# Nonlinear approximation in bounded orthonormal product bases

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joint work with Lutz Kämmerer and Daniel Potts

Workshop on Mathematical Signal and Image Analysis

21.03.2023

UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
CHEMNITZ

## General aim in the Fourier setting

Approximation (by using samples) of a (smooth) high-dimensional function $f \in L_2(\mathbb{T}^d, \mu)$

### by using samples

black-box sampling, so we choose the sampling nodes $\boldsymbol{x}$ adaptively

### Hilbert space $L_2(\mathbb{T}^d, \mu)$

domain $\mathbb{T}^d = \times_{j=1}^d \mathbb{T} \subset \mathbb{R}^d$

measure $\mu = \times_{j=1}^d \mu_j$

basis $\Phi_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^d \mathrm{e}^{2\pi \mathrm{i} k_j x_j}$

### smooth function $f$

$$f(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$$

### Truncation and approximation

$$S_I^{\mathcal{A}} f(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in I} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$$

### To compute

- approximated coefficients $\hat{f}_{\boldsymbol{k}} \approx c_{\boldsymbol{k}} \; \forall \boldsymbol{k} \in I$
- suitable sparse index set $I \subset \mathbb{Z}^d$

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## General aim for an arbitrary bounded orthonormal product basis (BOPB)

Approximation (by using samples) of a (smooth) high-dimensional function $f \in L_2(\mathcal{D}, \mu)$

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black-box sampling, so we choose the sampling nodes $\boldsymbol{x}$ adaptively

### Hilbert space $L_2(\mathcal{D}, \mu)$

domain $\mathcal{D} = \times_{j=1}^{d} \mathcal{D}_j \subset \mathbb{R}^d$

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$$f(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in \mathbb{N}^d} c_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\boldsymbol{x})$$

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Problem: How to find a good, $s$-sparse index set $I$?

- first idea:
  - choose a large search space $\Gamma \supset I, |\Gamma| \gg |I|$
  - compute all coefficients
  - choose indices corresponding to the $s$ largest coefficients (absolute value)
- but: unfeasible in practice for large $\Gamma$ ($\rightarrow$ "curse of dimensionality")
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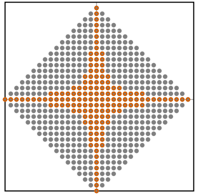

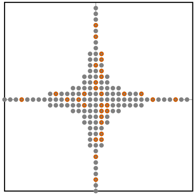

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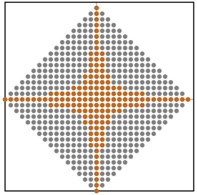

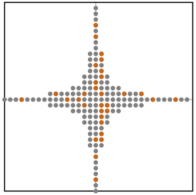

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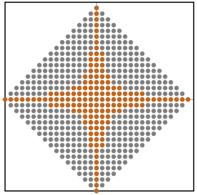

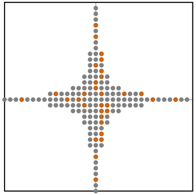

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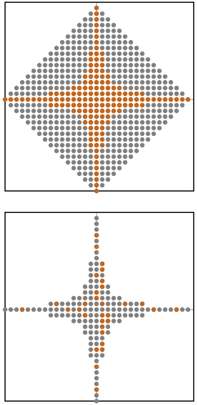

Question: Can we ensure to detect all important indices?

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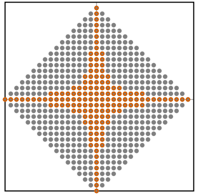
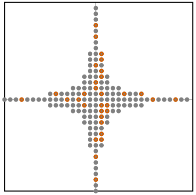
Question: Can we ensure to detect all important indices? $\rightarrow$ YES!





Previous works on sparse, high-dimensional approximation:

- our dimension-incremental approach in the Fourier setting:
  - sparse FFT using rank-1 lattices
    [Potts, Volkmer '16], [Kämmerer, Potts, Volkmer '21], [Kämmerer, Krahmer, Volkmer '22]
  - application to differential equations with high-dimensional random coefficients
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- other approximation methods in the Fourier setting:
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## Projected coefficients for the dimensions $\{1,\dots,t\}$

$$c_{\{1,\dots,t\},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) := \int_{\mathcal{D}_{\{1,\dots,t\}}} f(\boldsymbol{\xi},\tilde{\boldsymbol{x}})\,\overline{\Phi_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})}\,\mathrm{d}\mu_{\{1,\dots,t\}}(\boldsymbol{\xi}) \qquad \forall \boldsymbol{k}\in\mathbb{N}^t,\ \forall\tilde{\boldsymbol{x}}\in\mathcal{D}_{\{t+1,\dots,d\}}$$

Projected coefficients are an indicator for the importance of the indices $(\boldsymbol{k},\boldsymbol{h})$ with arbitrary $\boldsymbol{h}$, since

$$c_{\mathfrak{u},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) = \ldots = \sum_{\boldsymbol{h}\in\mathbb{N}^{d-t}} c_{(\boldsymbol{k},\boldsymbol{h})}\Phi_{\{t+1,\dots,d\},(\boldsymbol{k},\boldsymbol{h})}(\tilde{\boldsymbol{x}}).$$

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→ Works analogously for single dimensions $\{t\}$ and arbitrary $\mathfrak{u}\subset\{1,\dots,d\}$.

