The uniform sparse FFT

with application to PDEs with random coefficients

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joint work with Lutz Kämmerer and Daniel Potts

11.02.2022

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CHEMNITZ

Outline

We consider the PDE problem

$$\begin{aligned} -\nabla_{\boldsymbol{x}} \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla_{\boldsymbol{x}} u(\boldsymbol{x}, \boldsymbol{y})) &= f(\boldsymbol{x}), & \boldsymbol{x} &\in D,\, \boldsymbol{y} \in D_{\boldsymbol{y}} \\ u(\boldsymbol{x}, \boldsymbol{y}) &= 0, & \forall\, \boldsymbol{x} &\in \partial D,\, \boldsymbol{y} \in D_{\boldsymbol{y}}. \end{aligned}$$

- spatial variable $\boldsymbol{x} \in D \subset \mathbb{R}^{d_{\boldsymbol{x}}}$, typically with $d_{\boldsymbol{x}} = 1, 2, 3$
- random variable $\boldsymbol{y} = (y_j)_{j=1}^{d} \in D_{\boldsymbol{y}}$, typically very high-dimensional or even infinite-dimensional

- affine random coefficient [Cohen, DeVore, Schwab '10], [Dick, Kuo, Le Gia, Schwab '16], [Bachmayr, Cohen, Dahmen '18],[Gantner, Herrmann, Schwab '18], [Nguyen, Nuyens '21], ...

$$a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j=1}^{d} y_j\, \psi_j(\boldsymbol{x})$$

- periodic random coefficient

[Kaarnioja, Kuo, Sloan '20], [Kaarnioja, Kazashi, Kuo, Nobile, Sloan '20]

$$a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j=1}^{d} \Theta_j(\boldsymbol{y})\, \psi_j(\boldsymbol{x}),$$

with Θ_j periodic, e.g., $\Theta_j(\boldsymbol{y}) := \frac{1}{\sqrt{6}} \sin(2\pi y_j)$

- lognormal random coefficient [Graham, Kuo, Nichols, Scheichl, Schwab, Sloan '13], [Cheng, Hou, Yan, Zhang '13], [Bachmayr, Cohen, DeVore, Migliorati '17], [Nguyen, Nuyens '21], ...

$$a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \exp(b(\boldsymbol{x},\boldsymbol{y})), \qquad b(\boldsymbol{x},\boldsymbol{y}) = b_0(\boldsymbol{x}) + \sum_{j=1}^{d} y_j\, \psi_j(\boldsymbol{x})$$

- Common aim: Approximation of the solution $u(\boldsymbol{x}, \boldsymbol{y})$ or some quantity of interest, e.g., the expectation value $\mathbb{E}[F(\boldsymbol{y})]$ of some functional $F(\boldsymbol{y}) \coloneqq F[u(\cdot, \boldsymbol{y})]$, using samples of the function.
- Main problem: high-dimensional approximation!

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- Examples of other approaches:
  - Quasi-Monte Carlo methods

    [Kuo, Schwab, Sloan '15], [Dick, Le Gia, Schwab '16], [Nguyen, Nuyens '21], ...
  - collocation methods

    [Cheng, Hou, Yan, Zhang '13], [Ernst, Sprungk '14], [Zhang, Hu, Hou, Lin, Yan '14], ...
  - methods based on certain (tensorized) functions (e.g., Legendre polynomials) [Cohen, DeVore, Schwab '10], [Bachmayr, Cohen, Migliorati '17], ...

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- These approaches are often heavily influenced by the choice (or computation) of some weights, functions or kernels in advance!

- We aim for an approximation of $u(\boldsymbol{x}_0, \cdot)$ for fixed $\boldsymbol{x}_0 \in D$ using the sparse FFT (sFFT) based on rank-1 lattice sampling.

  [Potts, Volkmer '16], [Kämmerer, Krahmer, Volkmer '20], [Kämmerer, Potts, Volkmer '21]

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- The goal of the sFFT is the computation of the index set $\mathrm{I} = \operatorname{supp} \hat{p} \subset \mathbb{Z}^d$ in the given search space $\Gamma \supset \mathrm{I}$ and the coefficients $\hat{p}_{\boldsymbol{k}}, \boldsymbol{k} \in \mathrm{I}$, of the multivariate trigonometric polynomial

$$p(\boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathrm{I}} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}$$

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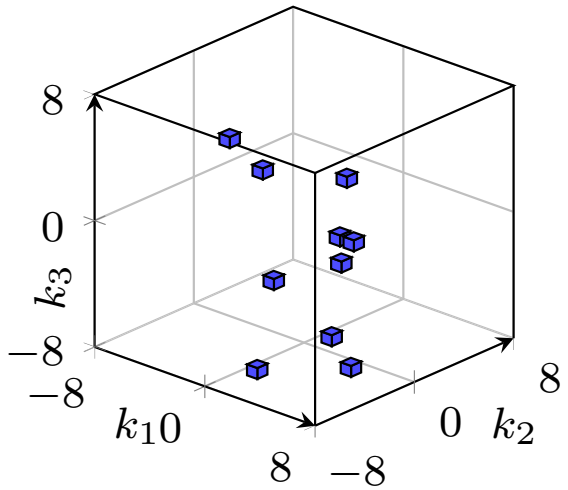
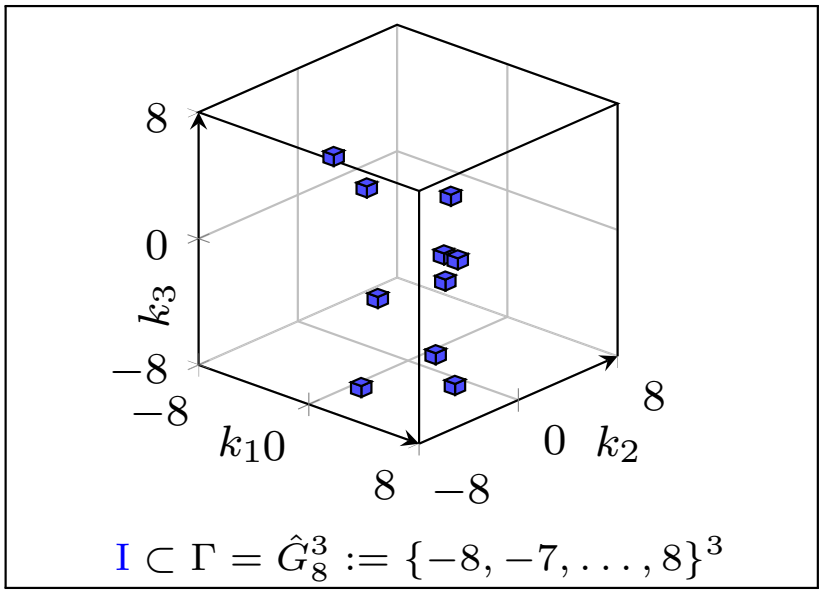
- Function approximation of $u(\boldsymbol{x}_0, \cdot)$ can be realized using the sFFT by assuming it to be a trigonometric polynomial p with some noise η, i.e.,

$$u(\boldsymbol{x}_0, \boldsymbol{y}) = p(\boldsymbol{y}) + \eta(\boldsymbol{y}).$$



$I \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}^3$



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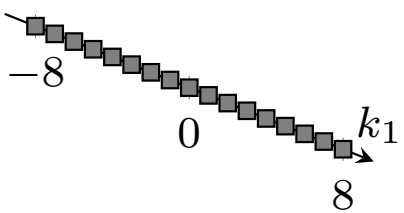
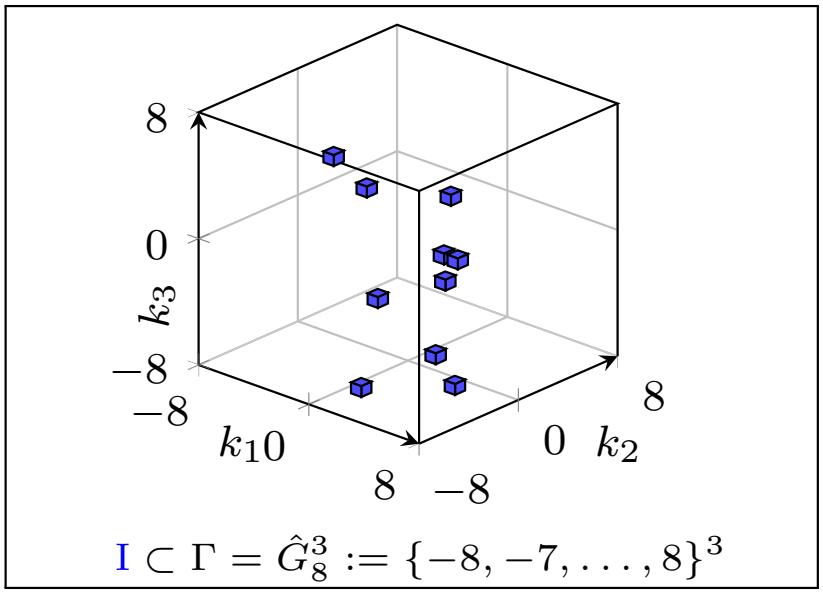


frequency candidates



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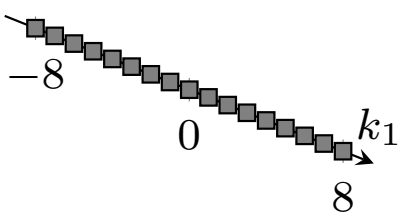
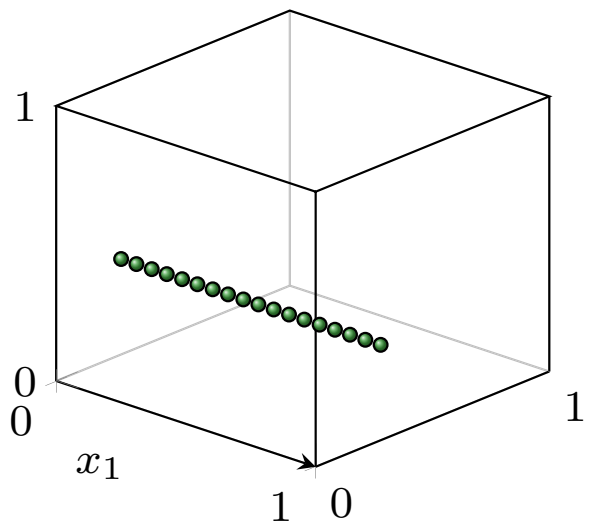
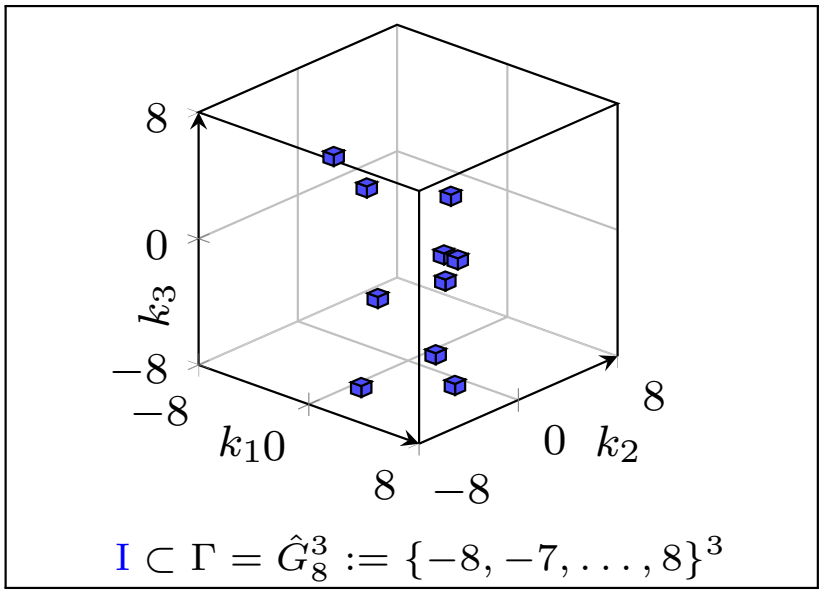


frequency candidates

construct
sampling set



sampling nodes $\left\{\left(\frac{\ell}{17}, x_2', x_3'\right)\right\}_{\ell=0}^{16}$



