| \leq \|f - S_I^{\Lambda} f | L_\infty(\mathbb{T}^d)\| \lesssim M^{-\beta} (\log M)^{d\beta+1}\, \|f|\mathcal{A}^{\beta}_{\mathrm{mix}}\|.$$

Numerical example

kink function $g_d\colon \mathbb{T}^d \to \mathbb{R}$,

$$g_d(\boldsymbol{x}) = \prod_{s=1}^{d} \left(\frac{5^{3/4} 15}{4\sqrt{3}} \max\left\{ \frac{1}{5} - \left(x_s - \frac{1}{2}\right)^2, 0 \right\} \right)$$





See also the references in

L. Kämmerer. Constructing spatial discretizations for sparse multivariate trigonometric polynomials that allow for a fast discrete Fourier transform. *ArXiv e-prints 1703.07230*, Nov. 2017.

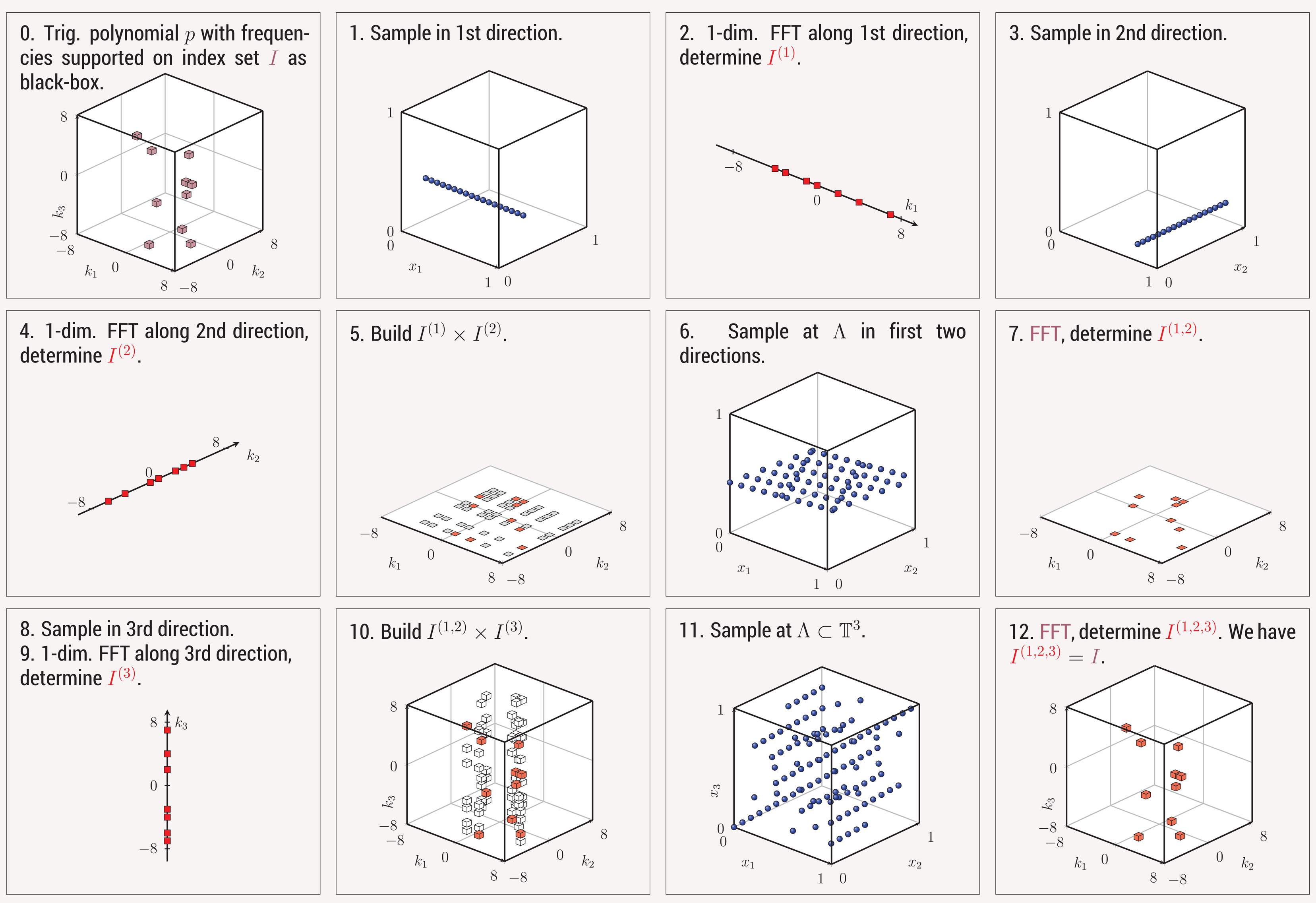
Unknown frequency index set I

Dimension-incremental sparse FFT method

Adaptively construct the index set of frequencies belonging to the approx. largest (or non-zero) Fourier coefficients in a **dimension incremental** way. Compute **projected Fourier coefficients** from samples along **multiple rank-1 lattices** and then determine **frequency locations**.