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## Key idea of the dimension-incremental algorithm

- Construct a candidate set with possible indices in the current dimension(s).
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- Build the index set: $\begin{cases} \text{projected coefficient large}: & \text{keep index} \\ \text{projected coefficient small}: & \text{neglect index} \end{cases}$

$\rightarrow$ Increase the dimension by combining different detected index sets of smaller dimension.

Approximation of the projected coefficients via, e.g., cubature formulas (nodes $\boldsymbol{\xi}_j$, weights $w_j$)

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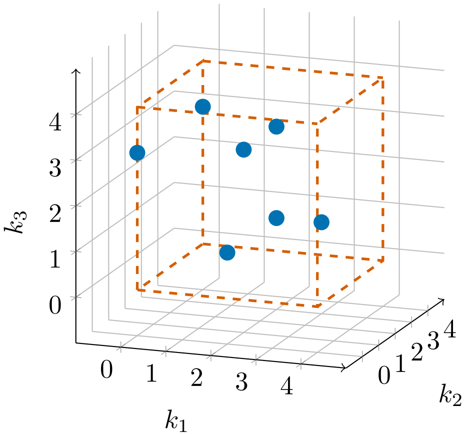
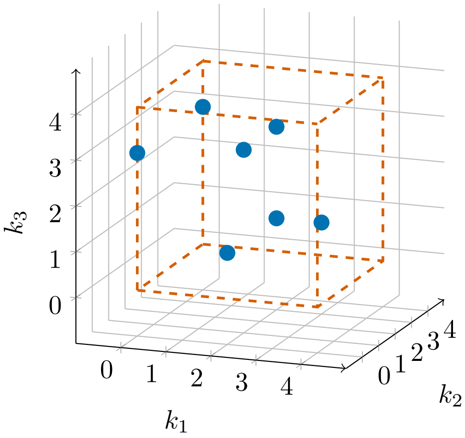



Figure: The desired but unknown index set $I$ and the search space $\Gamma = \{0, 1, 2, 3, 4\}^3$.

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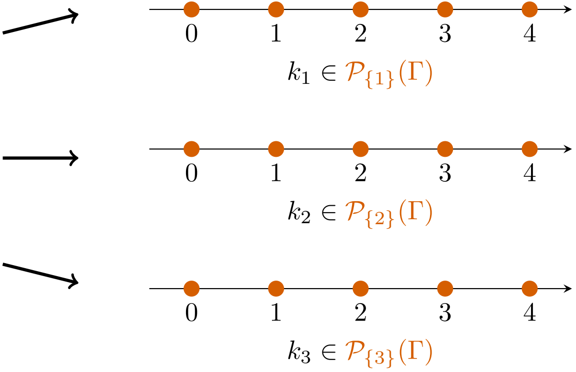
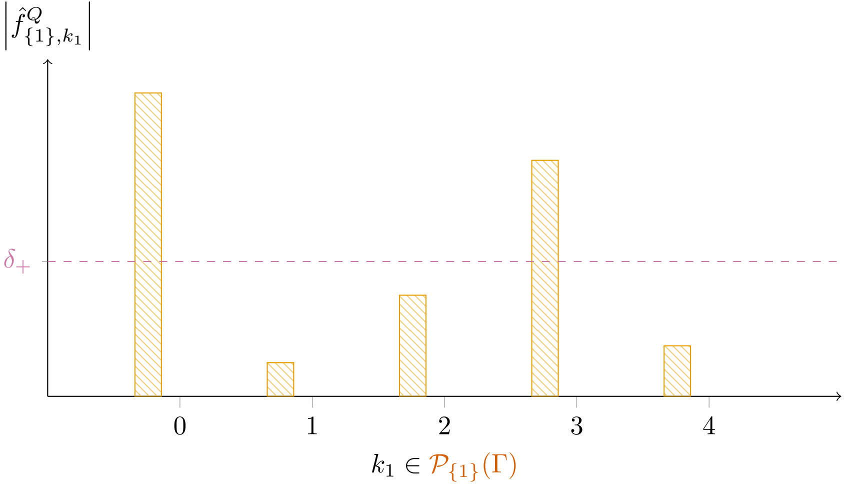



Figure: The one-dimensional candidate sets $\mathcal{P}_{\{j\}}(\Gamma)$.

