A dimension-incremental function approximation method for arbitrary bounded orthonormal product bases

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joint work with Lutz Kämmerer and Daniel Potts

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Applied Functional Analysis

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General aim in the Fourier setting

Approximation (by using samples) of a (smooth) high-dimensional function $f \in L_2(\mathbb{T}^d, \mu)$

by using samples

black-box sampling, so we choose the sampling nodes $\boldsymbol{x}$ adaptively

Hilbert space $L_2(\mathbb{T}^d, \mu)$

domain $\mathbb{T}^d = \times_{j=1}^d \mathbb{T} \subset \mathbb{R}^d$

measure $\mu = \times_{j=1}^d \mu_j$

basis $\Phi_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^d \mathrm{e}^{2\pi \mathrm{i} k_j x_j}$

smooth function f

$$f(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$$

Truncation and approximation

$$S_{\mathrm{I}}^{\mathcal{A}} f(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in \mathrm{I}} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}$$

To compute

- approximated coefficients $\hat{f}_{\boldsymbol{k}} \approx c_{\boldsymbol{k}} \; \forall \boldsymbol{k} \in \mathrm{I}$
- suitable sparse index set $\mathrm{I} \subset \mathbb{Z}^d$

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General aim for an arbitrary bounded orthonormal product basis (BOPB)

Approximation (by using samples) of a (smooth) high-dimensional function $f \in L_2(\mathcal{D}, \mu)$

by using samples

black-box sampling, so we choose the sampling nodes $\boldsymbol{x}$ adaptively

Hilbert space $L_2(\mathcal{D}, \mu)$

domain $\mathcal{D} = \times_{j=1}^{d} \mathcal{D}_j \subset \mathbb{R}^d$

measure $\mu = \times_{j=1}^{d} \mu_j$

basis $\Phi_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{d} \phi_{j,k_j}(x_j)$

smooth function f

$$f(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in \mathbb{N}^d} c_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\boldsymbol{x})$$

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- first idea:
  - choose a large search space $\Gamma \supset \mathrm{I}, |\Gamma| \gg |\mathrm{I}|$
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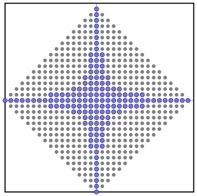

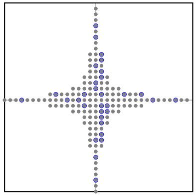

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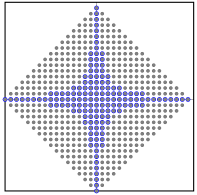

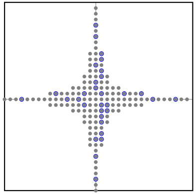

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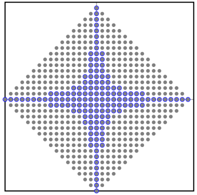

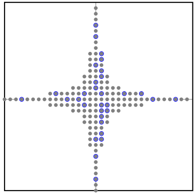

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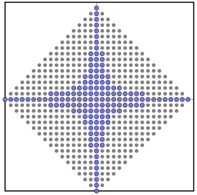

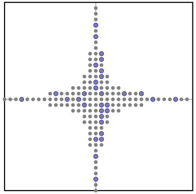

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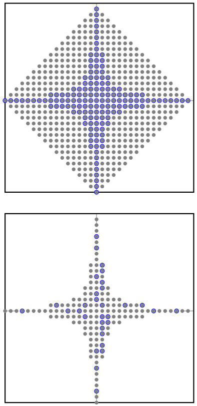
Question: Can we ensure to detect all important indices? $\rightarrow$ YES!