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$$\hat{p}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi \mathrm{i} \frac{\ell k_1}{17}}$$

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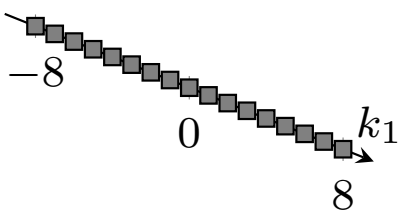
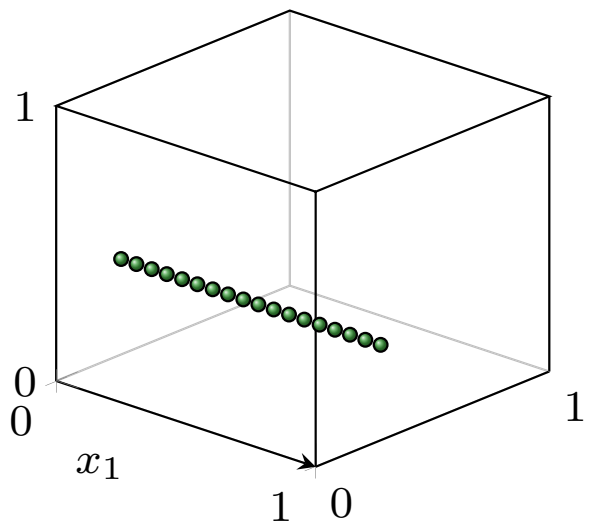


frequency candidates

1-dim. FFT



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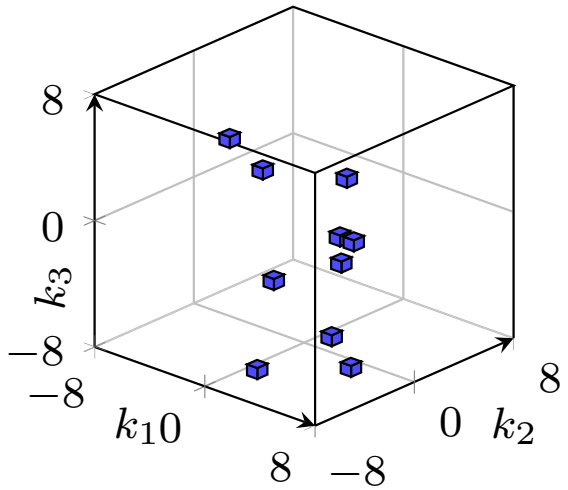

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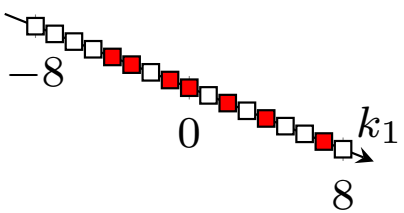
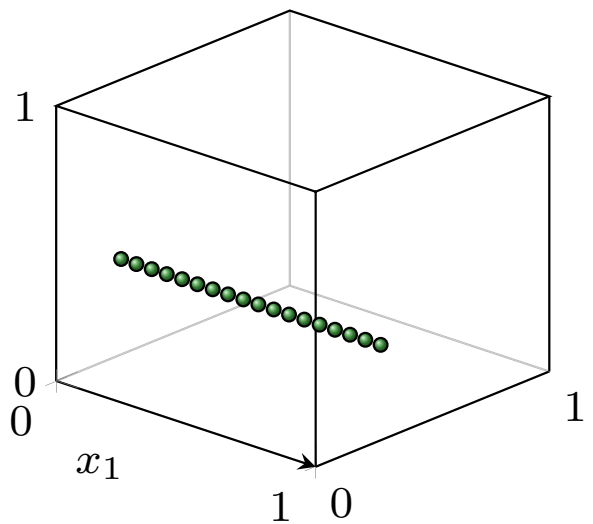


detected frequencies $\mathrm{I}^{(1)}$

1-dim. FFT



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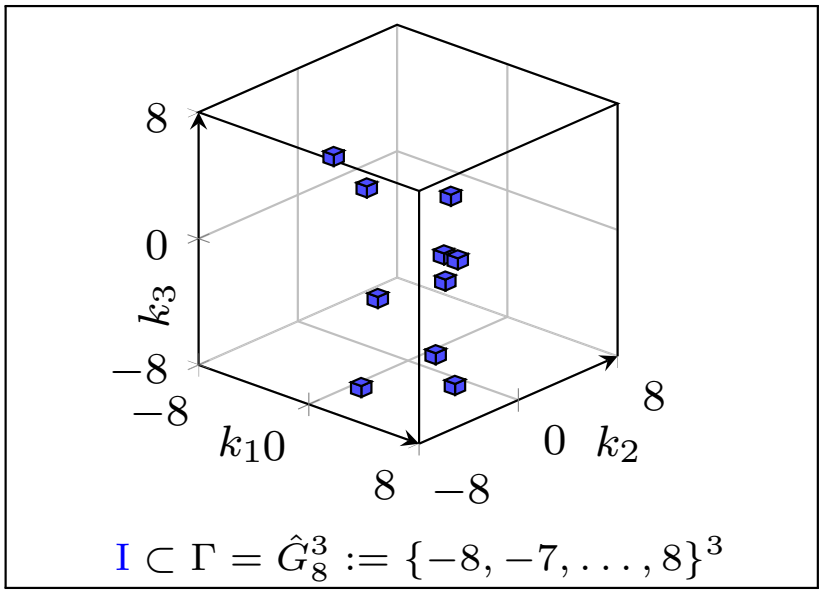

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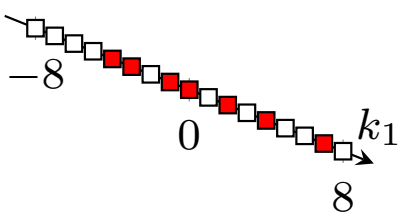

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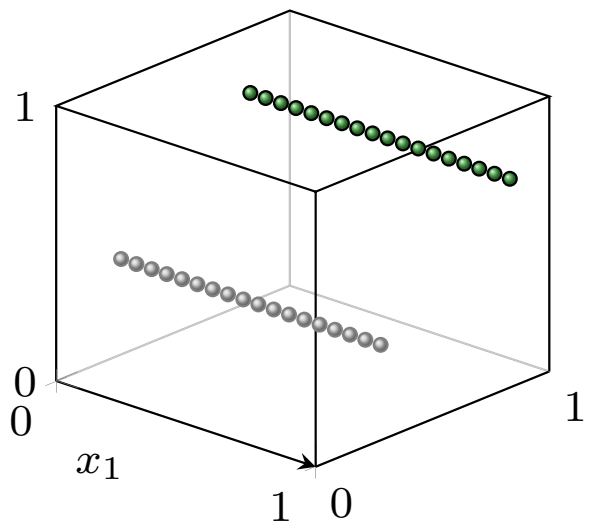

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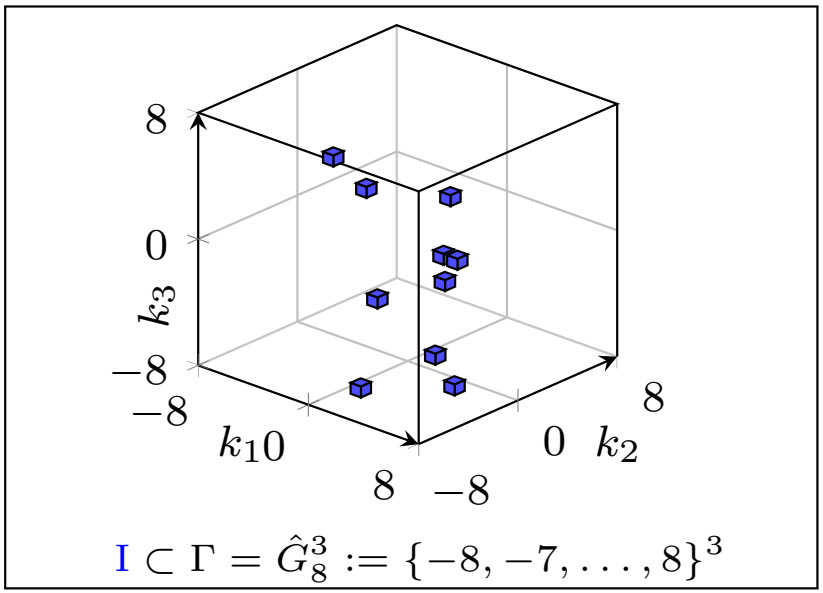
\+ repeat (r detection iterations)



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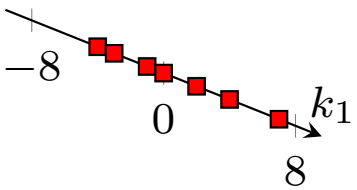

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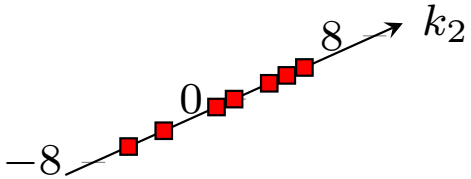
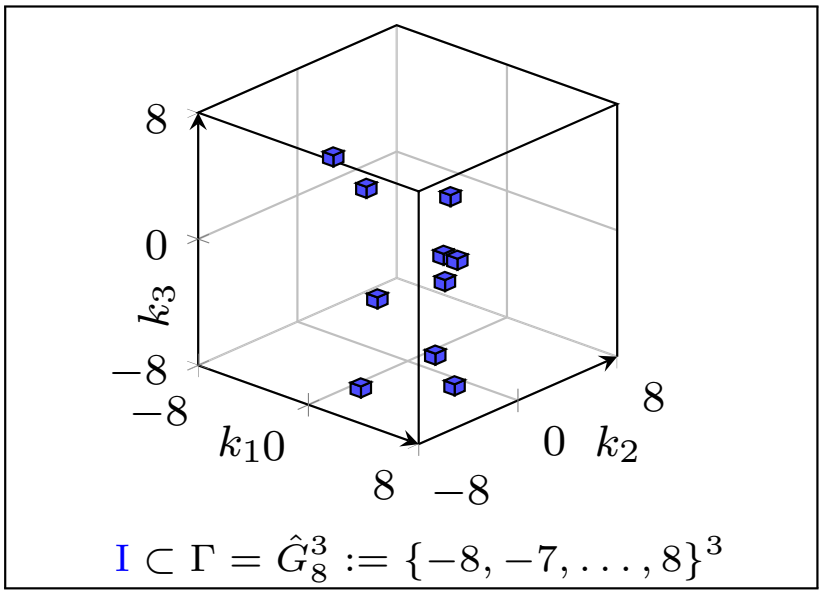


detected frequencies
$\mathrm{I}^{(2)}$



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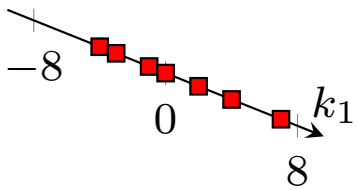