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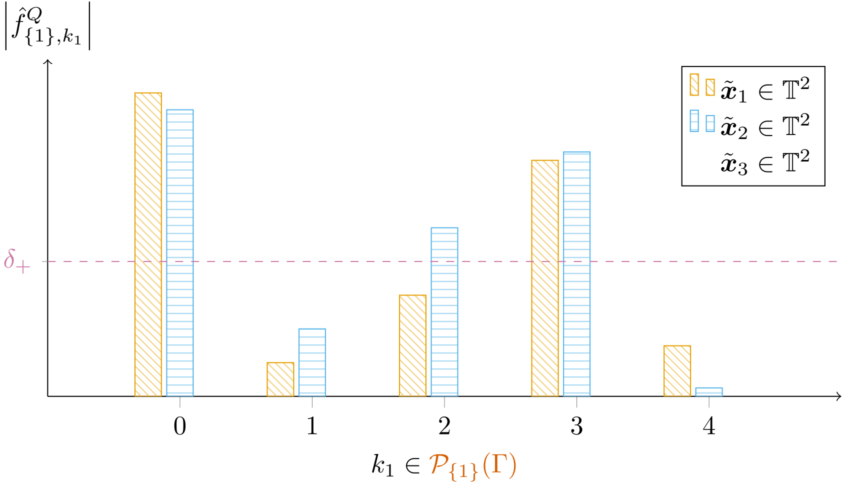


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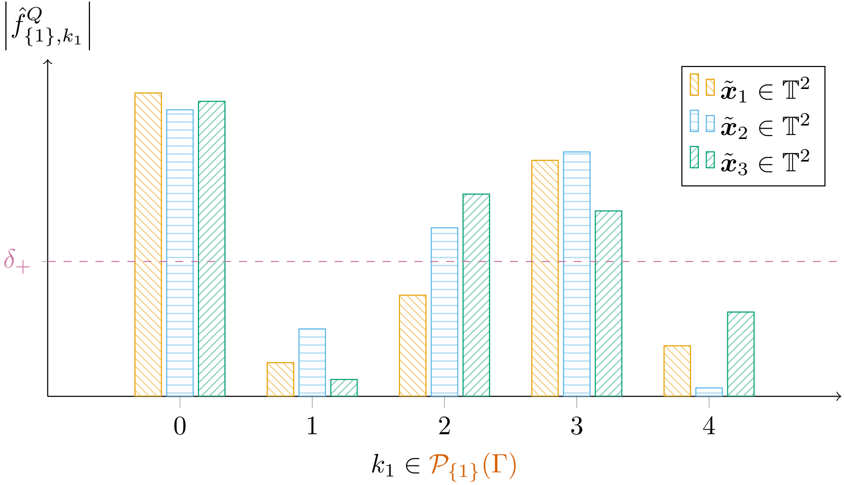


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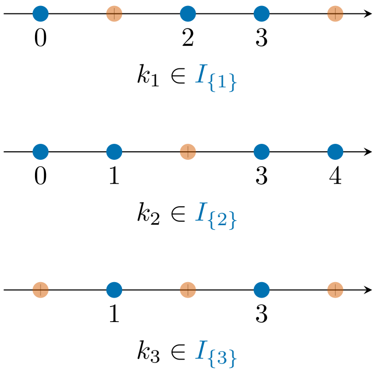
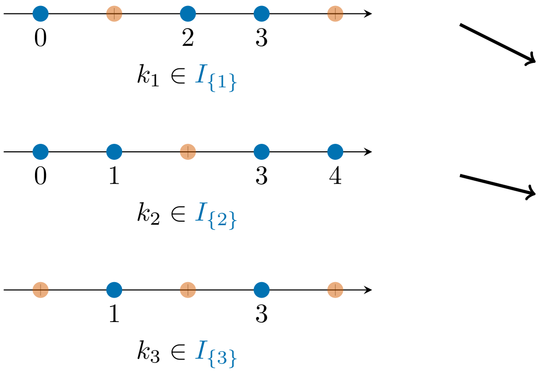



Figure: The one-dimensional index sets $I_{\{j\}}$.

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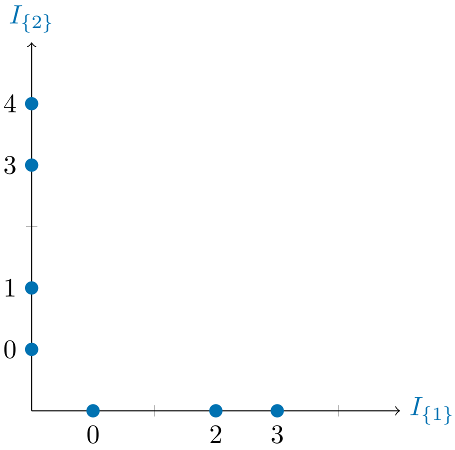
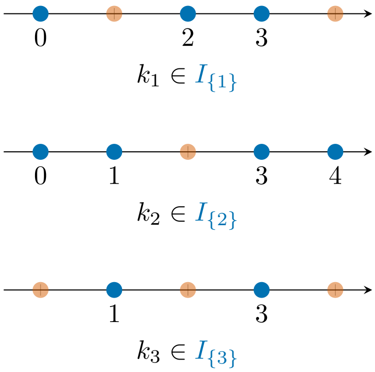



Figure: The two-dimensional candidate set $I_{\{1\}} \times I_{\{2\}}$.

