Tautological systems, homogeneous spaces and the holonomic rank problem

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Abstract

In [Rei14], it was shown that GKZ-systems can be endowed, for a suitable choice of parameters, with the structure of a mixed Hodge module. We generalize this result to tautological systems associated to homogeneous spaces as considered in [HLZ16] by constructing them in a functorial way. As an application, we solve the holonomic rank problem for such tautological systems.

Contents

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1 Introduction

The purpose of this paper is to investigate a certain type of differential system that one can naturally associate to group actions on smooth algebraic varieties, and more specifically to representations of algebraic groups. A well-known and widely studied case is when the group is an algebraic torus, in this case the corresponding differential systems are known as GKZ-systems (see, e.g., [RSSW21] for an overview on the algebraic aspects of this theory). More generally, one can attach to the data of an algebraic group (defined over $\mathbb{C}$) acting linearly on a finite-dimensional vector space, an invariant subvariety of this space and a homomorphism from the Lie algebra of the group into the complex numbers a so-called tautological system, which is a $\mathscr{D}$-module on the dual vector space. To the best of our knowledge, this construction goes back to [Hot98, Section 4.], but has been considered more recently in a series of papers by Bloch, Huang, Lian, Song, Yau and Zhu ([LSY13, BHL$^+$14, LY13, HLZ16]). One of their main motivations comes from mirror symmetry, understood in the classical sense of recovering enumerative geometry information (i.e. quantum cohomology of certain symplectic varieties) by period integral computations of their mirror families (this is at least the situation in the Calabi-Yau case, for other type of varieties one might need to look at oszillating integrals instead). While the case of toric varieties and complete intersections inside them can considered to be settled, at least under sufficient positivity assumptions, (see, e.g., [Giv98, Iri11, RS17]), a longstanding and still challenging problem is to establish mirror symmetry, expressed as an equivalence of $\mathscr{D}$-modules (possibly with additional structures, such as Hodge modules or irregular variants of them) for non-toric varieties. A prominent class of examples are homogeneous spaces. Without giving a comprehensive list of references for the known results on mirror symmetry for homogeneous spaces, let us mention [Rie08, MR20, LT17]. A common feature of these papers is that the mirror of a Fano manifold which is a homogeneous space for some group $G^\vee$ consists of a Landau-Ginzburg potential, which is constructed via Lie theoretic methods from the Langlands dual group G of $G^\vee$. When restricted to a torus inside G, such potential function can be expressed as a Laurent polynomial, and then the question of finding and studying an appropriate partial compactification of this mirror Laurent polynomial arises, pretty much similar to the toric situation considered in [RS17] or, from a very different point of view, in [CPS22].

Motivated by these results and problems, one is led to ask whether for any given homogeneous space $X = G/P$, it is possible to describe the differential system satisfied by periods of families of hyperplane sections, for an appropriate embedding of X into a projective space. Such a differential system would be the analogue of a GKZ-$\mathscr{D}$-module, and should yield (by dimensional reduction) the mirror $\mathscr{D}$-module considered in the papers mentioned above. Using this line of ideas, the paper [HLZ16] considers homogeneous spaces $X = G/P$, where G is semisimple, and P is a parabolic subgroup and studies tautological system arising from the representation of G in the space of sections of the anticanonical bundle ω_X^{-1}. Our main findings (see Theorem 1.2 below) gives a generalization of their results to the case where we consider any G-equivariant bundle $L \to X$. Notice however that one needs to impose rather strong conditions on the bundle L and the parameter Lie algebra homomorphism mentioned above in order to obtain a non-zero tautological system. If these conditions hold true, we show that the corresponding tautological system has a functorial description, that it underlies a pure (complex) Hodge module, and we determine its holonomic rank in terms of the (compactly supported) cohomology of a natural family of (complements of) hyperplane sections of X. The latter result is the analogue of the proof of the holonomic rank conjecture in [HLZ16, Theorem 1.4].

In the remainder of this introduction, we are going to describe our main results in more detail, and we give an overview on the content of this paper. The main playing character, called tautological system, is defined below, where we use, for a vector space V and its dual space $W := V^\vee$, the Fourier-Laplace transformation functor $\mathrm{FL}^V\colon \mathrm{Mod}(\mathscr{D}_V) \to \mathrm{Mod}(\mathscr{D}_W)$ (see Section 3.1 below for more details about Fourier-Laplace transformations on arbitrary vector bundles). For the moment, G' is any algebraic group, later we will however consider a group G acting transitively on a variety X, and G' will denote the product $\mathbb{C}^* \times G$ which we will let act on equivariant line bundles $L \to X$.