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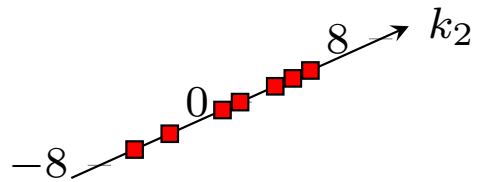

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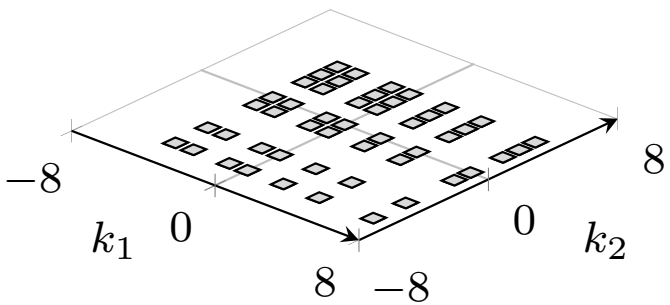
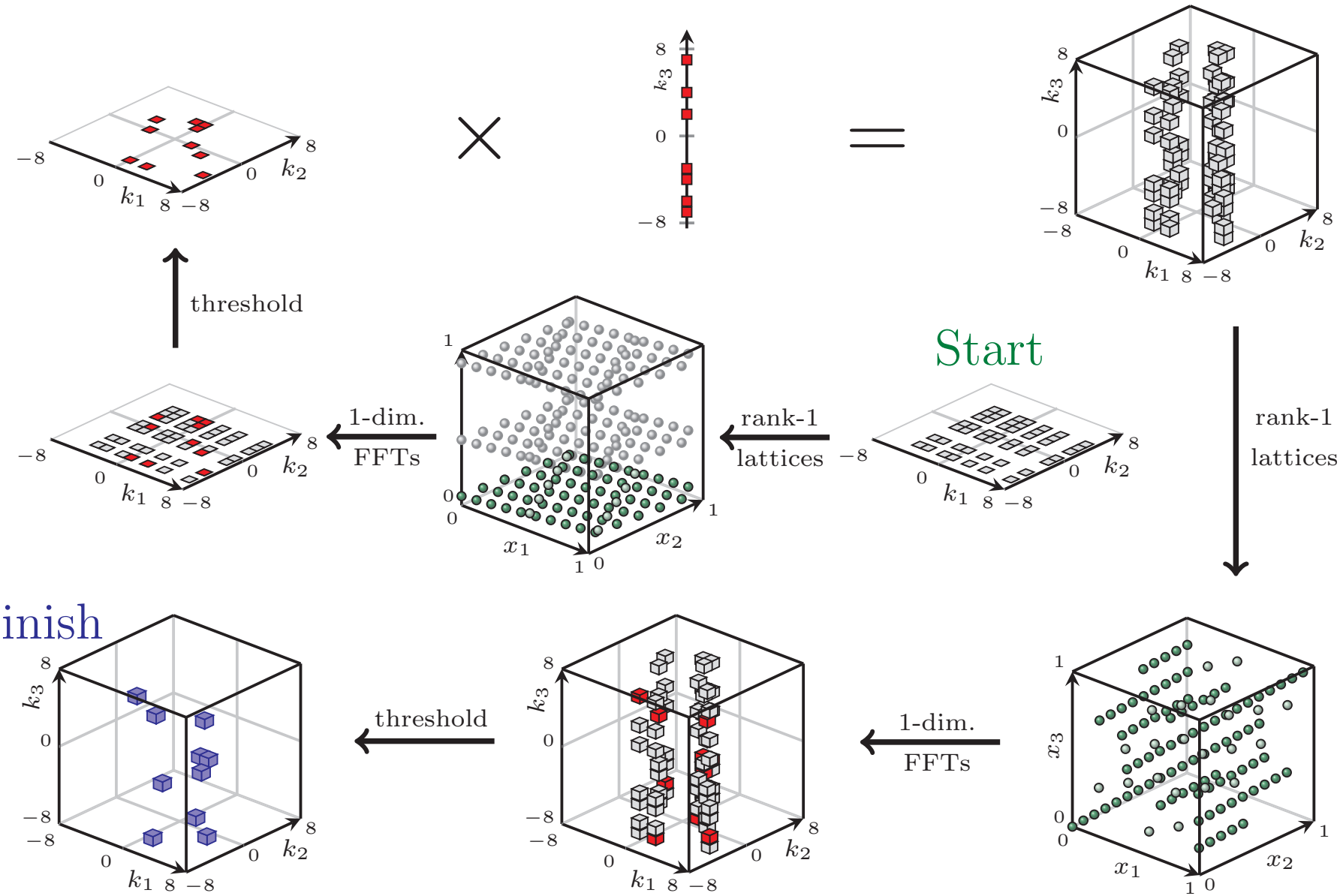


frequency candidates $\mathrm{I}^{(1)} \times \mathrm{I}^{(2)}$


Start
rank-1 lattices
1-dim. FFTs
threshold
1-dim. FFTs
threshold
Finish

- Back to our original problem:
  Approximating $u_{\boldsymbol{x}_0}(\boldsymbol{y}) \coloneqq u(\boldsymbol{x}_0, \boldsymbol{y})$ for a given $\boldsymbol{x}_0 \in \mathcal{D}$.
- Up to now:

$$u_{\boldsymbol{x}_0}(\boldsymbol{y}) \approx u_{\boldsymbol{x}_0}^{\mathtt{sFFT}}(\boldsymbol{y}) \coloneqq \sum_{\boldsymbol{k} \in \mathrm{I}_{\boldsymbol{x}_0}} c_{\boldsymbol{k}}^{\mathtt{sFFT}}(u_{\boldsymbol{x}_0})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{y}}$$

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- In practice: Consider $\{\boldsymbol{x}_g \in D,\, g = 1, ..., G\}$ instead of $\{\boldsymbol{x}_0 \in D\}$.
- Need G calls of the sFFT to compute $\mathrm{I}_{\boldsymbol{x}_g}$ and $c_{\boldsymbol{k}}^{\mathtt{sFFT}}(u_{\boldsymbol{x}_g})$, $g = 1, \ldots, G$.

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  One sample $u_{\boldsymbol{x}_g}(\boldsymbol{y}) \mathrel{\widehat{=}}$ one call of the PDE solver for fixed $\boldsymbol{y}$
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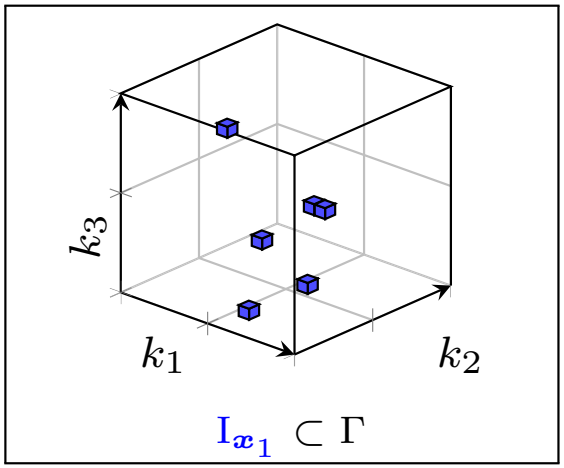
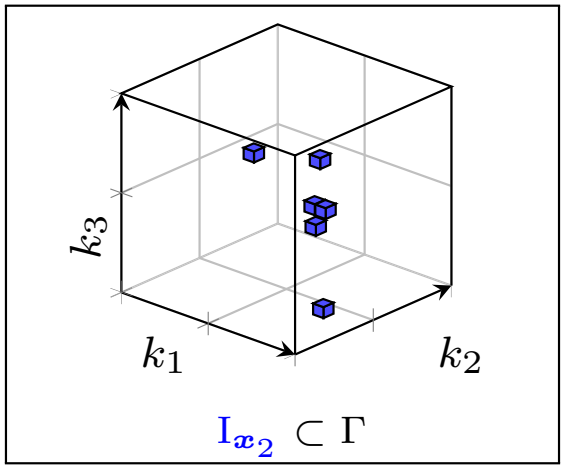
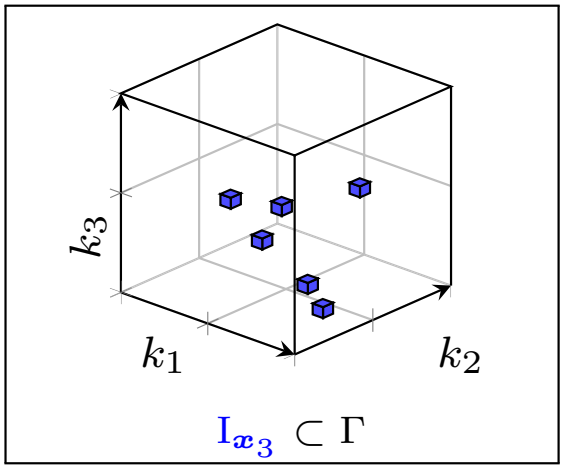
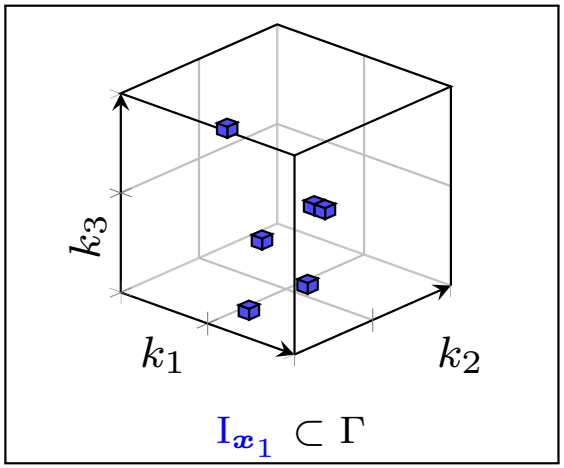
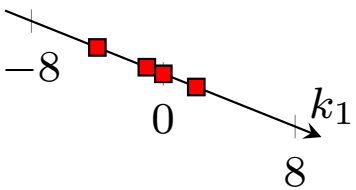
- Solution: the *uniform* sFFT!