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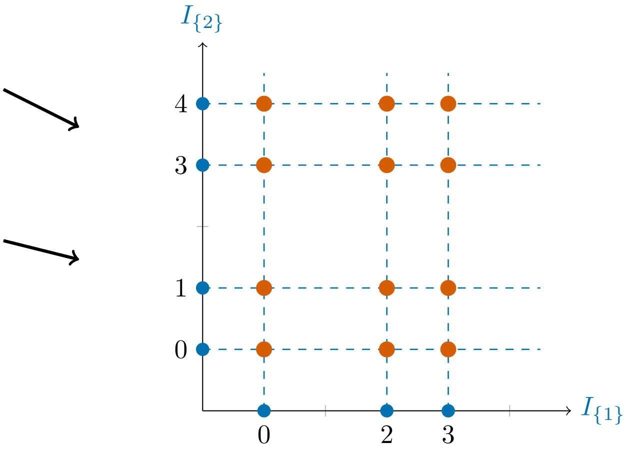


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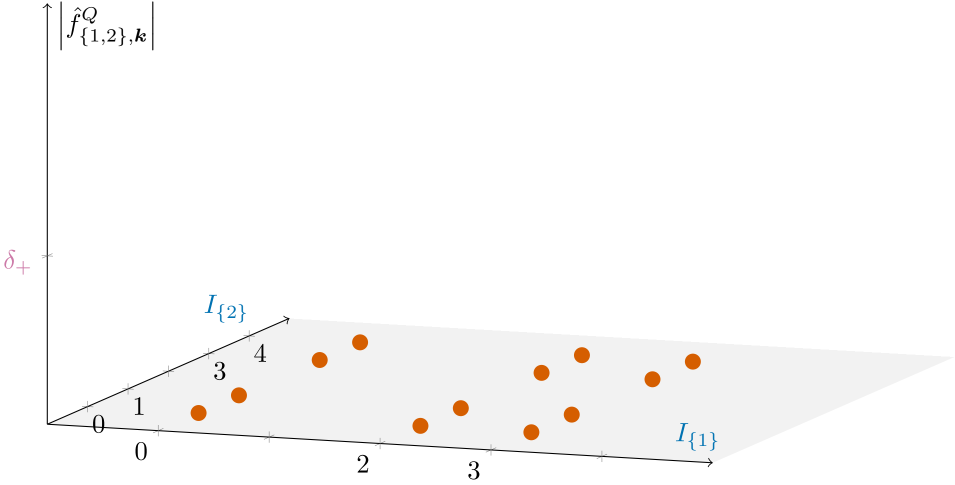
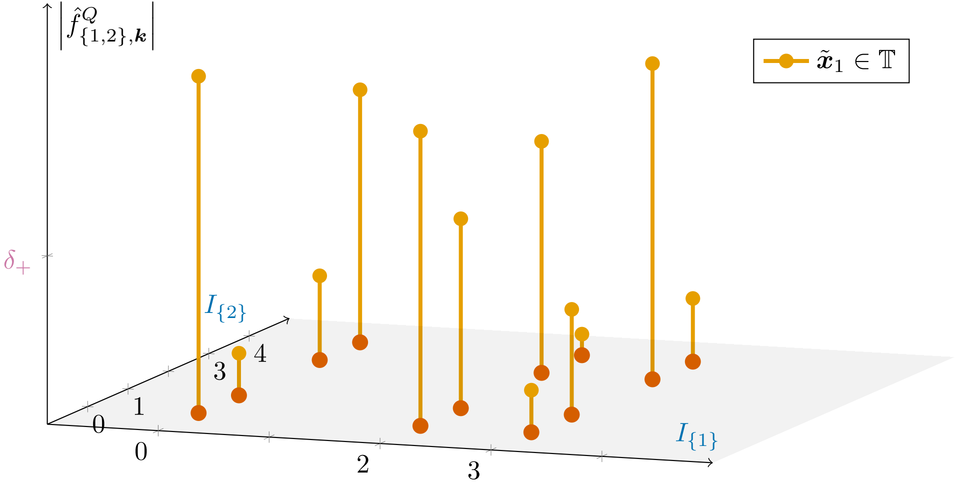



Figure: The two-dimensional detection step in the dimensions $\{1, 2\}$. (Only $r = 1$ detection iteration shown.)