Definition 1.1. *Let $\rho\colon G' \to \mathrm{GL}(V)$ be a finite-dimensional rational representation of an algebraic group and denote the induced Lie algebra representation by $\mathrm{d}\rho\colon \mathfrak{g}' \to \mathfrak{gl}(V)$. We write $W := V^\vee$ for the dual vector space. Let $\overline{Y}$ be a G'-invariant closed subvariety of V. For a Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathbb{C}$, define the left $\mathscr{D}_V$-module*

$$\hat{\tau}(\rho, \overline{Y}, \beta) := \mathscr{D}_V/(\mathscr{D}_V I + \mathscr{D}_V(Z_V(\xi) - \beta'(\xi) \mid \xi \in \mathfrak{g}')),$$

where $I \subseteq \mathbb{C}[V]$ is the vanishing ideal of $\overline{Y}$ and $\beta'(\xi) := \mathrm{trace}(\mathrm{d}\rho(\xi)) - \beta(\xi)$.

Its Fourier–Laplace transform

$$\tau(\rho, \overline{Y}, \beta) := \mathrm{FL}^V(\hat{\tau}(\rho, \overline{Y}, \beta))$$

is a left $\mathscr{D}_W$-module called the ***tautological system*** *associated to ρ, $\overline{Y}$ and β.*

The next statement summarizes our main results, in a slightly more restrictive setting than below in the main body of this article. To state them, assume that X is a smooth projective variety, and that G is a reductive and connected linear algebraic group that acts transitively on X. Suppose that $L \to X$ is an G-equivariant line bundle on X, with sheaf of sections $\mathscr{L}$, which we assume to be very ample. We put $G' := \mathbb{C}^* \times G$, and we define an action of G' act on L by letting the $\mathbb{C}^*$-factor act via inverse scaling in the fibres of L (see Definition 5.9 for a more precise and more general description). Let $V := H^0(X, \mathscr{L})^\vee$, then we obtain a representation $G' \to \mathrm{GL}(V)$. Moreover, since L is very ample, the linear system $|\mathscr{L}|$ yields an embedding $g\colon X \hookrightarrow \mathbb{P}V$. Let $\hat{X} \subset V$ be the affine cone, which is a G'-invariant subvariety. Write $h\colon \hat{X}\backslash 0 \hookrightarrow V$ for the corresponding locally closed embedding, and notice that there is an isomorphism $L^* \cong \hat{X}\backslash 0$, where L^* is the complement of the zero section of $L \to X$. Choose any Lie algebra homomorphism $\beta\colon \mathfrak{g}' = \mathbb{C}\mathbf{e} \oplus \mathfrak{g} \to \mathbb{C}$ with $\beta_{|\mathfrak{g}} = 0$ (this is automatic if G is semisimple, since then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), i.e., choose a number $\beta(\mathbf{e}) \in \mathbb{C}$. Then the following holds.

Theorem 1.2 (Theorem 7.4, Theorem 7.8 and Corollary 7.9)**.** *Let $\beta(\mathbf{e}) \notin \mathbb{Z}$. We have*

1. $$\tau(\rho, \hat{X}, \beta) = \begin{cases} \mathrm{FL}^V(h_+\mathcal{O}_{L^*}^{\ell/k}) & \text{if } \mathscr{L}^{\otimes \ell} \cong \omega_X^{\otimes(-k)} \text{ and } \beta(\mathbf{e}) = \ell/k, \\ 0 & \text{else,} \end{cases}$$

   where $\mathcal{O}_{L^*}^{\ell/k}$ is a smooth $\mathscr{D}_{L^*}$-module of rank 1 on L^* (and we denote by $\underline{\mathbb{C}}_{L^*}^{\ell/k}$ its associated local system) which underlies a pure complex Hodge module of weight $\dim(X) + 1$.

2. *If $\tau(\rho, \hat{X}, \beta) \neq 0$, then it underlies a pure complex Hodge module of weight* $\dim(X) + \dim(W)$.

3. *The holonomic rank of $\tau(\rho, \hat{X}, \beta)$ equals*

   $$\dim H_c^{\dim(X)}\left(X\backslash Z(\lambda), \underline{\mathbb{C}}_\lambda^{\ell/k}\right)$$

   for a generic $\lambda \in W = H^0(X, \mathscr{L})$, where $Z(\lambda)$ is the vanishing locus in X of the section λ, and where $\underline{\mathbb{C}}_\lambda^{\ell/k}$ is a local system $\lambda^*_{|X\backslash Z(\lambda)}\underline{\mathbb{C}}_{L^*}^{\ell/k}$.

The main point in obtaining these results is to rewrite the Fourier-Laplace transformation as an operation that involves only functors defined in the category of mixed Hodge modules. This is done using a strategy that already appeared in [Rei14], namely, using the Radon transformation for algebraic $\mathscr{D}_{\mathbb{P}^n}$-modules. We need, however, a variant here related to the fact that the parameter $\beta(\mathbf{e})$ mentioned above is not an integer. This twisted version of the Radon transformation was used in [RS20, Section 5.2], and we adapt the setup below in Section 3. Another ingredient of our construction (responsible for the restriction $\beta(\mathbf{e}) \notin \mathbb{Z}$) is a localization result for the Fourier-Laplace transform

$\hat{\tau}(\rho, \hat{X}, \beta)$ of the tautological system, treated in Section 7.2. It can be considered as a variant of a corresponding result in the GKZ-case in [SW09b].