k_3
k_1
k_2
$\mathrm{I}_{\boldsymbol{x}_1} \subset \Gamma$

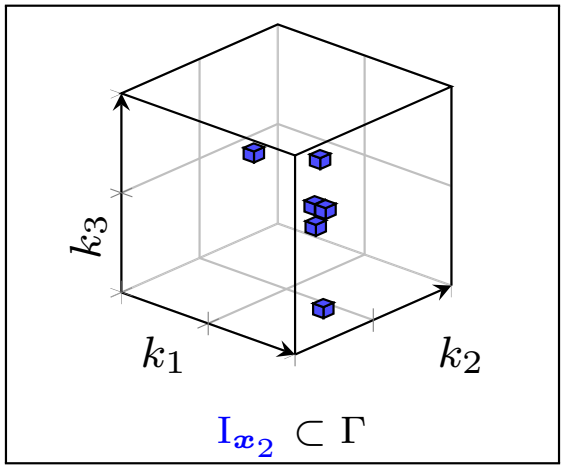

k_3
k_1
k_2
$\mathrm{I}_{\boldsymbol{x}_2} \subset \Gamma$

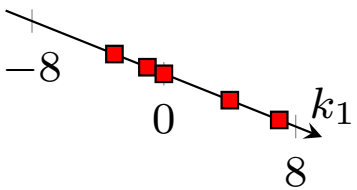

k_3
k_1
k_2
$\mathrm{I}_{\boldsymbol{x}_3} \subset \Gamma$



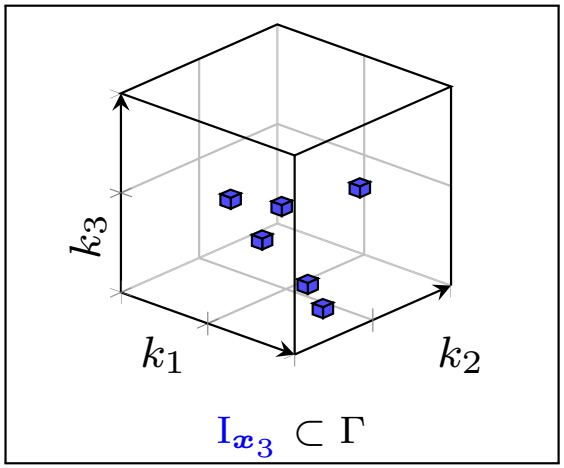


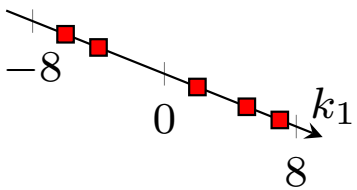
detected frequencies
$\mathrm{I}^{(1)}_{\boldsymbol{x}_1}$



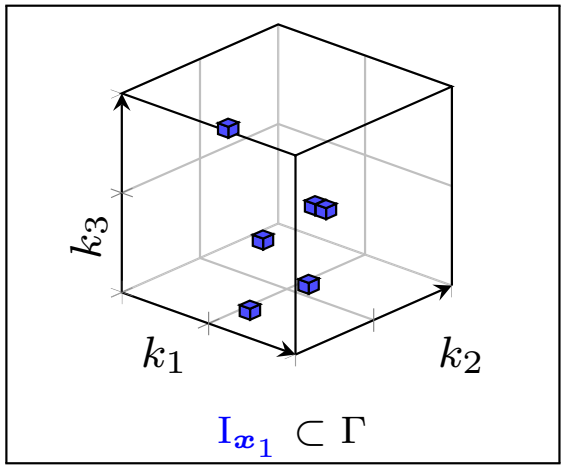


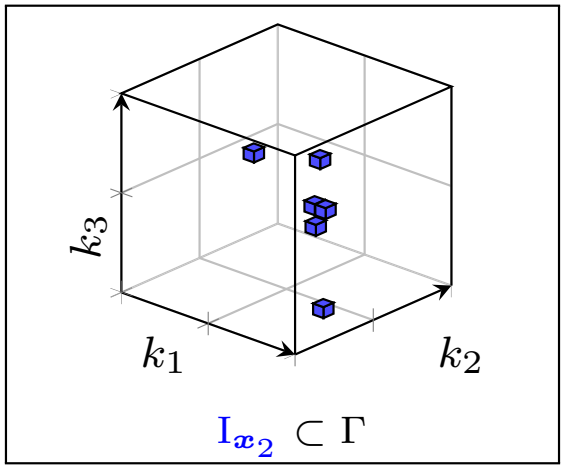
detected frequencies
$\mathrm{I}^{(1)}_{\boldsymbol{x}_2}$

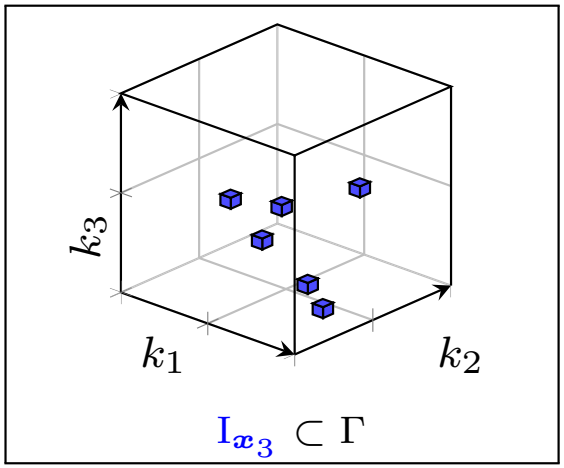


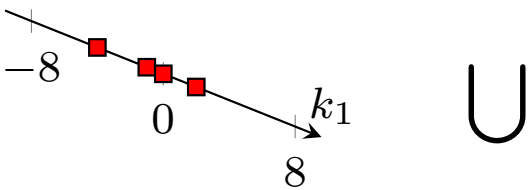


detected frequencies
$\mathrm{I}^{(1)}_{\boldsymbol{x}_3}$



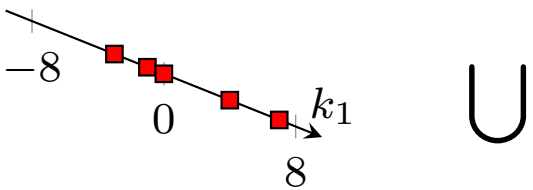






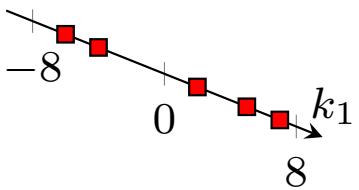
detected frequencies
$\mathrm{I}_{\boldsymbol{x}_1}^{(1)}$

$\cup$

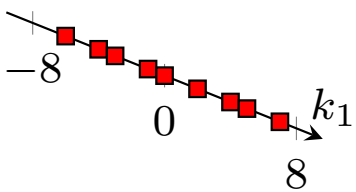


detected frequencies
$\mathrm{I}_{\boldsymbol{x}_2}^{(1)}$

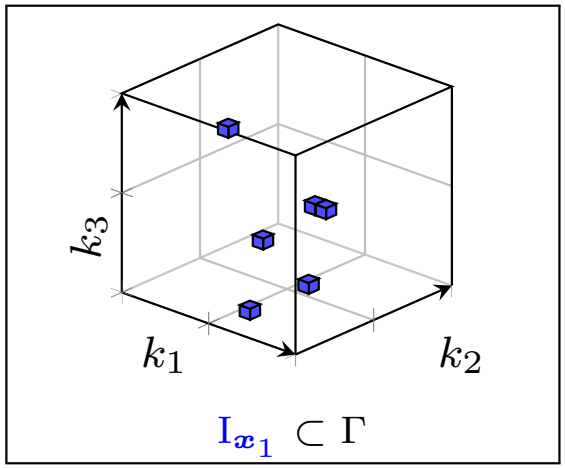
$\cup$



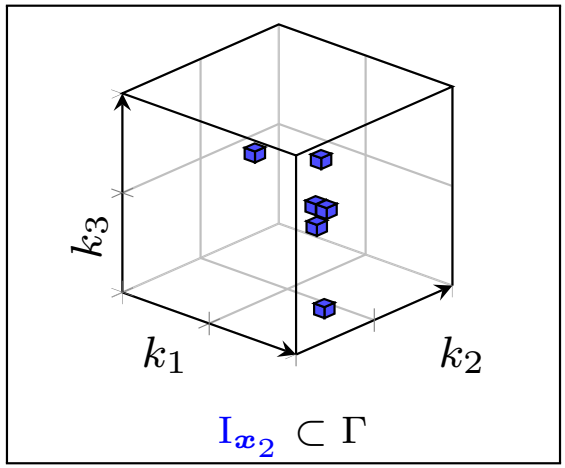
detected frequencies
$\mathrm{I}_{\boldsymbol{x}_3}^{(1)}$



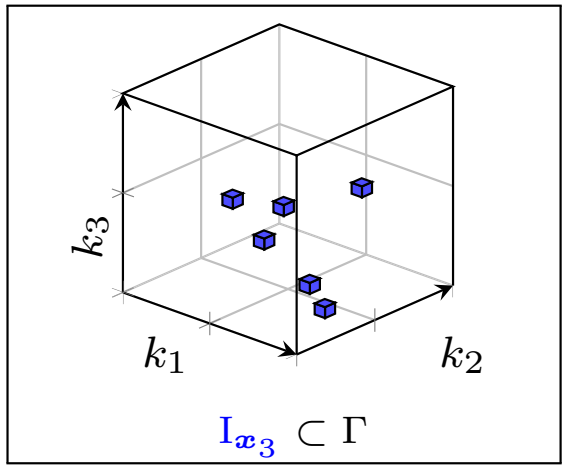
detected frequencies
$\mathrm{I}^{(1)} := \mathrm{I}_{\boldsymbol{x}_1}^{(1)} \cup \mathrm{I}_{\boldsymbol{x}_2}^{(1)} \cup \mathrm{I}_{\boldsymbol{x}_3}^{(1)}$



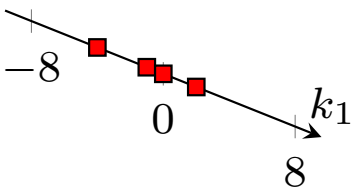
$\mathrm{I}_{\boldsymbol{x}_1} \subset \Gamma$



$\mathrm{I}_{\boldsymbol{x}_2} \subset \Gamma$

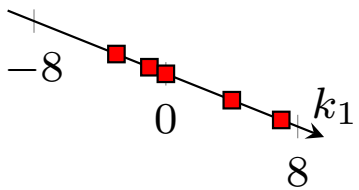


$\mathrm{I}_{\boldsymbol{x}_3} \subset \Gamma$



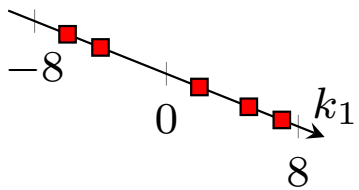
detected frequencies
$\mathrm{I}^{(1)}_{\boldsymbol{x}_1}$