Previous works on sparse, high-dimensional approximation:

- our dimension-incremental approach in the Fourier setting:
  - sparse FFT using rank-1 lattices
    [Potts, Volkmer '16], [Kämmerer, Potts, Volkmer '21], [Kämmerer, Krahmer, Volkmer '22]
  - application to differential equations with high-dimensional random coefficients
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- other approximation methods in the Fourier setting:
  e.g. [Iwen '13], [Indyk, Kapralov '14], [Choi, Christlieb, Wang '19], ...
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Projected coefficients for the dimensions $\{1, \dots, t\}$

$$c_{\{1,\dots,t\},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) \coloneqq \int_{\mathcal{D}_{\{1,\dots,t\}}} f(\boldsymbol{\xi}, \tilde{\boldsymbol{x}})\, \overline{\Phi_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})}\, \mathrm{d}\mu_{\{1,\dots,t\}}(\boldsymbol{\xi}) \qquad \forall \tilde{\boldsymbol{x}} \in \mathcal{D}_{\{t+1,\dots,d\}}$$

Projected coefficients are an indicator for the importance of the indices $(\boldsymbol{k}, \boldsymbol{h})$ with arbitrary $\boldsymbol{h}$, since

$$c_{\mathfrak{u},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) = \dots = \sum_{\boldsymbol{h}\in\mathbb{N}^{d-t}} c_{(\boldsymbol{k},\boldsymbol{h})} \Phi_{\{t+1,\dots,d\},(\boldsymbol{k},\boldsymbol{h})}(\tilde{\boldsymbol{x}}).$$

Example: Fourier setting with $d = 3$ and $t = 2$

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$\rightarrow$ Works analogously for single dimensions $\{t\}$ and arbitrary $\mathfrak{u} \subset \{1, \dots, d\}$.

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Key idea of the dimension-incremental algorithm

- Construct a candidate set with possible indices in the current dimension(s).
- Compute approximations of projected coefficients: use a reconstruction algorithm (e.g. via a cubature formula).
- Build the index set: keep indices with large projected coefficients and neglect indices with small projected coefficients.

→ Increase the dimension by combining different detected index sets of smaller dimension.

Example for a kind of reconstruction methods: cubature formulas (nodes ξ_j, weights w_j)

$$c_{\{1,\dots,t\},\boldsymbol{k}}(\tilde{\boldsymbol{x}}) := \int_{\mathcal{D}_{\{1,\dots,t\}}} f(\boldsymbol{\xi},\tilde{\boldsymbol{x}})\,\overline{\Phi_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})}\,\mathrm{d}\mu_{\{1,\dots,t\}}(\boldsymbol{\xi})$$
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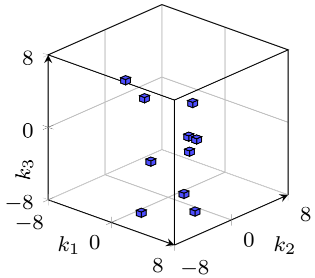
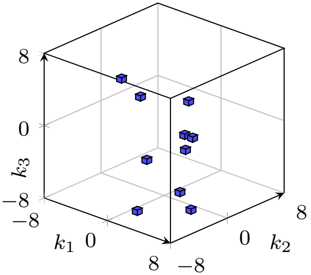


$\mathrm{I} \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \dots, 8\}^3$

index candidates

- choose candidates according to search space Γ
- construct sampling nodes using a random anchor $\tilde{\boldsymbol{x}} = (\tilde{x}_2, \tilde{x}_3)$
- compute approx. proj. coef. $\hat{f}_{\{1\},k_1}(\tilde{\boldsymbol{x}})$
- choose s indices by thresholding $|\hat{f}_{\{1\},k_1}(\tilde{\boldsymbol{x}})| \geq \delta_+$

sampling nodes $\{(\frac{\ell}{17}, \tilde{x}_2, \tilde{x}_3)\}_{\ell=0}^{16}$



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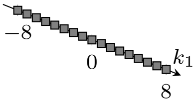

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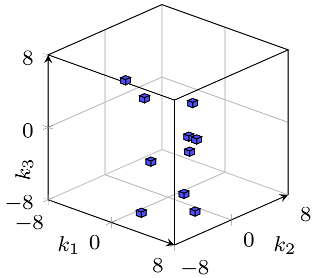

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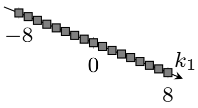

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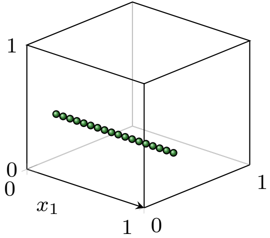
construct $\longrightarrow$ sampling set