Outline: We start by defining in Section 2 certain Hodge modules on line bundles $L \to X$ over smooth varieties (or rather on the complement of the zero section L^*). Their underlying $\mathscr{D}$-modules (denoted by $\mathcal{O}^\beta_{L^*}$) generalize the twisted structure sheaf $\mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t \cdot t + \beta)$ (which would correspond to the case where the variety is a point). Then we study their Fourier–Laplace transforms in Section 3 and show that they still underly a mixed Hodge module on the dual bundle.

In Section 4, we consider a quite general situation of a smooth algebraic variety Y endowed with the action of an algebraic group G'. From the vector fields defined by this group action, we construct a $\mathscr{D}_Y$-modules $\mathscr{N}^\beta_Y$ (also involving a Lie algebra homomorphism β). Especially important is the case where Y occurs as an orbit in a vector space V underlying a rational representation $\rho\colon G' \to \mathrm{GL}(V)$. Then according to the above discussion, one may consider the tautological system $\tau(\rho, \overline{Y}, \beta)$ and its Fourier transform $\hat{\tau}(\rho, \overline{Y}, \beta)$, and we relate in Corollary 4.11 the restriction of $\hat{\tau}(\rho, \overline{Y}, \beta)$ to Y with the intrinsically defined module $\mathscr{N}^\beta_Y$.

In Section 5 we address the issue mentioned earlier: Under which hypotheses does the definition of the tautological system $\tau(\rho, \overline{Y}, \beta)$ resp. of its Fourier transform $\hat{\tau}(\rho, \overline{Y}, \beta)$ define a non-zero $\mathscr{D}_W$- resp. $\mathscr{D}_V$-module? By the previous results, this question is almost equivalent to the vanishing resp. the non-vanishing of the module $\mathscr{N}^\beta_Y$. We develop therefore a framework, based on the formalism of Lie algebroids and their universal envelopping algebras, to study this problem. The main result in this section is Proposition 5.14 which gives a sufficient criterion for $\hat{\tau}(\rho, \overline{Y}, \beta)$ to be non-zero.

Section 6 discusses the more specific case where the variety Y is the total space of a line bundle over a variety X equipped with an action by a group G. Then L can be (in various ways depending the choice of a character) made into a G'-space, where $G' := \mathbb{C}^* \times G$. We compare the constructions from the previous chapters, and we give in Corollary 6.9 an alternative description, in this equivariant situation, of the module $\mathcal{O}^\beta_{L^*}$ from Section 2.

In Section 7, we apply all the previous results in the case where the variety X is a homogeneous space, and where the representation ρ is in the dual of the space of sections of an equivariant line bundle on X. The affine cone of X then takes the role of the G'-invariant space used in the definition of the tautological system $\tau(\rho, \hat{X}, \beta)$. The main result is then Theorem 7.8, showing that if β is not an integer, and is such that the $\tau(\rho, \hat{X}, \beta) \neq 0$, then it underlies pure complex Hodge module. It has a functorial description as a proper direct image of a family of hyperplane sections of X, and consequently, we can solve the holonomic rank problem as stated in [BHL$^+$14] and [HLZ16] in this generalized situation.

We conclude the paper by giving, in Section 8, two independent proofs of a formula for the complex parameter value $\beta(\mathbf{e}) \in \mathbb{C}$ for which the tautological system $\tau(\rho, \hat{X}, \beta)$ is non-zero, in the case of a semisimple group G.

Notations: Throughout, we work over $\mathbb{C}$. By *variety*, we mean an integral scheme of finite type over $\mathbb{C}$. When we talk about *points* on a variety, we mean closed points unless mentioned otherwise. Our convention for the projective space of a finite-dimensional vector space V is $\mathbb{P}V := \text{Proj Sym}\, V^\vee$, i.e., $\mathbb{P}V$ parameterizes one-dimensional *subspaces* of V.

For a smooth variety X, we let $\mathscr{D}_X$ be the sheaf of algebraic differential operators on X. If not mentioned otherwise, a $\mathscr{D}_X$-module is a quasi-coherent $\mathcal{O}_X$-module equipped with a left action by $\mathscr{D}_X$. The category of such modules is denoted by $\text{Mod}_{qc}(\mathscr{D}_X)$ and the corresponding bounded derived category by $D^b_{qc}(\mathscr{D}_X)$. Throughout, for a morphism $f\colon X \to Y$ between smooth varieties over $\mathbb{C}$, we denote by $f_+\colon D^b_{qc}(\mathscr{D}_X) \to D^b_{qc}(\mathscr{D}_Y)$ and $f^+\colon D^b_{qc}(\mathscr{D}_Y) \to D^b_{qc}(\mathscr{D}_X)$ the functors defined by

$$f_+\mathscr{M} := Rf_*(\mathscr{D}_{Y\leftarrow X} \otimes^{\mathbb{L}}_{\mathscr{D}_X} \mathscr{M}) \quad \text{and} \quad f^+\mathscr{N} := \mathscr{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}.$$