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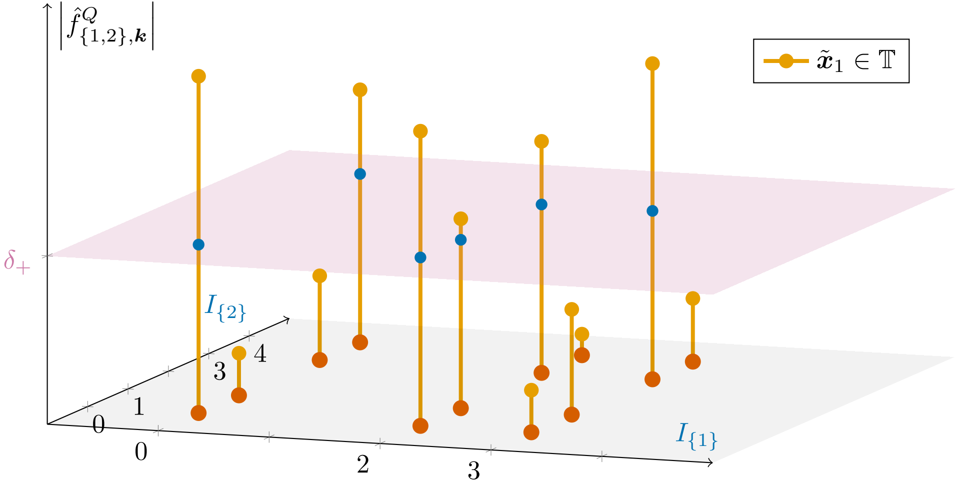


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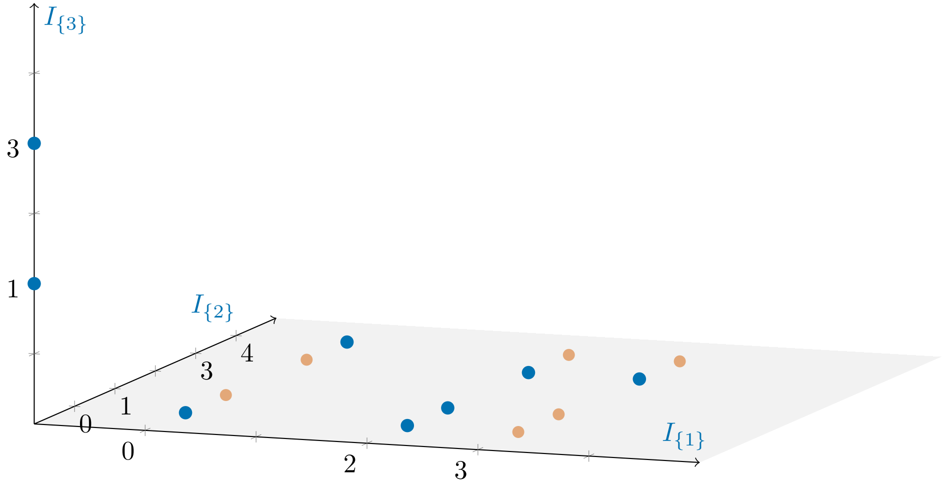
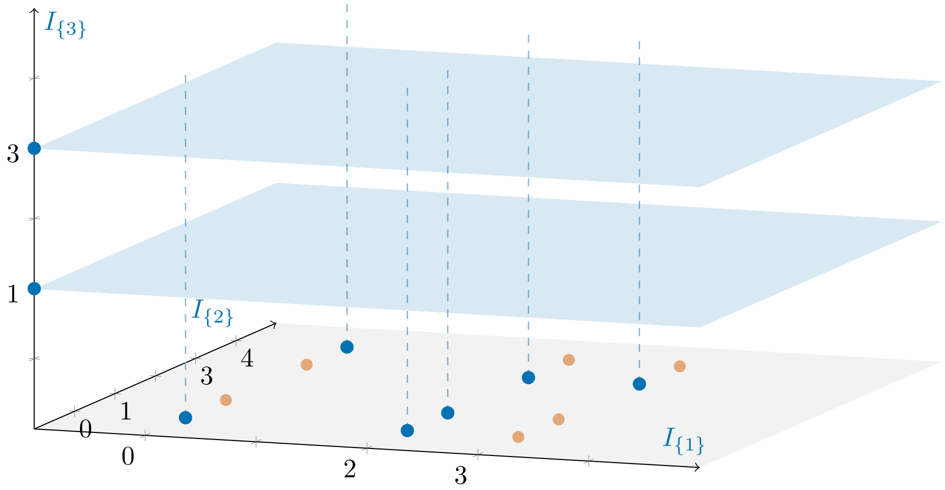



Figure: The two-dimensional index set $I_{\{1,2\}}$ and the three-dimensional candidate set $I_{\{1,2\}} \times I_{\{3\}}$.

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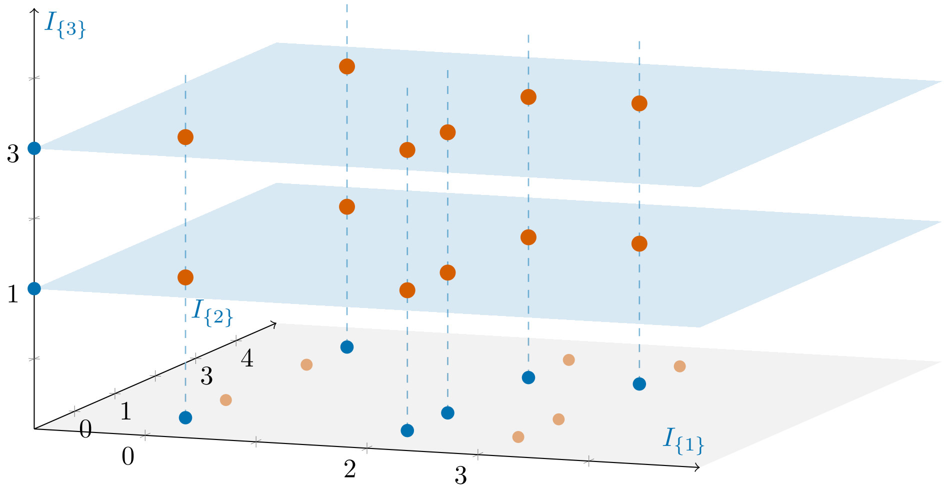