$\cup$

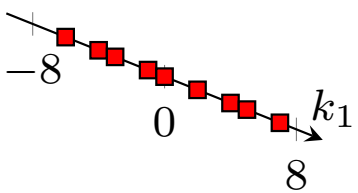


detected frequencies
$\mathrm{I}^{(1)}_{\boldsymbol{x}_2}$

$\cup$

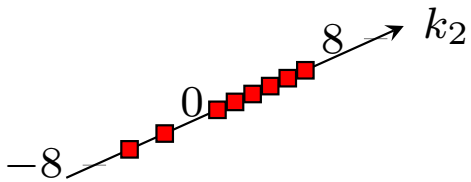


detected frequencies
$\mathrm{I}^{(1)}_{\boldsymbol{x}_3}$



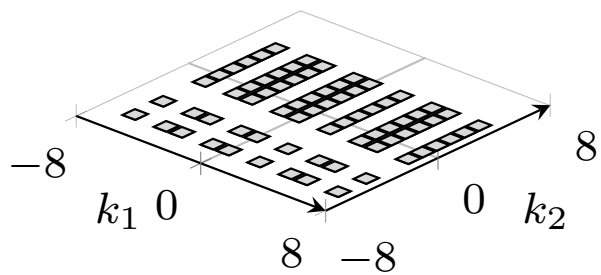
detected frequencies
$\mathrm{I}^{(1)} := \mathrm{I}^{(1)}_{\boldsymbol{x}_1} \cup \mathrm{I}^{(1)}_{\boldsymbol{x}_2} \cup \mathrm{I}^{(1)}_{\boldsymbol{x}_3}$

$\times$

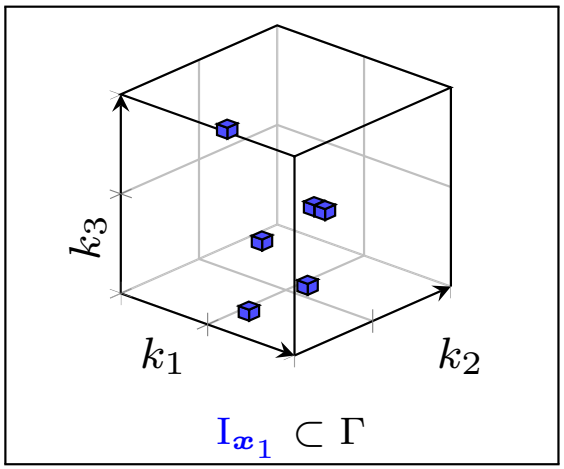


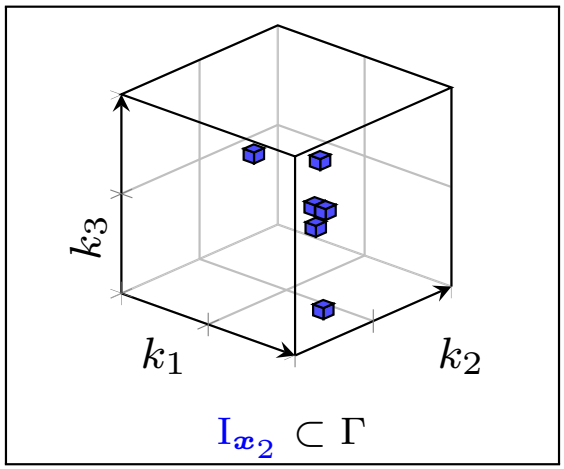
detected frequencies
$\mathrm{I}^{(2)} := \mathrm{I}^{(2)}_{\boldsymbol{x}_1} \cup \mathrm{I}^{(2)}_{\boldsymbol{x}_2} \cup \mathrm{I}^{(2)}_{\boldsymbol{x}_3}$

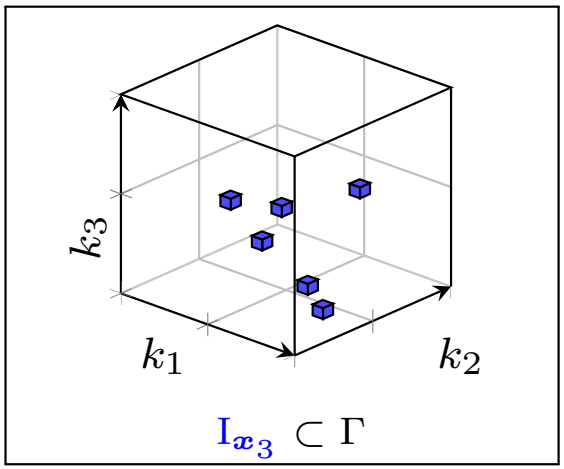
$=$

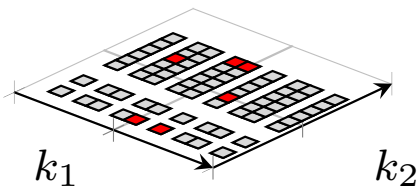


frequency candidates $\mathrm{I}^{(1)} \times \mathrm{I}^{(2)}$

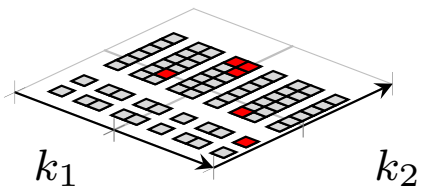




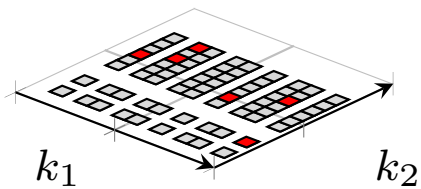




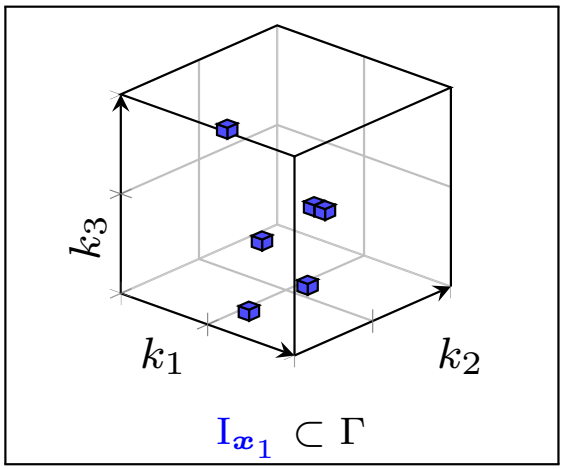
detected frequencies
$\mathrm{I}_{\boldsymbol{x}_1}^{(1,2)}$



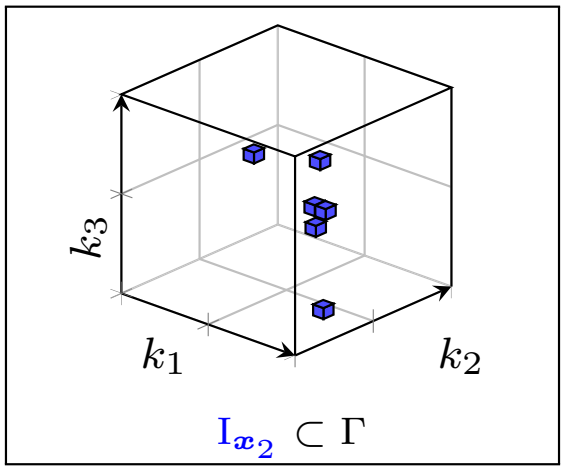
detected frequencies
$\mathrm{I}_{\boldsymbol{x}_2}^{(1,2)}$



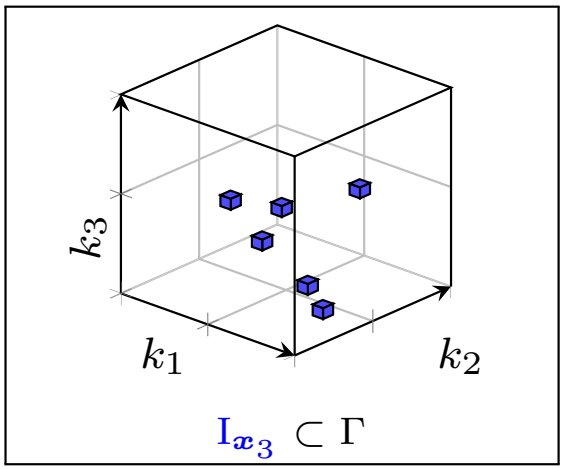
detected frequencies
$\mathrm{I}_{\boldsymbol{x}_3}^{(1,2)}$



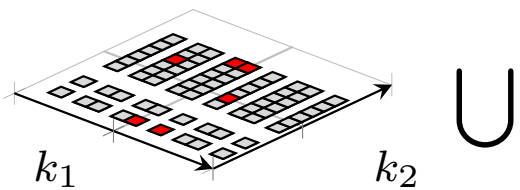
$\mathrm{I}_{\boldsymbol{x}_1} \subset \Gamma$



$\mathrm{I}_{\boldsymbol{x}_2} \subset \Gamma$

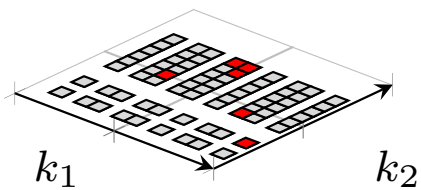


$\mathrm{I}_{\boldsymbol{x}_3} \subset \Gamma$



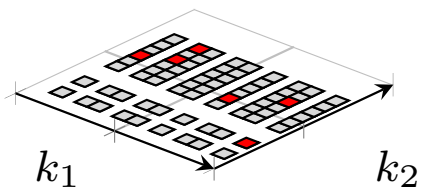
$\bigcup$

detected frequencies
$\mathrm{I}_{\boldsymbol{x}_1}^{(1,2)}$

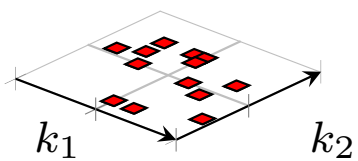


$\bigcup$

detected frequencies
$\mathrm{I}_{\boldsymbol{x}_2}^{(1,2)}$

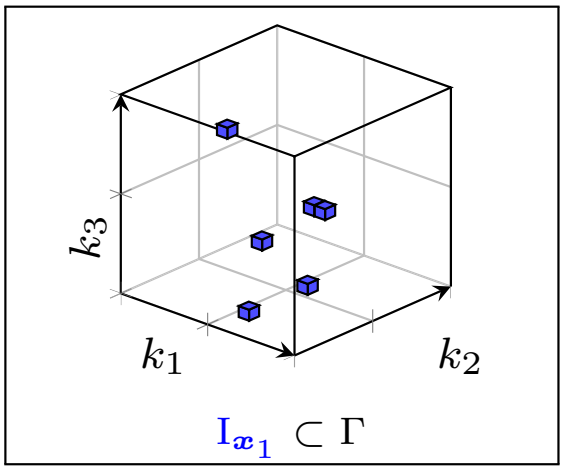


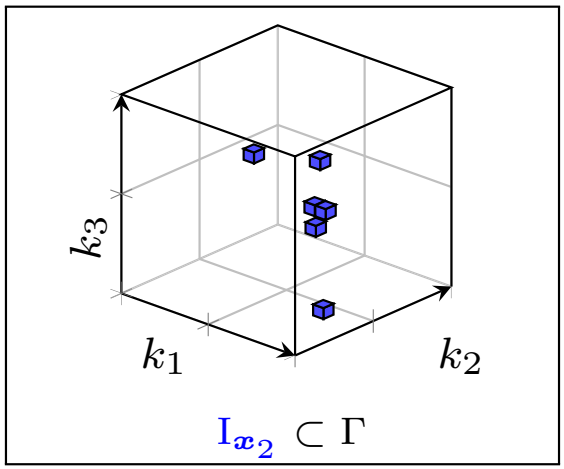
detected frequencies
$\mathrm{I}_{\boldsymbol{x}_3}^{(1,2)}$