For any variety X, we let $\text{MHM}(X)$ be the abelian category of algebraic ($\mathbb{Q}$-)mixed Hodge modules on X (as defined in [Sai88, Sai90]) and $D^b\text{MHM}(X)$ its bounded derived category. For any morphism $f\colon X \to Y$ the functors $f_+, f_\dagger$ resp. $f^\dagger[\dim(Y) - \dim(X)], f^+[\dim(X) - \dim(Y)]$ on $D^b_{qc}(\mathscr{D}_X)$ resp. $D^b_{qc}(\mathscr{D}_Y)$ lift to functors

$$f_*, f_!\colon D^b\text{MHM}(X) \to D^b\text{MHM}(Y) \qquad \text{resp.} \qquad f^*, f^!\colon D^b\text{MHM}(Y) \to D^b\text{MHM}(X).$$

Any object $\mathcal{M} \in \text{MHM}(X)$ is a tuple $\mathcal{M} = (\mathscr{M}, F_\bullet, W_\bullet, K)$ where $\mathscr{M} \in \text{Mod}_h(\mathscr{D}_X)$ and $W_\bullet\mathscr{M}$ is its weight filtration. We denote by $\text{HM}(X, w)$ (or simply $\text{HM}(X)$ if w is clear from the context) the full subcategory of objects such that $\text{Gr}_l(\mathscr{M}) = 0$ for all $l \neq w$, these are the pure Hodge modules of weight w.

We will need an extension of the notion of ($\mathbb{Q}$-)mixed Hodge modules to the category of complex mixed Hodge modules. It can be constructed by first defining $\mathbb{R}$-mixed Hodge modules, see [Moc15, Section 13.5]. Then a filtered $\mathscr{D}$-module $(\mathscr{M}, F_\bullet)$ is said to underly a complex mixed Hodge module iff it is a direct summand of an $\mathbb{R}$-mixed Hodge module ([DS13, Definition 3.2.1.]). We denote the corresponding abelian category by $\text{MHM}(X, \mathbb{C})$, by $D^b\text{MHM}(X, \mathbb{C})$ its bounded derived category and by $\text{HM}(X, \mathbb{C}, w)$ (or $\text{HM}(X, \mathbb{C})$ for short) the category of pure complex Hodge modules of weight w. Many of the known constructions for $\mathbb{R}$-mixed Hodge modules carry over to the categories $\text{MHM}(X, \mathbb{C})$ and $\text{HM}(X, \mathbb{C})$ since they are stable under taking direct sums. Notice that [DV22, Section 7.1 and Appendix A] contains a more detailed discussion of complex Hodge modules. Particularly important will be the following smooth Hodge modules. For any variety X, write $a_X : X \to \{pt\}$ for the map to the point. We denote by ${}^H\mathbb{C}_{pt}$ the trivial complex Hodge structure of dimension 1. Then we define

$${}^H\underline{\mathbb{C}}_X := a_X^*\, {}^H\mathbb{C}_{pt}[\dim(X)],$$

this is an object in $\text{MHM}(X, \mathbb{C})$. Notice that our notation differs from the convention in [Sai88, Sai90], where the ($\mathbb{Q}$-)constant Hodge module of rank 1 is denoted by ${}^p\mathbb{Q}^H_X$.

We will further need a particular smooth (but non-constant) complex Hodge module on the one-dimensional torus $\mathbb{C}^*$. Namely, for any $\beta \in \mathbb{R}$, we denote by $\mathcal{O}^\beta_{\mathbb{C}^*}$ the $\mathscr{D}_{\mathbb{C}^*}$-module

$$\mathcal{O}^\beta_{\mathbb{C}^*} = \mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t t + \beta).$$

We further write ${}^H\underline{\mathbb{C}}^\beta_{\mathbb{C}^*}$ for the complex Hodge module with underlying $\mathscr{D}$-module equal to $\mathcal{O}^\beta_{\mathbb{C}^*}$ (placed in cohomological degree zero), and where $\text{Gr}^F_p\, \mathcal{O}^\beta_{\mathbb{C}^*} = 0$ for $p \neq 0$ and $\text{Gr}^W_i\, \mathcal{O}^\beta_{\mathbb{C}^*} = 0$ for $i \neq 1$. Its corresponding perverse sheaf is $\mathbb{V}[1]$, where $\mathbb{V}$ is the local system of rank 1 on $\mathbb{C}^*$ given by the monodromy with eigenvalue $e^{2\pi\sqrt{-1}\beta}$. Again, in the conventions of [Sai88, Sai90] this object would have been denoted by ${}^p\mathbb{C}_X^{H,\beta}$.

