UNIVERSITY OF TECHNOLOGY
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Learning the solution of differential equations by sparse high-dimensional approximation

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Setting

Differential problem

$$Lu = f \qquad \boldsymbol{x} \in \Omega \subset \mathbb{R}^d$$

Solvability: We can access samples $u(\boldsymbol{x})$ for a given f and any $\boldsymbol{x} \in \Omega$ at any time (via classical differential equation solvers).
Operator learning: We are interested in the solution mapping $\mathcal{G}(f) = u$.

Parametrization of f

$$f(\boldsymbol{x}) \approx \sum_{j=1}^{n} a_j A_j(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega$$

Functions: We use fixed functions $A_j, j = 1, \ldots, n$, e.g., B-splines or trigonometric polynomials.
Coefficients: We identify f by its coefficients $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

Basis expansion of u

$$u(\boldsymbol{x}, \boldsymbol{a}) \coloneqq \sum_{\boldsymbol{k} \in \mathbb{N}^{d+n}} c_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{a}) \qquad (\boldsymbol{x}, \boldsymbol{a}) \in \mathcal{D}.$$

Bounded orthonormal product basis: The $\{\Phi_{\boldsymbol{k}}(\cdot),\ \boldsymbol{k} \in \mathbb{N}^{d+n}\}$ are bounded, orthonormal and of tensor-product structure, e.g., trigonometric or Chebyshev polynomials.

High-dim. approximation

$$S_I^{\mathcal{A}} u(\boldsymbol{x}, \boldsymbol{a}) \coloneqq \sum_{\boldsymbol{k} \in I} \hat{u}_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{a})$$

Index set: $I \subset \mathbb{N}^{d+n}$ is unknown, but s-sparse, i.e., $|I| = s$.
Coefficients: $\hat{u}_{\boldsymbol{k}} \in \mathbb{C}$ are approximations of the true coefficients $c_{\boldsymbol{k}}$.

Main challenge

Unknown index set: If I was known, we could compute the $\hat{u}_{\boldsymbol{k}}$ directly and efficiently.
Search space: We consider a reasonable search space $\Gamma \subset \mathbb{N}^{d+n}$ and assume $I \subset \Gamma$. However, we have $|\Gamma| \gg |I|$.
Curse of dimensionality: Computing all $\hat{u}_{\boldsymbol{k}}$ with $\boldsymbol{k} \in \Gamma$ is computationally unfeasible!

Method

Algorithm Input

- search space $\Gamma \subset \mathbb{N}^{d+n}$
- sparsity $s \in \mathbb{N}$
- detection threshold $\delta > 0$
- detection iterations $r \in \mathbb{N}$
- target function u as black box ($\rightarrow$ PDE solver)

Dimension-incremental Algorithm[1]

- Works by detecting "good" index sets in lower dimensions and combining them.
- Utilizes cosine-transformed multiple rank-1 lattices.[2]
- Complexities (with $\tilde{d} = d + n$ and superposition dimension d_s):
  - Sampling compl.: $\mathcal{O}\left(\tilde{d} r^3 s^2 2^{d_s} \log(rs)\right)$
  - Computational compl.: $\mathcal{O}\left(\tilde{d} r^3 s^2 d_s^2 2^{d_s} \log^5(rs)\right)$

Algorithm Output

- detected index set $I \subset \Gamma$ with $|I| = s$
- approximated coefficients $\hat{u}_{\boldsymbol{k}}$ with $|\hat{u}_{\boldsymbol{k}}| \geq \delta$.

Numerical experiments (sparsity $s = 1000$)

Poisson equation (1D)

Differential equation:

$$-\frac{d^2}{dx^2} u(x) = f(x) \qquad x \in (0,1)$$
$$u(0) = u(1) = 0$$

Parametrization:

$$f(x) \approx \sum_{\ell=-(n-1)/2}^{(n-1)/2} a_\ell \mathrm{e}^{2\pi \mathrm{i} \ell x}$$

Remarks: We set $n = 9$, restricted $a_\ell \in [-1, 1]$ and used the analytical solution instead of a solver.
Transfer learning:

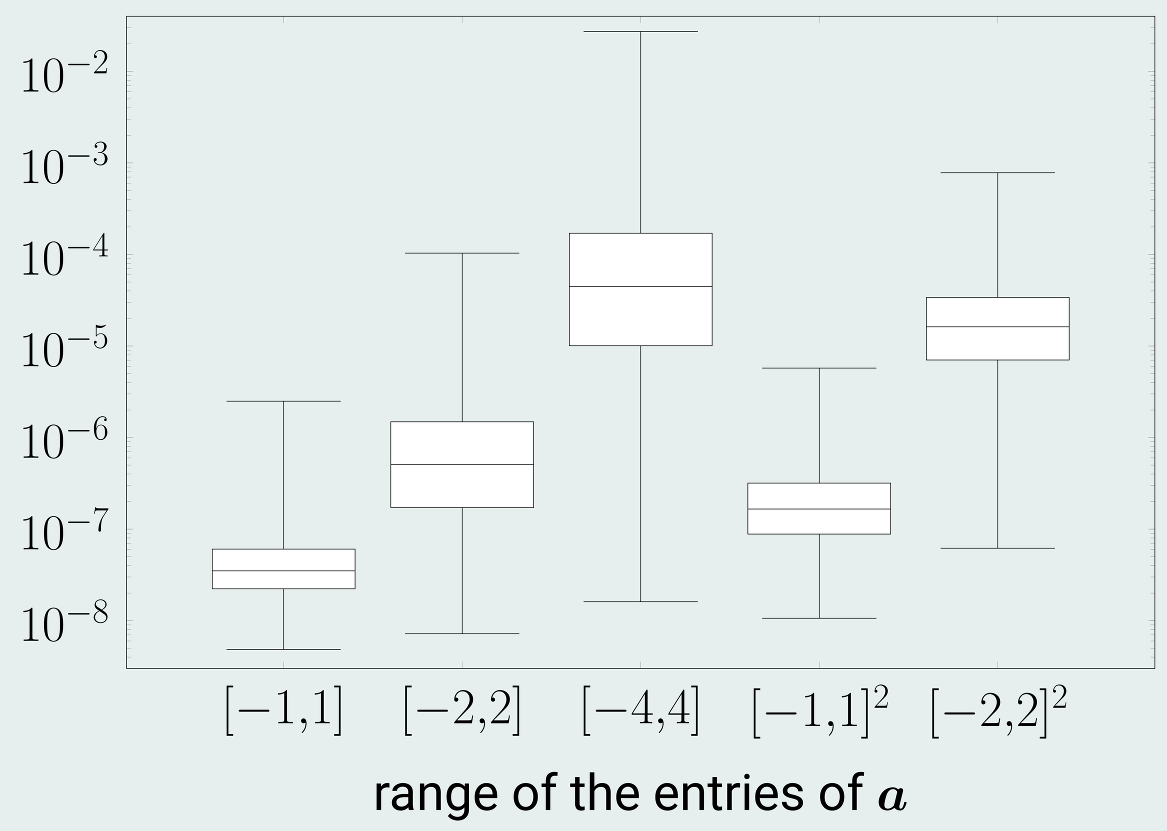

Figure: The relative approximation error for 10000 randomly drawn $\boldsymbol{a}$.

Poisson equation (2D)

Differential equation:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega$$
$$u(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in \delta\Omega$$

Parametrization:

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{\ell} \in J} a_{\boldsymbol{\ell}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{\ell} \boldsymbol{x}}$$

Remarks: We used a FEM (1893 nodes) and set $J = \{-1, 0, 1\}^2$.

Average relative approximation error: $\approx 10^{-4}$
Structural information:

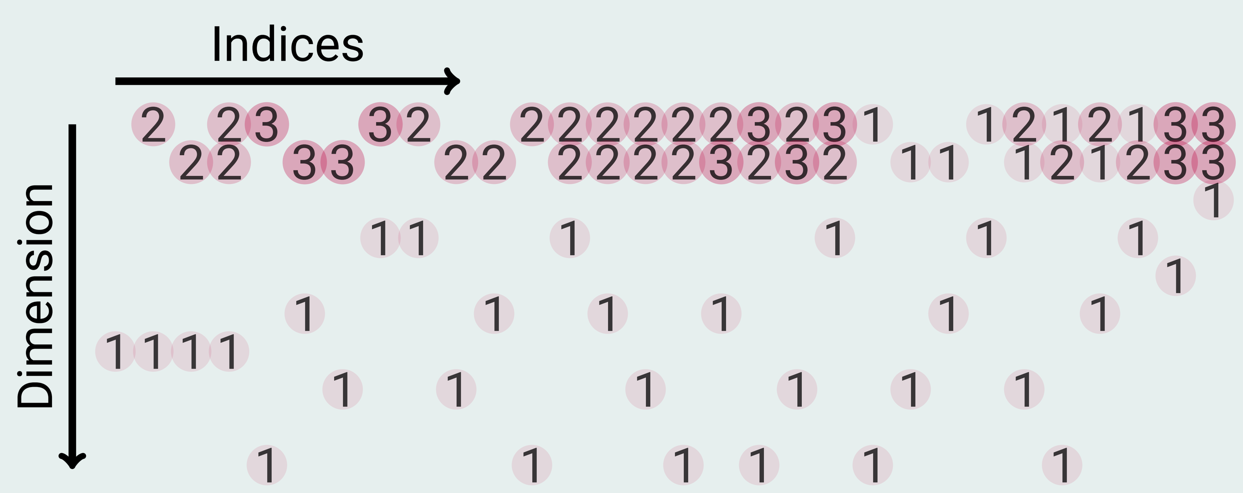

Figure: The first 30 indices detected.

Heat equation

Differential equation:

$$\partial_t u = \alpha^2 \partial_{xx} u \qquad x, t \in (0,1)$$
$$u(x, 0) = f(x) \qquad x \in (0,1)$$
$$u(0, t) = 0 \qquad t \in (0,1)$$
$$u(1, t) = 0 \qquad t \in (0,1)$$

Parametrization:

$$f(x) \approx \sum_{\ell=1}^{n} a_\ell \sin(\ell \pi x)$$

Remarks: We set $n = 9$ and used the MATLAB® function pdepe.

Average relative approximation error: $\approx 10^{-3}$
Structural information:

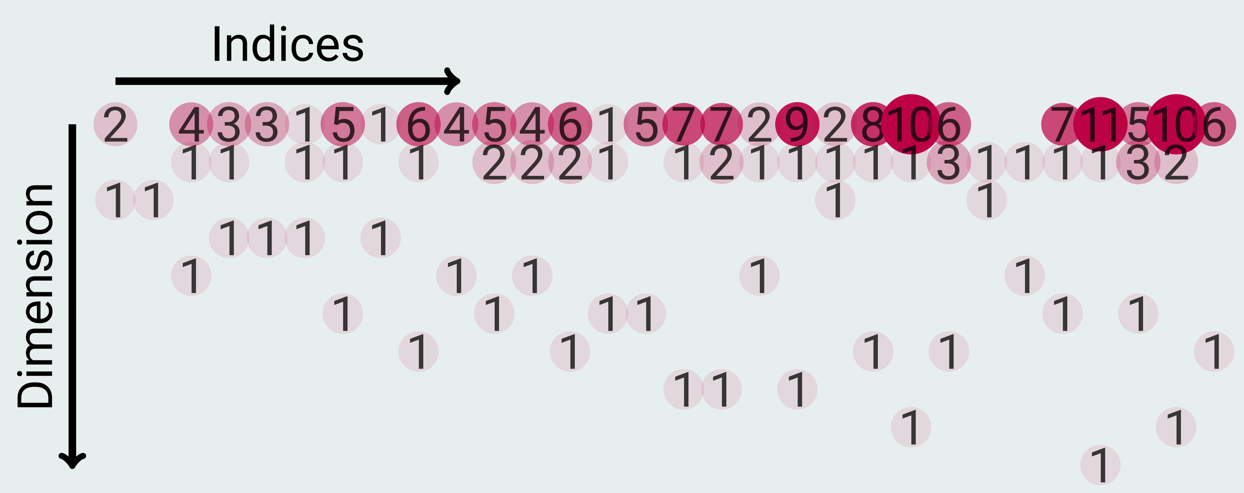

Figure: The first 30 indices detected.

[1] L. Kämmerer, D. Potts and F. Taubert. Nonlinear approximation in bounded orthonormal product bases. *Sampl. Theory Signal Process. Data Anal., 2023.*
[2] L. Kämmerer. An efficient spatial discretization of spans of multivariate Chebyshev polynomials. *arXiv preprint, 2024.*
[3] D. Potts and F. Taubert. Operator learning based on sparse high-dimensional approximation. *arXiv preprint, 2024.*

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