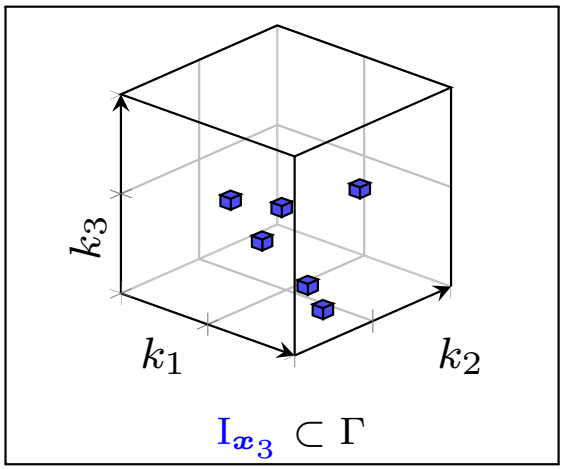


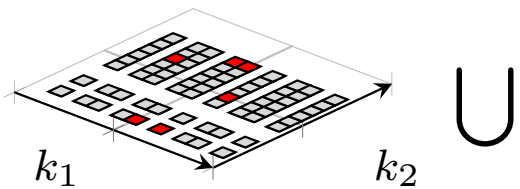
detected frequencies

$$\mathrm{I}^{(1,2)} := \mathrm{I}_{\boldsymbol{x}_1}^{(1,2)} \cup \mathrm{I}_{\boldsymbol{x}_2}^{(1,2)} \cup \mathrm{I}_{\boldsymbol{x}_3}^{(1,2)}$$



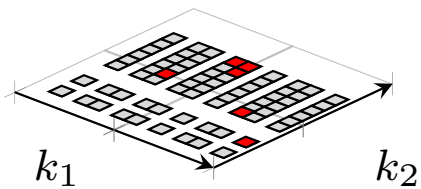






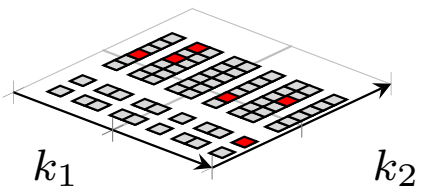
$\bigcup$

detected frequencies
$\mathrm{I}_{\boldsymbol{x}_1}^{(1,2)}$

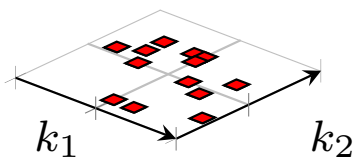


$\bigcup$

detected frequencies
$\mathrm{I}_{\boldsymbol{x}_2}^{(1,2)}$



detected frequencies
$\mathrm{I}_{\boldsymbol{x}_3}^{(1,2)}$



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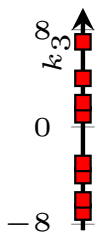
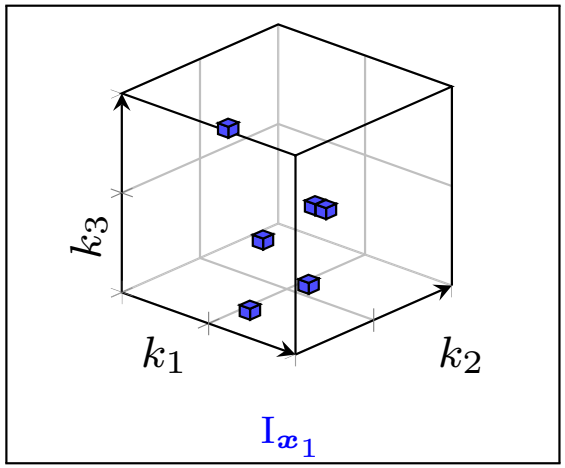


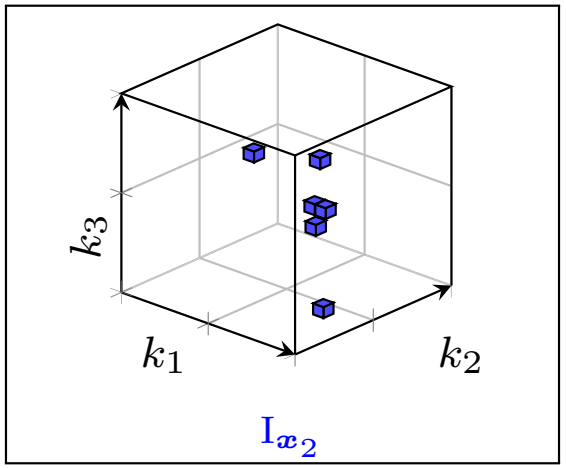
$=$

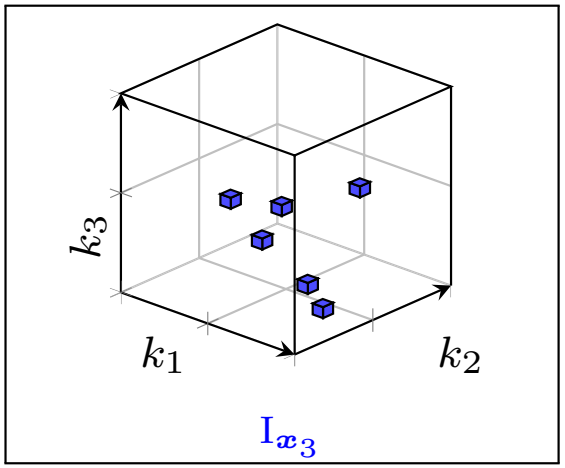
$\cdots$

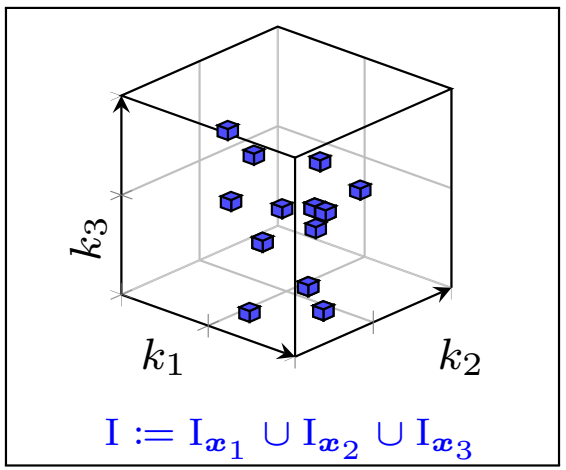
detected frequencies
$\mathrm{I}^{(1,2)} := \mathrm{I}_{\boldsymbol{x}_1}^{(1,2)} \cup \mathrm{I}_{\boldsymbol{x}_2}^{(1,2)} \cup \mathrm{I}_{\boldsymbol{x}_3}^{(1,2)}$

detected frequencies
$\mathrm{I}^{(3)} := \mathrm{I}_{\boldsymbol{x}_1}^{(3)} \cup \mathrm{I}_{\boldsymbol{x}_2}^{(3)} \cup \mathrm{I}_{\boldsymbol{x}_3}^{(3)}$









- Result:

$$u_{\boldsymbol{x}_g}(\boldsymbol{y}) \approx u_{\boldsymbol{x}_g}^{\mathtt{usFFT}}(\boldsymbol{y}) := \sum_{\boldsymbol{k}\in\mathrm{I}} c_{\boldsymbol{k}}^{\mathtt{usFFT}}\left(u_{\boldsymbol{x}_g}\right) \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \qquad g = 1, \ldots, G$$

- Output:
  - index set $\mathrm{I} \in \Gamma$ with $\mathrm{I}_{\boldsymbol{x}_g} \subset \mathrm{I}$ for all $g = 1, ..., G$
  - approximations $c_{\boldsymbol{k}}^{\mathtt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, for all $g = 1, ..., G$

- Output:
  - index set $\mathrm{I} \in \Gamma$ with $\mathrm{I}_{\boldsymbol{x}_g} \subset \mathrm{I}$ for all $g = 1, ..., G$
  - approximations $c_{\boldsymbol{k}}^{\texttt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, for all $g = 1, ..., G$

- samples $\hat{=}$ amount of PDE solutions needed:

$$\mathcal{O}\left(d\,s\,\max(s, N_\Gamma)\,\log^2\frac{d\,s\,G\,N_\Gamma}{\delta} + \max(sG, N_\Gamma)\,\log\frac{d\,s\,G}{\delta}\right)$$

- Output:
  - index set $\mathrm{I} \in \Gamma$ with $\mathrm{I}_{\boldsymbol{x}_g} \subset \mathrm{I}$ for all $g = 1, ..., G$
  - approximations $c_{\boldsymbol{k}}^{\texttt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, for all $g = 1, ..., G$

- samples $\hat{=}$ amount of PDE solutions needed:

$$\mathcal{O}\left(d\, s\, \max(s, N_\Gamma)\, \log^2 \frac{d\, s\, G\, N_\Gamma}{\delta} + \max(sG, N_\Gamma)\, \log \frac{d\, s\, G}{\delta}\right)$$

- computational complexity of the usFFT:

$$\mathcal{O}\left(d^2\, s^2\, G^2\, N_\Gamma\, \log^3 \frac{d\, s\, G\, N_\Gamma}{\delta}\right) \quad \text{with high probability } 1 - \delta$$

$$\mathcal{O}\left(d^2\, s^3\, G^2\, N_\Gamma\, \log^3 \frac{d\, s\, G\, N_\Gamma}{\delta}\right) \quad \text{worst case}$$

- Output:
  - index set $\mathrm{I} \in \Gamma$ with $\mathrm{I}_{\boldsymbol{x}_g} \subset \mathrm{I}$ for all $g = 1, ..., G$
  - approximations $c_{\boldsymbol{k}}^{\texttt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, for all $g = 1, ..., G$

- samples $\hat{=}$ amount of PDE solutions needed:

$$\mathcal{O}\left(d\, s\, \max(s, N_\Gamma)\, \log^2 \frac{d\, s\, G\, N_\Gamma}{\delta} + \max(sG, N_\Gamma)\, \log \frac{d\, s\, G}{\delta}\right)$$

- computational complexity of the usFFT:

$$\mathcal{O}\left(d^2\, s^2\, G^2\, N_\Gamma\, \log^3 \frac{d\, s\, G\, N_\Gamma}{\delta}\right) \quad \text{with high probability } 1 - \delta$$

$$\mathcal{O}\left(d^2\, s^3\, G^2\, N_\Gamma\, \log^3 \frac{d\, s\, G\, N_\Gamma}{\delta}\right) \quad \text{worst case}$$