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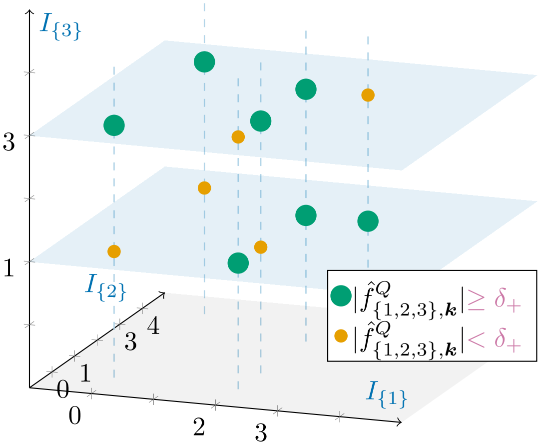


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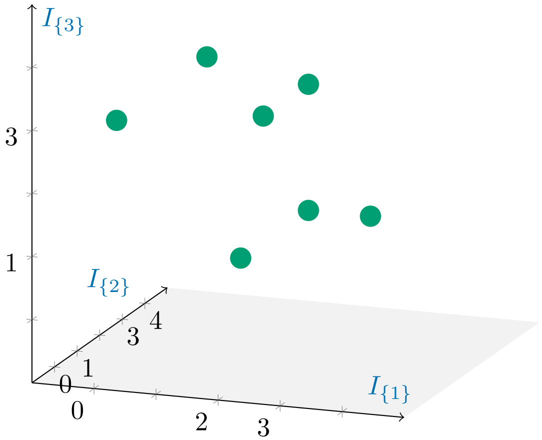


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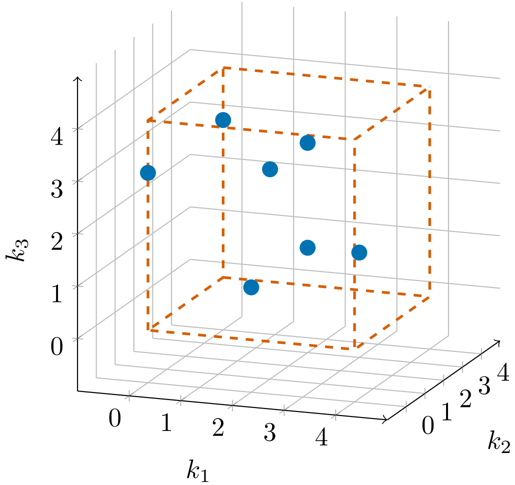


Figure: The correctly detected index set $I = I_{\{1,2,3\}}$.

Previously: Only recovery results for sparse trigonometric polynomials.
Now:

**Theorem [Kämmerer, Potts, T. '22]**

- given: function $f \in L_2(\mathcal{D},\mu)$, BOPB constant $B$, threshold $\delta > 0$, failure prob. $\varepsilon \in (0,1)$
- index set: $I_{3\delta} := \{\boldsymbol{k} \in \mathbb{N}^d : |c_{\boldsymbol{k}}| \geq 3\delta\}$
- cubature method: bound on proj. error $\delta_\Psi < \delta$, weight constant $C > 0$, failure prob. $\leq \varepsilon/(3d)$
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  - detection threshold $\delta_+ < 3B^{-1} \min_{\boldsymbol{k}\in I_{3\delta}} |c_{\boldsymbol{k}}| - \delta_\Psi$
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- **Then, with probability $1-\varepsilon$, the output index set $I$ of the Algorithm contains $I_{3\delta}$.**

**Improvements**: works for **function approximation** in **any** BOPB!
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## Test function (from [Potts, Volkmer '16], [Kämmerer, Krahmer, Volkmer '21])

$$f(\boldsymbol{x}) := \prod_{j\in\{1,3,8\}} N_2(x_j) + \prod_{j\in\{2,5,6,10\}} N_4(x_j) + \prod_{j\in\{4,7,9\}} N_6(x_j)$$

$f \in L_2(\mathbb{T}^{10}, \mu)$, with the B-Spline of order $m \in \mathbb{N}$

$$N_m(x) := C_m \sum_{k\in\mathbb{Z}} \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m (-1)^k \mathrm{e}^{2\pi \mathrm{i} k x}$$