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2 Mixed Hodge modules on line bundles

On $\mathbb{C}^*$, we consider the $\mathscr{D}_{\mathbb{C}^*}$-module

$$\mathcal{O}_{\mathbb{C}^*}^\beta := \mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*} \cdot (\partial_t t + \beta)$$

for every $\beta \in \mathbb{C}$, which underlies a polarizable pure complex Hodge module ${}^H\underline{\mathbb{C}}_{\mathbb{C}^*}^\beta$ when $\beta \in \mathbb{R}$. For a smooth variety X, we may then define for every $\beta \in \mathbb{C}$ the $\mathscr{D}_{\mathbb{C}^* \times X}$-module

$$\mathcal{O}_{\mathbb{C}^* \times X}^\beta := \mathcal{O}_{\mathbb{C}^*}^\beta \boxtimes \mathcal{O}_X = p^+\mathcal{O}_{\mathbb{C}^*}^\beta \otimes_{\mathcal{O}_{\mathbb{C}^* \times X}} q^+\mathcal{O}_X, \tag{1}$$

where $p\colon \mathbb{C}^* \times X \to \mathbb{C}^*$ and $q\colon \mathbb{C}^* \times X \to X$ are the projections onto the first and second factor, respectively.

In this section, we are interested in line bundles $\pi\colon L \to X$ on a smooth variety and we wish to define on $L^* \subseteq L$, the complement of the zero section of π, a $\mathscr{D}_{L^*}$-module $\mathcal{O}_{L^*}^\beta$ by gluing the above construction over a trivializing open cover of X.

We immediately face the following issue: If $\widetilde{\psi}$ is an automorphism of $\mathbb{C}^* \times X$ given by $\widetilde{\psi}(\lambda, p) = (\alpha(p) \cdot \lambda, p)$ for some $\alpha \in \Gamma(X, \mathcal{O}_X^\times)$, then $\widetilde{\psi}_+\mathcal{O}_{\mathbb{C}^* \times X}^\beta$ is in general not isomorphic to $\mathcal{O}_{\mathbb{C}^* \times X}^\beta$. This leads to several complications:

(i) If L is isomorphic to the trivial line bundle $\mathbb{C} \times X$, then to define $\mathcal{O}_{L^*}^\beta$ we need to choose an isomorphism with $\mathbb{C} \times X$, and the resulting $\mathscr{D}$-module depends on this choice (with the notable exception of the case $\beta \in \mathbb{Z}$, in which case we get $\mathcal{O}_{L^*}^\beta \cong \mathcal{O}_{L^*}$).

(ii) For a non-trivial line bundle $L \ncong \mathbb{C} \times X$ and a trivializing open cover $X = \bigcup_{i \in I} U_i$, in order to glue all $\mathcal{O}_{\mathbb{C}^* \times U_i}^\beta$, $i \in I$, to a $\mathscr{D}_{L^*}$-module, we need to identify $\mathcal{O}_{\mathbb{C}^* \times U_{ij}}^\beta$ with $\widetilde{\psi}_{ij,+}\mathcal{O}_{\mathbb{C}^* \times U_{ij}}^\beta$, where $\widetilde{\psi}_{ij}\colon \mathbb{C}^* \times U_{ij} \to \mathbb{C}^* \times U_{ij}$ is the transition map over the intersection $U_{ij} := U_i \cap U_j$. However, it turns out that these two $\mathscr{D}_{\mathbb{C}^* \times U_{ij}}$-modules are in general isomorphic (for a suitable choice of transition maps $\widetilde{\psi}_{ij}$) only if $\beta \in \frac{1}{k}\mathbb{Z}$ and L is the k-th tensor power of another line bundle F on X.

(iii) Even if $\mathcal{O}_{\mathbb{C}^* \times U_{ij}}^\beta$ and $\psi_{ij}^+\mathcal{O}_{\mathbb{C}^* \times U_{ij}}^\beta$ are isomorphic, i.e., $\beta \in \frac{1}{k}\mathbb{Z}$ and $L = F^{\otimes k}$, then there may be several isomorphisms between these $\mathscr{D}_{\mathbb{C}^* \times U_{ij}}$-modules. leading to non-isomorphic $\mathscr{D}_{L^*}$-modules after gluing, corresponding to the possible choices of the k-th root F of L. In fact, one can show that the glued $\mathscr{D}_{L^*}$-modules is independent of the choices only in the case that there are no k-torsion line bundles on X.

Given all of these constraints, our approach is as follows: We fix one non-trivial line bundle $\pi^F\colon F \to X$ to begin with. Then, for the tensor powers $L = F^{\otimes k}$ of this line bundle, we define $\mathscr{D}_{L^*}$-modules $\mathcal{O}_{L^*}^{\beta}$ for all $\beta \in \mathbb{C}$ with $k\beta \in \mathbb{Z}$ by appropriately gluing (1) over trivializing open subsets of X. The constructed $\mathscr{D}$-module $\mathcal{O}_{L^*}^{\beta}$ still depends on the choice of F as the k-th root of L. Only in the case that there is no torsion in the Picard group of X will the constructed $\mathscr{D}_{L^*}$-modules $\mathcal{O}_{L^*}^{\beta}$ always be intrinsic to the line bundles L on which they are defined.