- Output:
  - index set $\mathrm{I} \in \Gamma$ with $\mathrm{I}_{\boldsymbol{x}_g} \subset \mathrm{I}$ for all $g = 1, ..., G$
  - approximations $c_{\boldsymbol{k}}^{\texttt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, for all $g = 1, ..., G$

- samples $\hat{=}$ amount of PDE solutions needed: $G \lesssim ds$

$$\mathcal{O}\left(d\,s\,\max(s, N_\Gamma)\,\log^2\frac{d\,s\,G\,N_\Gamma}{\delta} + \max(sG, N_\Gamma)\,\log\frac{d\,s\,G}{\delta}\right)$$

- computational complexity of the usFFT:

$$\mathcal{O}\left(d^2\,s^2\,G^2\,N_\Gamma\,\log^3\frac{d\,s\,G\,N_\Gamma}{\delta}\right) \quad \text{with high probability } 1-\delta$$

$$\mathcal{O}\left(d^2\,s^3\,G^2\,N_\Gamma\,\log^3\frac{d\,s\,G\,N_\Gamma}{\delta}\right) \quad \text{worst case}$$

Consider the problem

$$\begin{aligned} -\nabla_{\boldsymbol{x}} \cdot (a(\boldsymbol{x},\boldsymbol{y})\nabla_{\boldsymbol{x}} u(\boldsymbol{x},\boldsymbol{y})) &= 1 && \boldsymbol{x} \in D,\, \boldsymbol{y} \in D_{\boldsymbol{y}} \\ u(\boldsymbol{x},\boldsymbol{y}) &= 0 && \forall \boldsymbol{x} \in \partial D,\, \boldsymbol{y} \in D_{\boldsymbol{y}} \end{aligned}$$

with $D = (0,1)^2$, $D_{\boldsymbol{y}} = [-1,1]^{20}$, $\boldsymbol{y} \sim \mathcal{U}\left([-1,1]^{20}\right)$ and the random coefficient

$$a(\boldsymbol{x},\boldsymbol{y}) = 1 + \sum_{j=1}^{20} y_j \psi_j(\boldsymbol{x})$$

with

$$\psi_j(\boldsymbol{x}) := \frac{0.9}{\zeta(2)} j^{-2} \cos(2\pi m_1(j) x_1) \cos(2\pi m_2(j) x_2), \qquad \boldsymbol{x} \in D,\, j \geq 1.$$

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$m_1(j)$	0	1	0	1	2	0	1	2	3	0	1	2	3	4	...
$m_2(j)$	1	0	2	1	0	3	2	1	0	4	3	2	1	0	...

Example taken from
M. Eigel, C. J. Gittelson, C. Schwab and E. Zander.
Adaptive stochastic Galerkin FEM.
Comput. Methods Appl. Mech. Engrg., 270: 247–269, 2014.

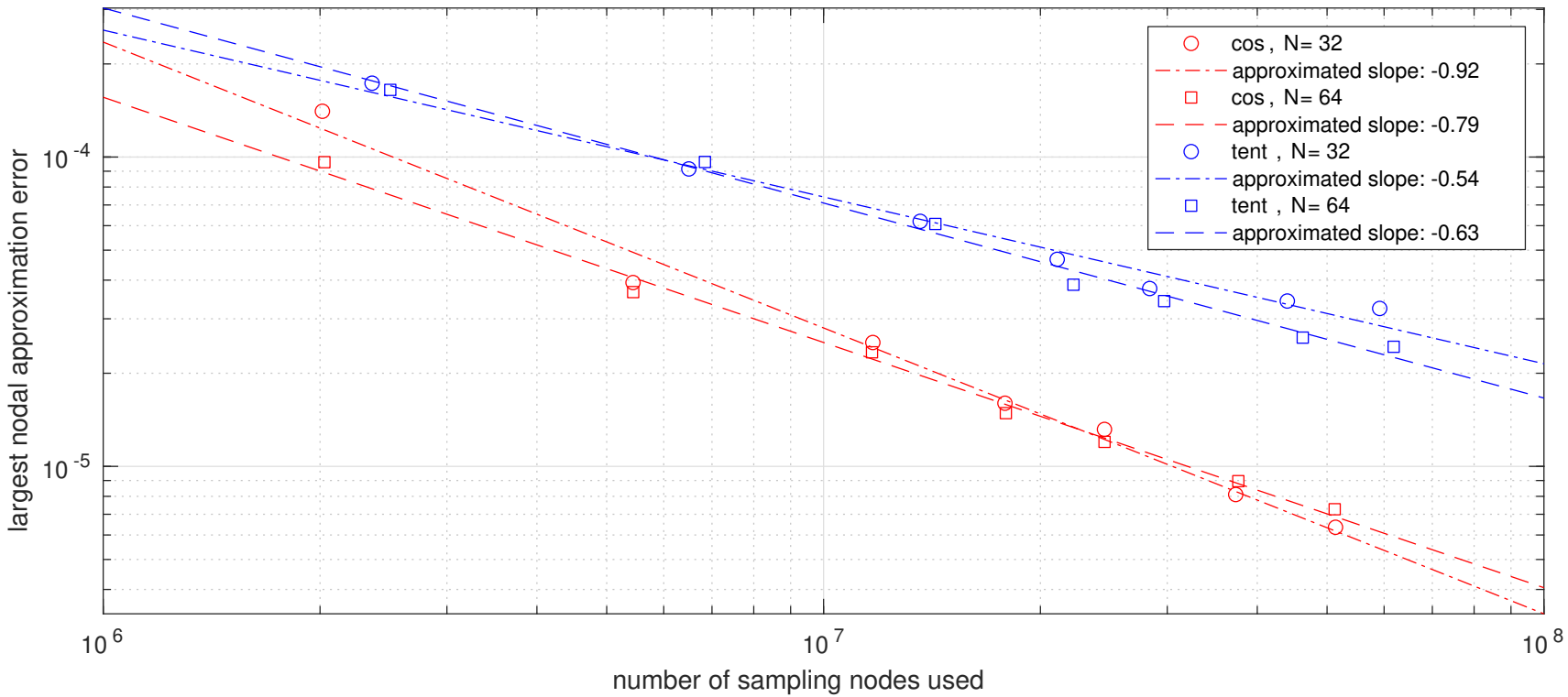


Figure: Largest error err_2^η w.r.t. the nodes $\boldsymbol{x}_g$ for different parameter settings, i.e., $s = 100, 250, 500, 750, 1000, 1500, 2000$, for the affine example.

$$\mathrm{err}_2^\eta(\boldsymbol{x}_g) := \sqrt{\frac{1}{n_{\text{test}}} \sum_{j=1}^{n_{\text{test}}} \left| \check{u}\left(\boldsymbol{x}_g, \boldsymbol{y}^{(j)}\right) - u^{\texttt{usFFT}}\left(\boldsymbol{x}_g, \boldsymbol{y}^{(j)}\right) \right|^2}$$

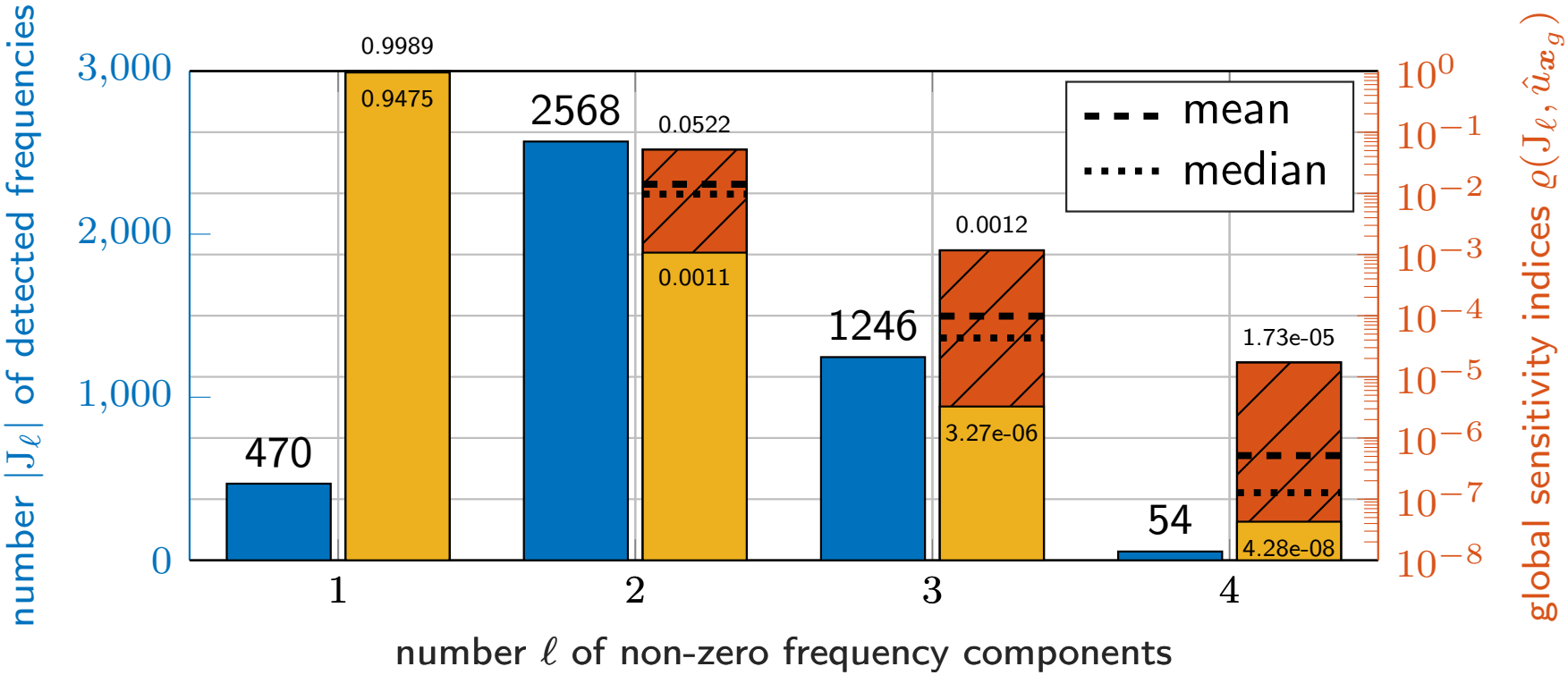


Figure: Analysis of the approximation for the affine example with $s = 2000,\ N = 32$.

$$\varrho(\mathrm{J}, \tilde{u}^{\mathtt{usFFT}}_{\boldsymbol{x}_g}) := \frac{\sigma^2(\tilde{u}^{\mathtt{usFFT}}_{\boldsymbol{x}_g,\mathrm{J}})}{\sigma^2(\tilde{u}^{\mathtt{usFFT}}_{\boldsymbol{x}_g})} = \frac{\sum_{\boldsymbol{k}\in\mathrm{J}\setminus\{\mathbf{0}\}} |c^{\mathtt{usFFT}}_{\boldsymbol{k}}(\tilde{u}_{\boldsymbol{x}_g})|^2}{\sum_{\boldsymbol{k}\in\mathrm{I}\setminus\{\mathbf{0}\}} |c^{\mathtt{usFFT}}_{\boldsymbol{k}}(\tilde{u}_{\boldsymbol{x}_g})|^2} \in [0,1],$$