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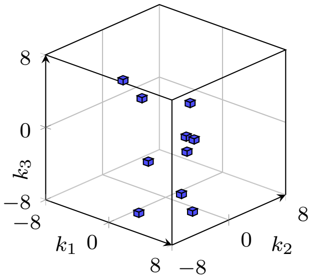

$\mathrm{I} \subset \Gamma = \hat{G}_8^3 := \{-8, -7, \ldots, 8\}^3$

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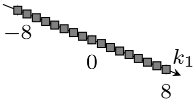
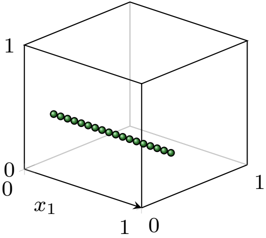


index candidates

reconstruction
⟵
algorithm



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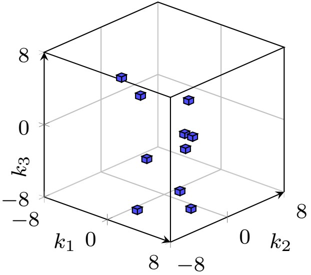

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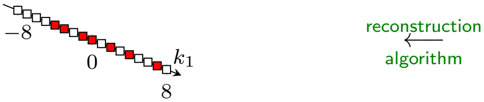
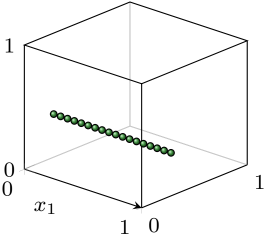


detected indices $\mathrm{I}^{(1)}$



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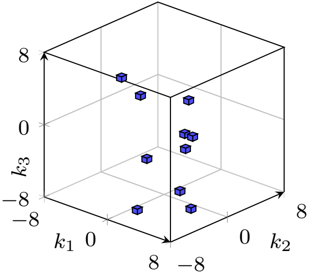

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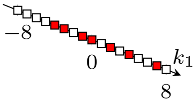
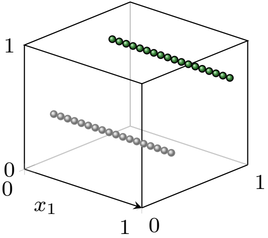


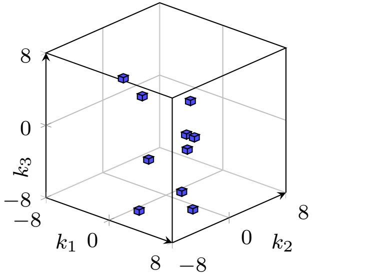
detected indices $\mathrm{I}^{(1)}$

reconstruction ⟵ algorithm



sampling nodes $\{(\frac{\ell}{17}, \tilde{x}_2, \tilde{x}_3)\}_{\ell=0}^{16}$

+ repeat with other $\tilde{\boldsymbol{x}}$ (r detection iterations)



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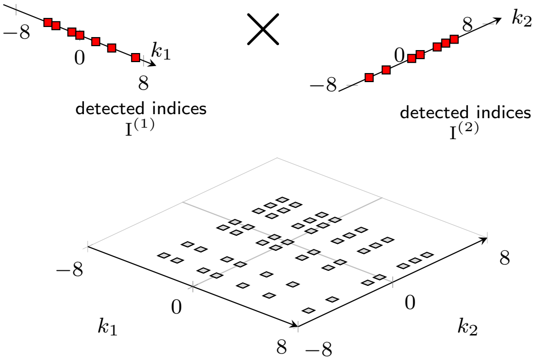
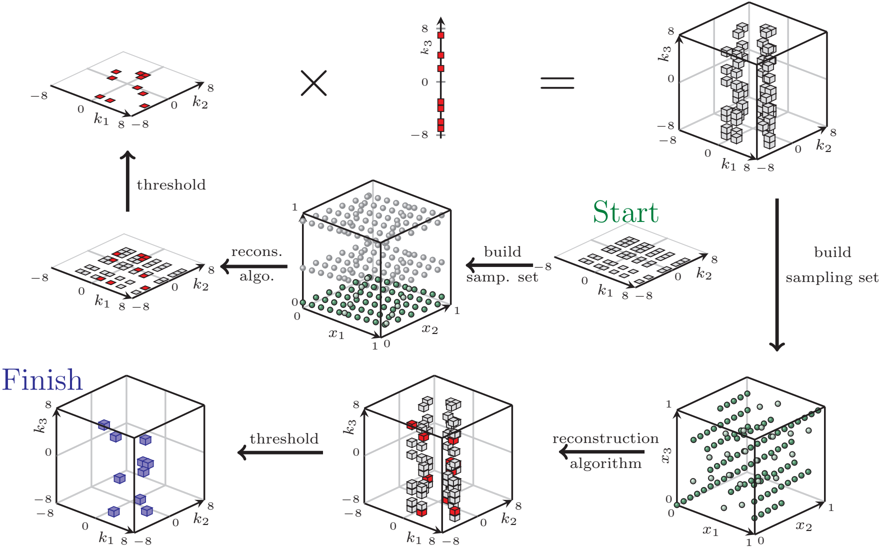