2.1 The $\mathscr{D}$-module $\mathcal{O}_{L^*}^{\beta}$

Throughout, let X be a smooth variety and let $\pi^F\colon F \to X$ be a line bundle. Fix $k \in \mathbb{Z}$ and let $\pi\colon L := F^{\otimes k} \to X$ be the k-th tensor power of the line bundle F. For negative k, this means the line bundle dual to $F^{\otimes(-k)}$. In particular, we have $L = F^{\vee}$ if $k = -1$. For $k = 0$, we have the trivial bundle $L = F^{\otimes 0} = \mathbb{C} \times X$. We denote the sheaf of sections of the line bundles F and L by $\mathscr{F}$ and $\mathscr{L}(= \mathscr{F}^{\otimes k})$, respectively.

Let $L^* \subseteq L$ denote the complement of the zero section of $\pi\colon L \to X$ (and similarly for other line bundles). In this section, we prove:

Proposition 2.1. *Let $\beta \in \mathbb{C}$ with $k\beta \in \mathbb{Z}$. Up to isomorphism, there exists a unique $\mathscr{D}_{L^*}$-module $\mathcal{O}_{L^*}^{\beta}$ with the following local description over open subsets of X trivializing the line bundle F: If $s \in \Gamma(U, \mathscr{F})$ is a local non-vanishing section of $F \to X$, then under*

$$\widetilde{\psi}_{s^k}\colon \mathbb{C}^* \times U \xrightarrow{\cong} \pi^{-1}(U)^*, \qquad (\lambda, p) \mapsto \lambda \cdot s^k(p),$$

we have

$$\widetilde{\psi}_{s^k}^{+}\big(\mathcal{O}_{L^*}^{\beta}\big|_{\pi^{-1}(U)^*}\big) \cong \mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t t + \beta) \boxtimes \mathcal{O}_U.$$

We will mainly be concerned with the line bundle L, and its k-th root F should be viewed an additional datum necessary for the definition of $\mathcal{O}_{L^*}^{\beta}$. Later on, we will specialize to cases in which the Picard group of X is torsion-free. In this case, F is up to isomorphism already uniquely determined by the line bundle L admitting a k-th root.

We start with the case $k = 0$, i.e., the case that $\pi\colon L \to X$ is the trivial line bundle $F^{\otimes 0} = \mathbb{C} \times X$. Then $L^* = \mathbb{C}^* \times X$ and for any $\beta \in \mathbb{C}$, we define the $\mathscr{D}_{L^*}$-module

$$\begin{aligned} \mathcal{O}_{\mathbb{C}^* \times X}^{\beta} &:= \mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t t + \beta) \boxtimes \mathcal{O}_X \\ &= p^{+}\big(\mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t t + \beta)\big) \otimes_{\mathcal{O}_{\mathbb{C}^* \times X}} q^{+}\mathcal{O}_X, \end{aligned} \tag{2}$$

where $p\colon \mathbb{C}^* \times X \to \mathbb{C}^*$ and $q\colon \mathbb{C}^* \times X \to X$ are the projections onto the first and second factor, respectively.

From now on, let $k \in \mathbb{Z} \setminus \{0\}$. We construct $\mathcal{O}_{L^*}^{\beta}$ by gluing the local construction. For this, we choose an open cover $X = \bigcup_{i \in I} U_i$ trivializing the line bundle $\pi^F\colon F \to X$ and therefore also its k-th tensor power L. We will for convenience assume that each U_i is affine and admits a local coordinate system. The choice of non-vanishing local sections $s_i \in \Gamma(U_i, \mathscr{F})$ determines trivializing isomorphisms

$$\psi_{s_i}\colon \mathbb{C} \times U_i \xrightarrow{\cong} \pi^{F,-1}(U_i), \qquad (\lambda, p) \mapsto \lambda \cdot s_i(p)$$

of line bundles over U_i, where we view the section s_i as a morphism $U_i \to F$ with $\pi^F \circ s_i = \mathrm{id}_{U_i}$.

Then, over the intersections $U_{ij} := U_i \cap U_j$, the transition maps

$$\psi_{s_i/s_j}\colon \mathbb{C} \times U_{ij} \xrightarrow[\cong]{\psi_{s_i}|_{\mathbb{C}\times U_{ij}}} \pi^{F,-1}(U_{ij}) \xrightarrow[\cong]{\psi_{s_j}^{-1}|_{\pi^{F,-1}(U_{ij})}} \mathbb{C} \times U_{ij} \tag{3}$$

are of the form $\psi_{s_i/s_j}(\lambda, p) = (\alpha_{ij}(p) \cdot \lambda, p)$ for $\alpha_{ij} := s_j/s_i \in \Gamma(U_{ij}, \mathcal{O}_{U_{ij}}^{\times})$. These $(\alpha_{ij})_{i,j\in I}$ can also be viewed as a Čech cocyle $\alpha \in \check{C}^1(\mathcal{U}, \mathcal{O}_X^{\times})$, $\mathcal{U} := (U_i)_{i\in I}$, representing the isomorphism class of the line bundle $\mathscr{F}$ in $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^{\times}) \cong \check{H}^1(\mathcal{U}, \mathcal{O}_X^{\times})$.