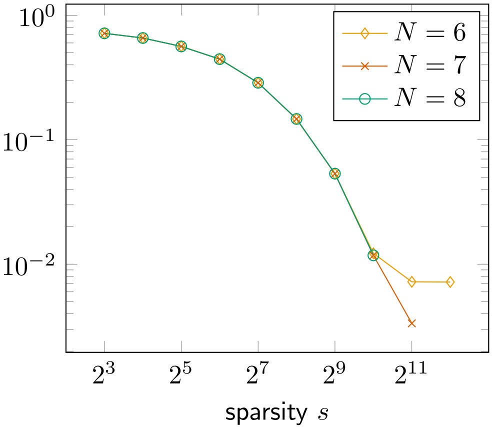
## Parameters

- hyperbolic cross search space:
  $\Gamma = \left\{\boldsymbol{k} \in \mathbb{Z}^d : \prod_{j=1}^d \max(1, 2\,|k_j|) \leq 2^N\right\}$
- detection iterations $r = 5$
- detection threshold $\delta_+ = 10^{-12}$

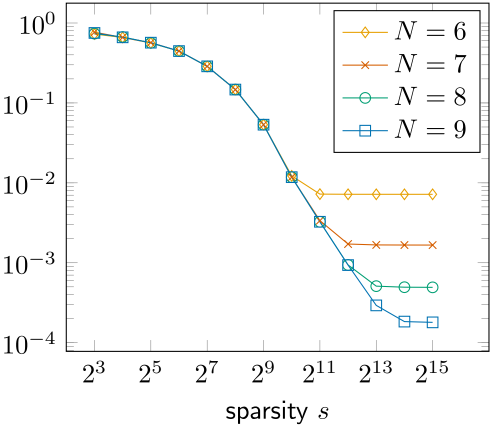
## Approximation error ("How good is $I$?")

- rel. $L_2(\mathbb{T}^{10}, \mu)$ error:

$$\frac{\|f - S_I f\|_{L_2(\mathbb{T}^{10})}}{\|f\|_{L_2(\mathbb{T}^{10})}}$$

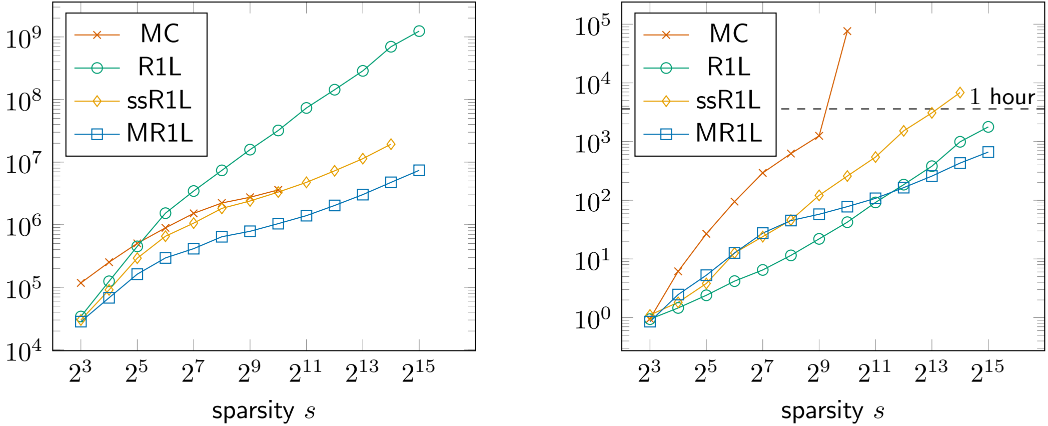



(a) approx. error for MC




(b) approx. error for R1L, ssR1L & MR1L

Figure: Approximation results for the 10-dimensional periodic test function




(a) amount of samples for $N = 8$

(b) computation time (in seconds) for $N = 8$

Figure: Approximation results for the 10-dimensional periodic test function

## Test function (from [Volkmer '17], [Potts, Volkmer '17])

$$f(\boldsymbol{x}) := \prod_{j\in\{1,3,4,7\}} B_2(x_j) + \prod_{j\in\{2,5,6,8,9\}} B_4(x_j)$$

$f \in L_2([-1,1]^9, \mu_{\mathrm{Cheb}})$, with $B_2$ and $B_4$ shifted, scaled and dilated B-Splines of order $2$ and $4$