index candidates $\mathrm{I}^{(1)} \times \mathrm{I}^{(2)}$


Start
build
sampling set
reconstruction
algorithm
threshold
Finish
build
samp. set
recons.
algo.
threshold

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Now:

Theorem [Kämmerer, Potts, T. '22]

- given: function $f \in L_2(\mathcal{D}, \mu)$, BOPB constant B, threshold $\delta > 0$, failure prob. $\varepsilon \in (0,1)$
- index set: $\mathrm{I}_{3\delta} := \{\boldsymbol{k} \in \mathbb{N}^d : |c_{\boldsymbol{k}}| \geq 3\delta\}$
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Improvements: works for **function approximation** in **any** BOPB!

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Proof as a blueprint:

- Show, that a single approximated projected coefficient is small (= forgotten) with probability $q < 1$ (due to approximation and random anchor $\tilde{\boldsymbol{x}}$) though it should be large.
- Iteration over r gives us the probability $q^r < 1$ of missing *one* index.
- Apply union bounds to bound the probability of missing *any* index (still depends on q^r).
- Show bound on r such that failure probability for each step becomes smaller than $\varepsilon/(3d)$. Then:

$$\begin{aligned}\mathbb{P}\left(\bigcap_{t=1}^{d} E_{1,t} \cap \bigcap_{t=2}^{d} E_{2,t} \cap \bigcap_{t=2}^{d} E_{3,t}\right) &= 1 - \mathbb{P}\left(\bigcup_{t=1}^{d} E_{1,t}^{\mathrm{c}} \cup \bigcup_{t=2}^{d} E_{2,t}^{\mathrm{c}} \cup \bigcup_{t=2}^{d} E_{3,t}^{\mathrm{c}}\right) \\ &\geq 1 - \sum_{t=1}^{d} \mathbb{P}(E_{1,t}^{\mathrm{c}}) - \sum_{t=2}^{d} \mathbb{P}(E_{2,t}^{\mathrm{c}}) - \sum_{t=2}^{d} \mathbb{P}(E_{3,t}^{\mathrm{c}}) = 1 - \varepsilon.\end{aligned}$$

$E_{1,t}$ = one-dimensional detections work successfully
$E_{2,t}$ = construction of sampling set and reconstruction work successfully
$E_{3,t}$ = higher-dimensional detections work successfully

Test function (from [Potts, Volkmer '16], [Kämmerer, Krahmer, Volkmer '21])

$$f(\boldsymbol{x}) := \prod_{j\in\{1,3,8\}} N_2(x_j) + \prod_{j\in\{2,5,6,10\}} N_4(x_j) + \prod_{j\in\{4,7,9\}} N_6(x_j)$$

$f \in L_2(\mathbb{T}^{10}, \mu)$, with the B-Spline of order $m \in \mathbb{N}$

$$N_m(x) := C_m \sum_{k\in\mathbb{Z}} \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m (-1)^k \mathrm{e}^{2\pi \mathrm{i} kx}$$