Consider the problem

$$\begin{aligned} -\nabla_{\boldsymbol{x}} \cdot (a(\boldsymbol{x},\boldsymbol{y})\nabla_{\boldsymbol{x}} u(\boldsymbol{x},\boldsymbol{y})) &= f(\boldsymbol{x}) && \boldsymbol{x} \in D,\, \boldsymbol{y} \in D_{\boldsymbol{y}} \\ u(\boldsymbol{x},\boldsymbol{y}) &= 0 && \forall \boldsymbol{x} \in \partial D,\, \boldsymbol{y} \in D_{\boldsymbol{y}} \end{aligned}$$

with $f(\boldsymbol{x}) = \sin(1.3\pi x_1 + 3.4\pi x_2)\cos(4.3\pi x_1 - 3.1\pi x_2)$, $D = (0,1)^2$, $D_{\boldsymbol{y}} = \mathbb{R}^{10}$, $\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ and the random coefficient

$$\log a(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{10} \frac{y_j}{j} \sin(2\pi j x_1)\cos(2\pi(11-j)x_2).$$

Example taken and modified from
M. Cheng, T. Y. Hou, M. Yan and Z. Zhang.
A data-driven stochastic method for elliptic PDEs with random coefficients.
SIAM/ASA J. Uncertain. Quantif., 1: 452-493, 2013.

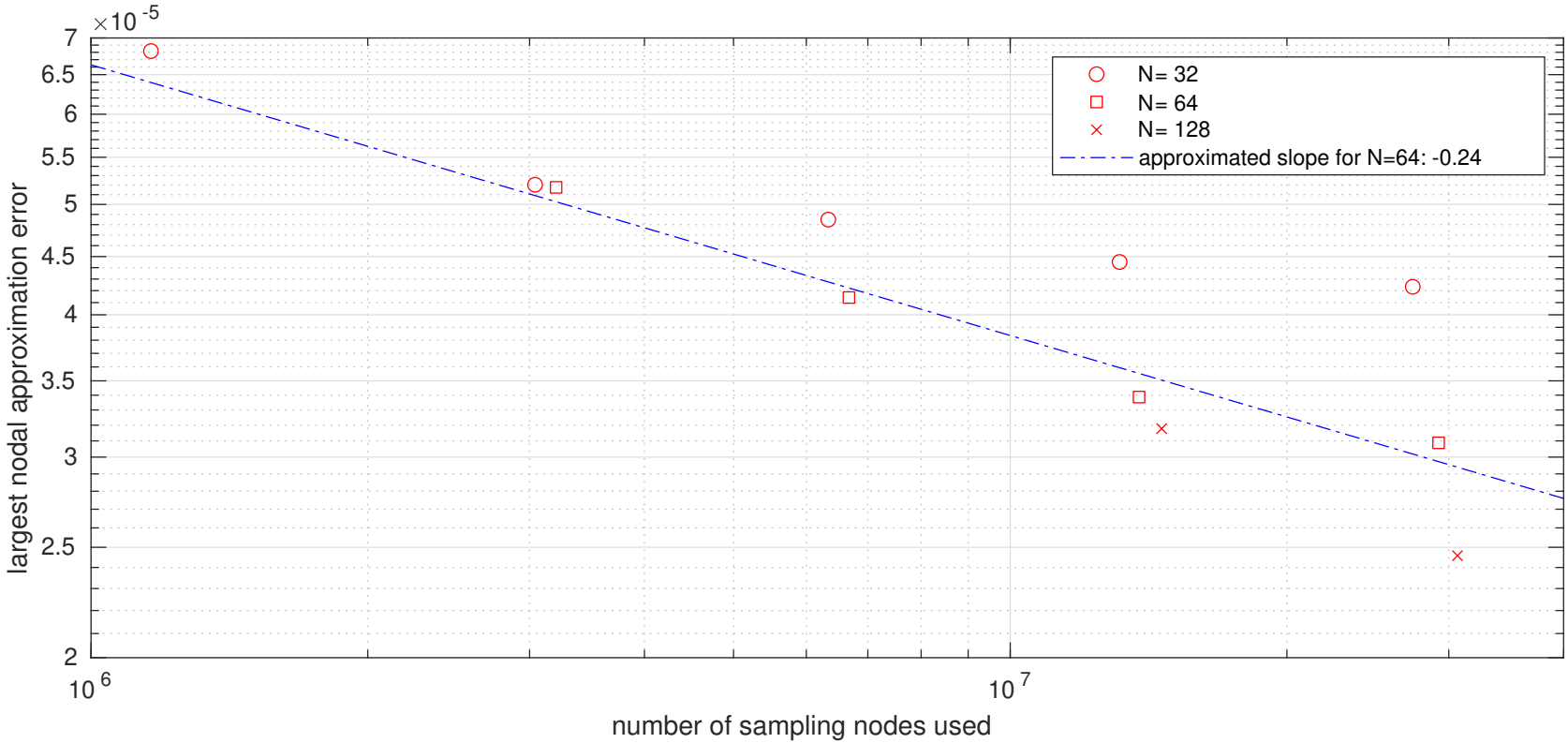


Figure: Largest error err_2^η w.r.t. the nodes $\boldsymbol{x}_g$ for different parameter settings, i.e., $s = 100, 250, 500, 1000, 2000$, for the lognormal example.

$$\mathrm{err}_2^\eta(\boldsymbol{x}_g) := \sqrt{\frac{1}{n_{\text{test}}} \sum_{j=1}^{n_{\text{test}}} \left| \breve{u}\left(\boldsymbol{x}_g, \boldsymbol{y}^{(j)}\right) - u^{\texttt{usFFT}}\left(\boldsymbol{x}_g, \boldsymbol{y}^{(j)}\right) \right|^2}$$

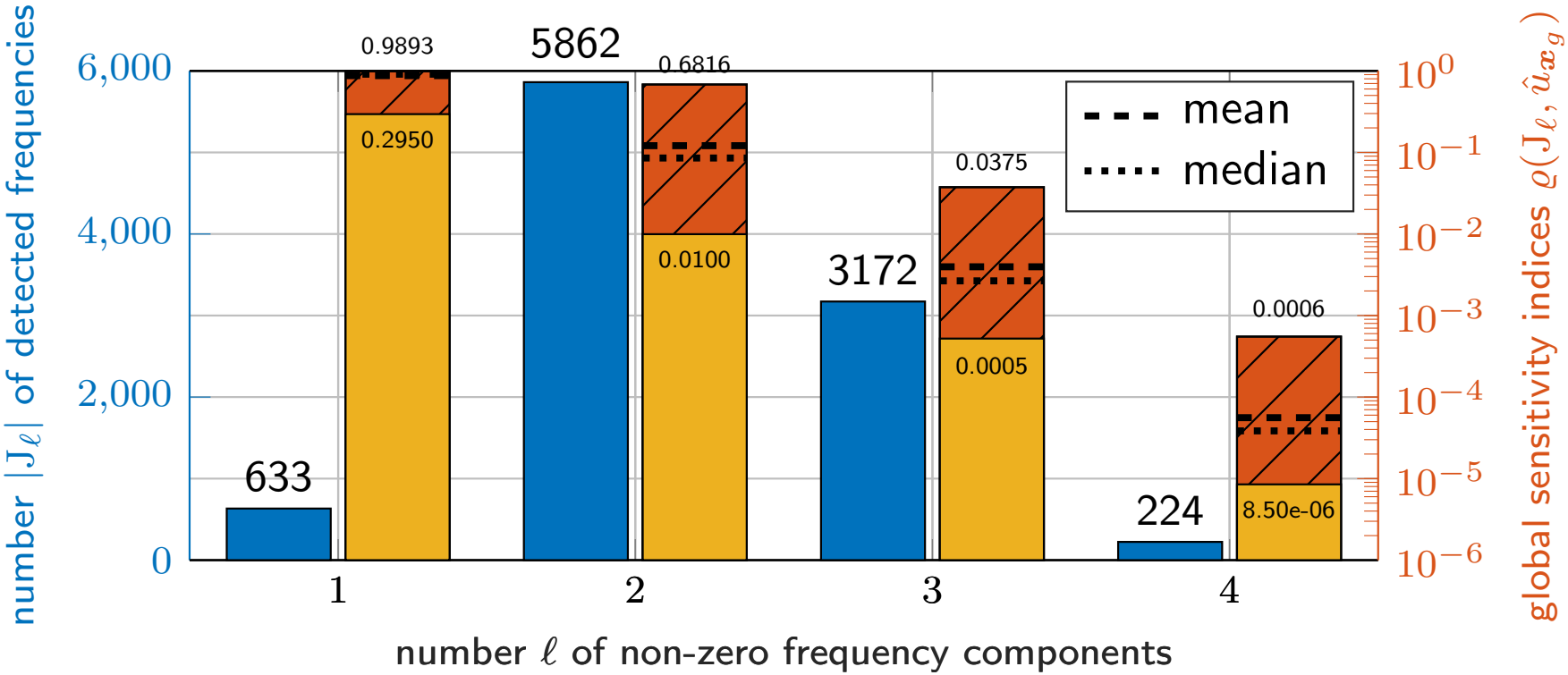


Figure: Analysis of the approximation for the lognormal example with $s = 2000,\ N = 32$.

$$\varrho(\mathrm{J}, \tilde{u}_{\boldsymbol{x}_g}^{\mathtt{usFFT}}) := \frac{\sigma^2(\tilde{u}_{\boldsymbol{x}_g,\mathrm{J}}^{\mathtt{usFFT}})}{\sigma^2(\tilde{u}_{\boldsymbol{x}_g}^{\mathtt{usFFT}})} = \frac{\sum_{\boldsymbol{k}\in\mathrm{J}\setminus\{\mathbf{0}\}} |c_{\boldsymbol{k}}^{\mathtt{usFFT}}(\tilde{u}_{\boldsymbol{x}_g})|^2}{\sum_{\boldsymbol{k}\in\mathrm{I}\setminus\{\mathbf{0}\}} |c_{\boldsymbol{k}}^{\mathtt{usFFT}}(\tilde{u}_{\boldsymbol{x}_g})|^2} \in [0,1],$$

- Main advantages of the usFFT:
  - fully adaptive, no critical a priori choice needed
  - sample efficient (in terms of sampling locations)
  - approximation gives insight on the influence and interactions of the y_j
  - adapts easily to other domains, boundary conditions, ...
  - non-intrusive and parallelizable

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- Lutz Kämmerer, Daniel Potts, Fabian Taubert
  The uniform sparse FFT with application to PDEs with random coefficients
  ArXiv e-prints, 2021. arXiv:2109.04131 [math.NA]

- Thank you for your attention!