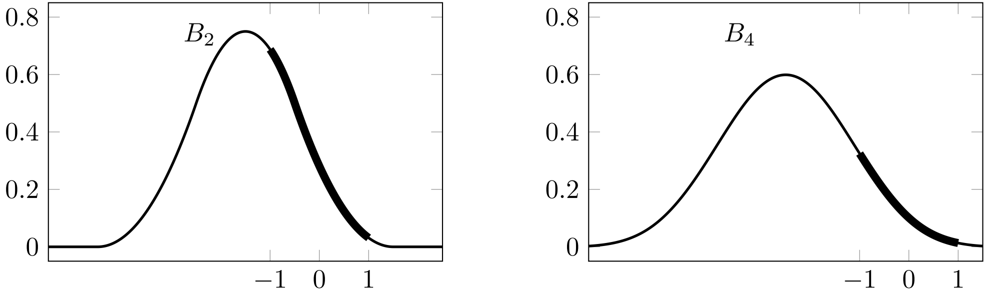
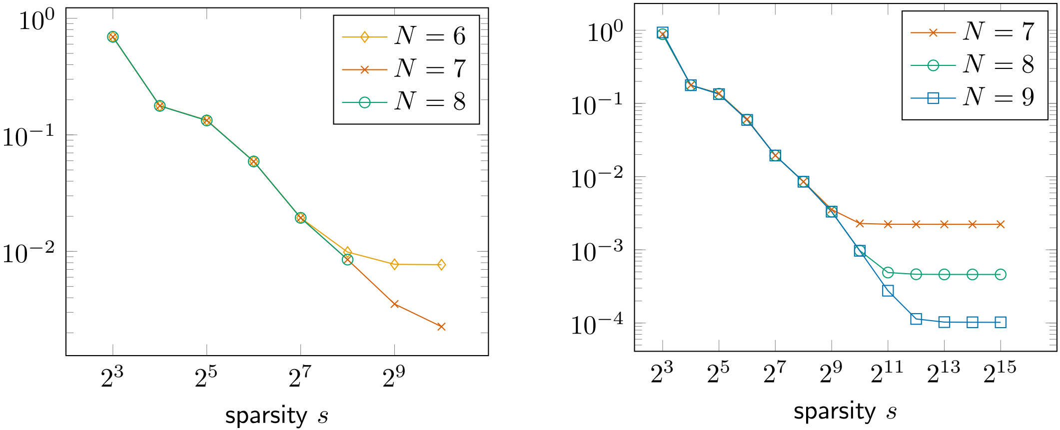



Figure: B-Splines $B_2$ and $B_4$ and the considered domain $[-1,1]$

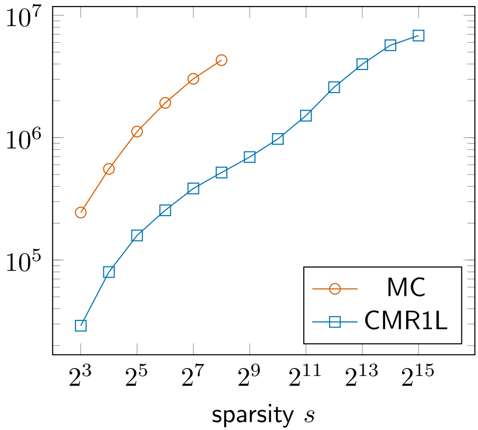



(a) approx. error for MC



(b) approx. error for CMR1L

Figure: Approximation results for the 9-dimensional non-periodic test function

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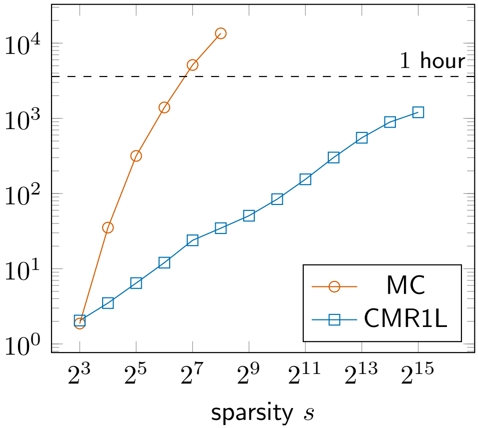


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Figure: Approximation results for the $9$-dimensional non-periodic test function

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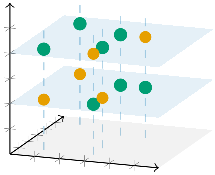
  - generalization of the dimension-incremental method to arbitrary BOPB
  - generalized algorithm also works
    - for various search spaces $\Gamma$
    - with other dimension-incremental strategies (dyadic, data-driven, ...)
    - ...



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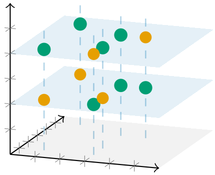

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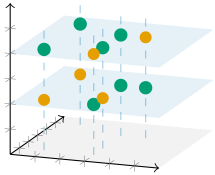

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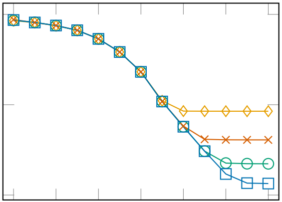

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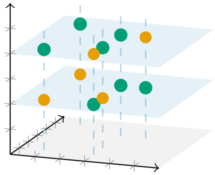

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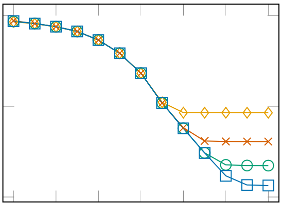

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**Thank you for your attention!**

**Questions? Ideas? Suggestions?**