- MATLAB implementation available
- samples: $\mathcal{O}\big(d\,|I|^2 N (\log|I|)^2 |\log\varepsilon|\big)$ w.h.p., arithm. operations: $\mathcal{O}\big(d^2 |I|^2 N (\log|I|)^3 |\log\varepsilon|\big)$ w.h.p. (if $I \subset ([-N,N]^d \cap \mathbb{Z}^d)$ and $|I| \gtrsim N$)

Steps for reconstruction of 3-dim. trigonometric polynomial

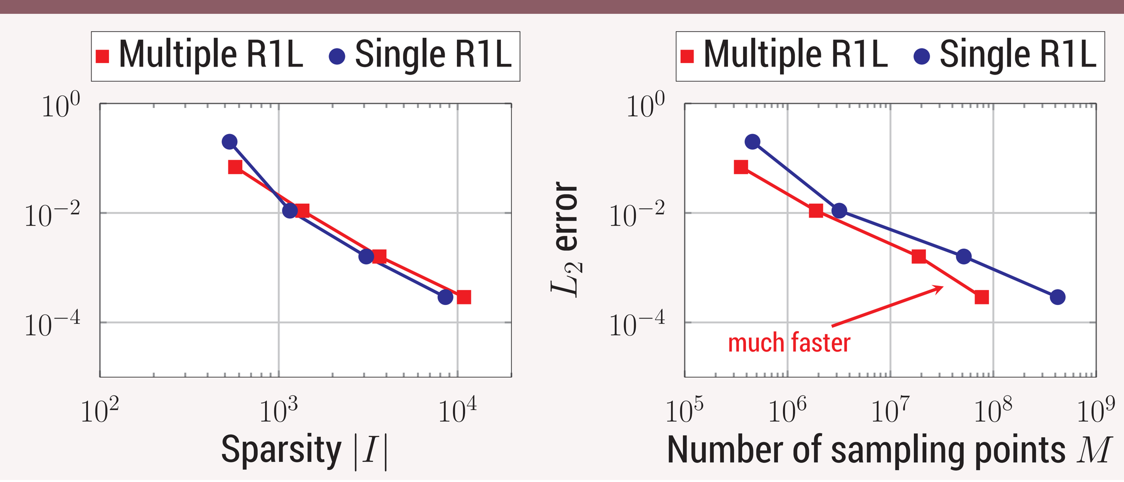


Example: Approximation of 10-dimensional test function

$$f\colon \mathbb{T}^{10} \to \mathbb{R}$$

$$f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} B_2(x_t) + \prod_{t\in\{2,5,6,10\}} B_4(x_t) + \prod_{t\in\{4,7,9\}} B_6(x_t)$$

B_m is B-Spline of order m



See also the references in

L. Kämmerer, D. Potts, T. Volkmer. High-dimensional sparse FFT based on sampling along multiple rank-1 lattices. *ArXiv e-prints 1711.05152*, Nov. 2017.

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