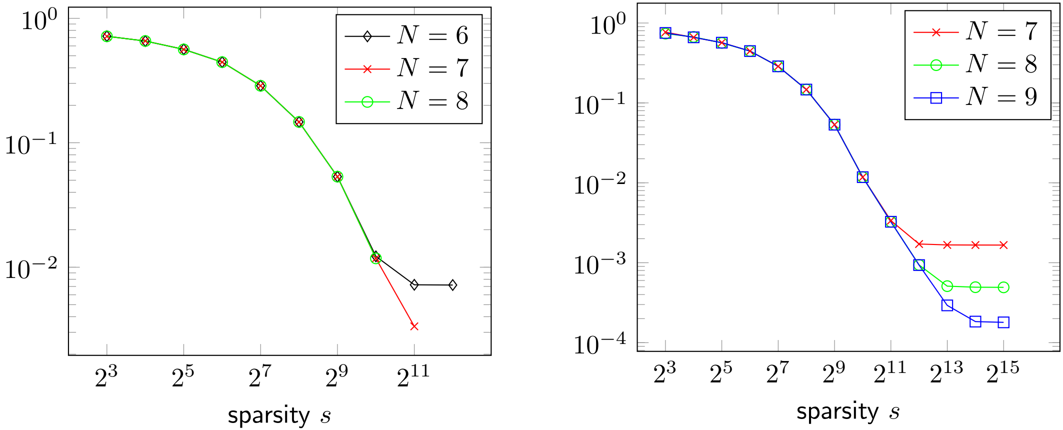
Parameters

- $\Gamma = \left\{\boldsymbol{k} \in \mathbb{Z}^d : \prod_{j=1}^d \max\left(1, 2\,|k_j|\right) \leq 2^N\right\}$
- detection iterations $r = 5$
- detection threshold $\delta_+ = 10^{-12}$

Approximation error

- rel. $L_2(\mathbb{T}^{10}, \mu)$ error:

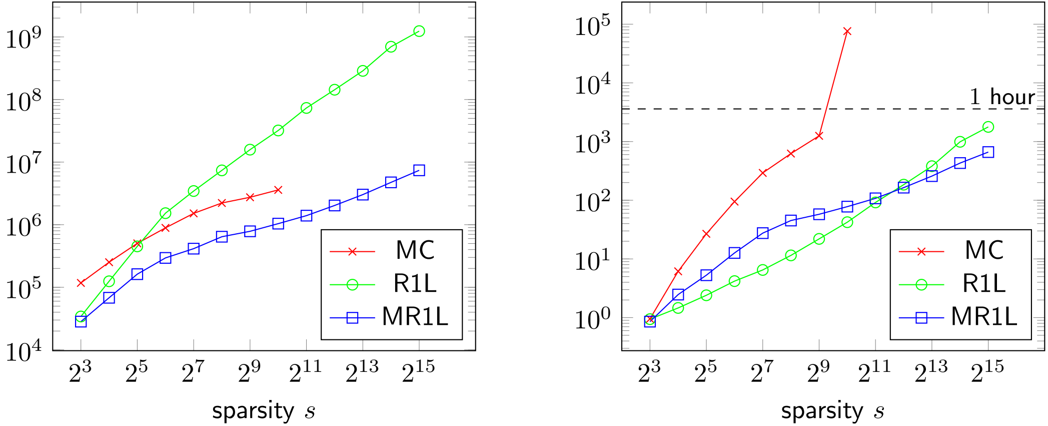
$$\frac{\|f - S_{\mathrm{I}} f\|_{L_2(\mathbb{T}^{10})}}{\|f\|_{L_2(\mathbb{T}^{10})}}$$



(a) approx. error for MC

(b) approx. error for R1L & MR1L

Figure: Approximation results for the 10-dimensional periodic test function



(a) amount of samples for $N = 8$

(b) computation time (in seconds) for $N = 8$

Figure: Approximation results for the 10-dimensional periodic test function

Test function (from [Volkmer '17], [Potts, Volkmer '17])

$$f(\boldsymbol{x}) := \prod_{j\in\{1,3,4,7\}} B_2(x_j) + \prod_{j\in\{2,5,6,8,9\}} B_4(x_j)$$

$f \in L_2([-1,1]^9, \mu_{\text{Cheb}})$, with B_2 and B_4 shifted, scaled and dilated B-Splines of order 2 and 4

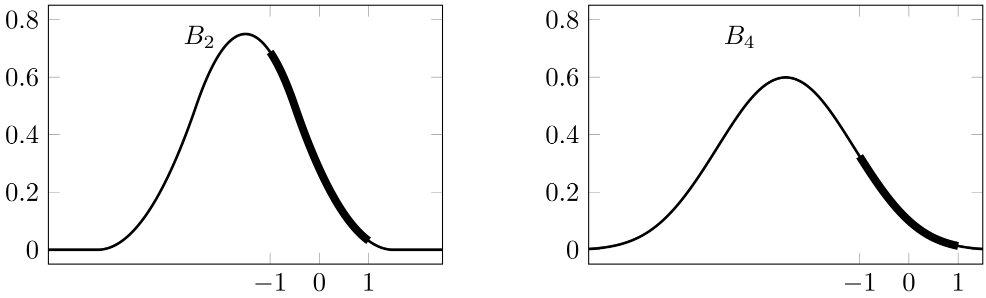
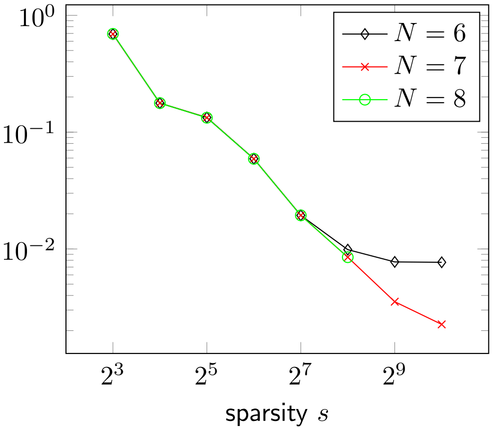


Figure: B-Splines B_2 and B_4 and the considered domain $[-1, 1]$



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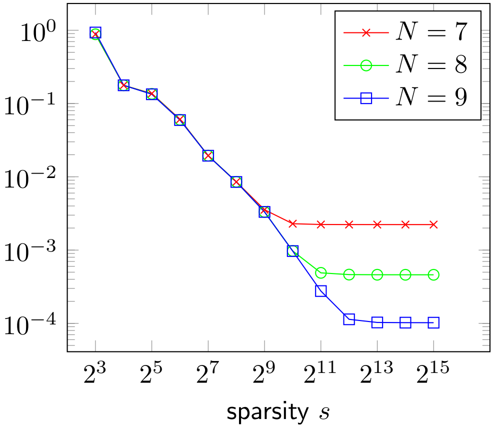

(b) approx. error for CMR1L

Figure: Approximation results for the 9-dimensional non-periodic test function

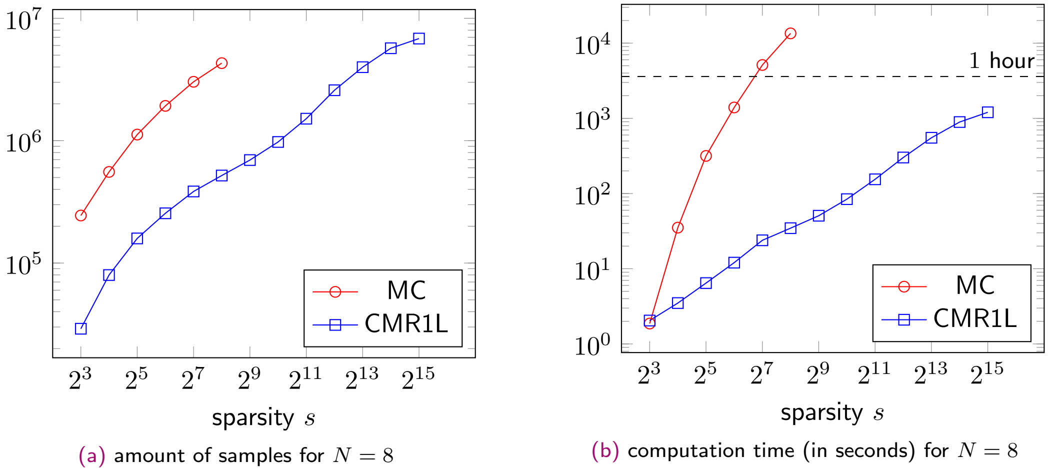

Figure: Approximation results for the 9-dimensional non-periodic test function

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  - first theoretical detection guarantee of the dimension-incremental method for function approximation
  - proof technique can be used as a blueprint when using other reconstruction methods too

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- Thank you for your attention!

Conclusion

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