The k-th tensor power $L = F^{\otimes k}$ also trivializes over the open sets U_i and the non-vanishing local sections $s_i^k \in \Gamma(U_i, \mathscr{L})$ induce trivializing isomorphisms

$$\psi_{s_i^k}\colon \mathbb{C} \times U_i \xrightarrow{\cong} \pi^{-1}(U_i), \qquad (\lambda, p) \mapsto \lambda \cdot s_i^k(p).$$

They restrict to isomorphisms on the complements of the zero sections, which we denote by

$$\widetilde{\psi}_{s_i^k}\colon \mathbb{C}^* \times U_i \xrightarrow{\cong} \pi^{-1}(U_i)^*.$$

As in (3), the transition maps for L (resp. L^*) over the intersections $U_{ij} := U_i \cap U_j$ are of the form

$$\psi_{s_i^k/s_j^k}\colon \mathbb{C} \times U_{ij} \xrightarrow{\cong} \pi^{-1}(U_{ij}) \xrightarrow{\cong} \mathbb{C} \times U_{ij}, \quad \text{resp. } \widetilde{\psi}_{s_i^k/s_j^k}\colon \mathbb{C}^* \times U_{ij} \xrightarrow{\cong} \pi^{-1}(U_{ij})^* \xrightarrow{\cong} \mathbb{C}^* \times U_{ij},$$

mapping (λ, p) to $(\alpha_{ij}^k(p) \cdot \lambda, p)$, where we recall that $\alpha_{ij} := s_i/s_j \in \Gamma(U_{ij}, \mathcal{O}_{U_{ij}}^{\times})$.

If the collection of non-vanishing local sections $(s_i)_{i\in I}$ is fixed, we will for simplicity denote

$$\psi_i := \psi_{s_i^k}, \qquad \psi_{ij} := \psi_{s_i^k/s_j^k}, \qquad \widetilde{\psi}_i := \widetilde{\psi}_{s_i^k}, \qquad \widetilde{\psi}_{ij} := \widetilde{\psi}_{s_i^k/s_j^k}.$$

Let $\beta \in \frac{1}{k}\mathbb{Z}$. For $i \in I$, we define the $\mathscr{D}_{\pi^{-1}(U_i)^*}$-module

$$\mathcal{M}_i^{\beta} := \widetilde{\psi}_{i,+}\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_i},$$

where the right hand side was defined in (2). We wish to glue these $\mathcal{M}_i^{\beta}$ to a $\mathscr{D}_{L^*}$-module, which we will call $\mathcal{O}_{L^*}^{\beta}$.

We recall what needs to be checked in order to glue. If there exists a $\mathscr{D}_{L^*}$-module $\mathcal{M}$ with $\mathcal{M}|_{\pi^{-1}(U_i)^*} \cong \mathcal{M}_i^{\beta}$, then

$$\mathcal{M}_i^{\beta}\big|_{\pi^{-1}(U_{ij})^*} \cong \mathcal{M}|_{\pi^{-1}(U_{ij})^*} \cong \mathcal{M}_j^{\beta}\big|_{\pi^{-1}(U_{ij})^*}.$$

Conversely, in order to glue $(\mathcal{M}_i^{\beta})_{i\in I}$ to a $\mathscr{D}_{L^*}$-module, we must identify under the transition maps the restrictions of $\mathcal{M}_i^{\beta}$ and $\mathcal{M}_j^{\beta}$ to $\pi^{-1}(U_{ij})^*$. More precisely, we need to construct isomorphisms of $\mathscr{D}_{\mathbb{C}^*\times U_{ij}}$-modules

$$\mu_{ij}\colon \widetilde{\psi}_{ij,+}\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_{ij}} \xrightarrow{\cong} \mathcal{O}^{\beta}_{\mathbb{C}^*\times U_{ij}} \qquad \text{for } i, j \in I \tag{4}$$

satisfying the compatibility conditions

$$\begin{cases} \mu_{j\ell}|_U \circ \big((\widetilde{\psi}_{j\ell}|_U)_+\mu_{ij}|_U\big) = \mu_{i\ell}|_U & \text{for } i, j, \ell \in I,\ U := \mathbb{C}^* \times (U_i \cap U_j \cap U_\ell), \\ \mu_{ii} = \mathrm{id} & \text{for } i \in I. \end{cases} \tag{5}$$

In order to construct the gluing data (μ_{ji}), the following explicit description is key:

Lemma 2.2. *Let U be a smooth affine variety which admits a local coordinate system $(x_1, \dots, x_n)$. Let $\widetilde{\psi}$ be an automorphism on $\mathbb{C}^* \times U$ given by $(\lambda, p) \mapsto (\alpha(p) \cdot \lambda, p)$ for some $\alpha \in \Gamma(U, \mathcal{O}_U^\times)$. Then, for all $\beta \in \mathbb{C}$, we have*

$$\mathcal{O}^\beta_{\mathbb{C}^* \times U} = \mathscr{D}_{\mathbb{C}^* \times U} / \mathscr{D}_{\mathbb{C}^* \times U} \Big(\partial_t t + \beta,\ \partial_{x_1}, \dots, \partial_{x_n} \Big),$$

$$\widetilde{\psi}_+ \mathcal{O}^\beta_{\mathbb{C}^* \times U} = \mathscr{D}_{\mathbb{C}^* \times U} / \mathscr{D}_{\mathbb{C}^* \times U} \left(\partial_t t + \beta,\ \partial_{x_\ell} - (\beta + 1) \alpha^{-1} \frac{\partial \alpha}{\partial x_\ell} \;\Big|\; \ell = 1, \dots, n \right).$$

In terms of these cyclic descriptions, any $\mathscr{D}_{\mathbb{C}^* \times U}$-isomorphism $\widetilde{\psi}_+ \mathcal{O}^\beta_{\mathbb{C}^* \times U} \xrightarrow{\cong} \mathcal{O}^\beta_{\mathbb{C}^* \times U}$ is given by right-multiplication with some $h \in \Gamma(U, \mathcal{O}_U)^\times$ satisfying

$$h^{-1} \frac{\partial h}{\partial x_\ell} = (\beta + 1) \cdot \alpha^{-1} \frac{\partial \alpha}{\partial x_\ell} \qquad \text{for } \ell = 1, \dots, n. \tag{6}$$

Proof. The description of $\mathcal{O}^\beta_{\mathbb{C}^* \times U}$ follows from $\mathcal{O}_U = \mathscr{D}_U / \mathscr{D}_U(\partial_{x_1}, \dots, \partial_{x_n})$. The automorphism $\widetilde{\psi} \colon \mathbb{C}^* \times U \to \mathbb{C}^* \times U$ is in coordinates given by mapping $(t, x_1, \dots, x_n) \mapsto (\alpha \cdot t, x_1, \dots, x_n)$. With this, one easily computes $\widetilde{\psi}_+ \mathcal{O}^\beta_{\mathbb{C}^* \times U}$ to be the claimed cyclic module.

A $\mathscr{D}_{\mathbb{C}^* \times U}$-module isomorphism from the cyclic module $\widetilde{\psi}_+ \mathcal{O}^\beta_{\mathbb{C}^* \times U}$ to the cyclic module $\mathcal{O}^\beta_{\mathbb{C}^* \times U}$ must be given by right-multiplication with an invertible function $h \in \Gamma(\mathbb{C}^* \times U, \mathcal{O}_{\mathbb{C}^* \times U})^\times$. Here, we are using that every operator in $\Gamma(\mathbb{C}^* \times U, \mathcal{O}^\beta_{\mathbb{C}^* \times U})$ is represented by a function h which must be invertible if the operator is. With the explicit descriptions of the cyclic modules from above, one sees that right-multiplication by h is a well-defined isomorphism between the two $\mathscr{D}_{\mathbb{C}^* \times U}$-modules if and only if

$$\frac{\partial h}{\partial t} = 0 \quad \text{and} \quad \frac{\partial h}{\partial x_\ell} = (\beta + 1) \alpha^{-1} \frac{\partial \alpha}{\partial x_\ell} h \quad \text{for } \ell = 1, \dots, n.$$

The first condition is fulfilled if and only if $h \in \Gamma(U, \mathcal{O}_U)^\times$ and the second condition is precisely (6). □

For $k \neq 0$, we write $\beta \in \frac{1}{k}\mathbb{Z}$ as $\beta = \frac{m}{k} - 1$ with $m \in \mathbb{Z}$. Given Lemma 2.2, we can now define the gluing isomorphisms μ_{ij} in (4) to be given by right-multiplication with $\alpha^m_{ij} = s^m_i / s^m_j$, since then the equation (6) is satisfied for $h = \alpha^m_{ij}$ with respect to $\widetilde{\psi} = \widetilde{\psi}_{ij} = \widetilde{\psi}_{s^k_i / s^k_j}$. Moreover, these isomorphisms satisfy the compatibility cocycle condition (5), since the transition maps of L satisfy $\widetilde{\psi}_{j\ell}\big|_U \circ \widetilde{\psi}_{ij}\big|_U = \widetilde{\psi}_{i\ell}\big|_U$ on $U := \mathbb{C}^* \times (U_i \cap U_j \cap U_\ell)$ for all $i, j, \ell \in I$.

Definition 2.3. *Let $F \to X$ be a line bundle on a smooth variety X and consider its k-th tensor power $L = F^{\otimes k} \to X$ for some $k \in \mathbb{Z}$. Let $\beta \in \mathbb{C}$ with $m := k\beta + 1 \in \mathbb{Z}$. We define $\mathcal{O}^\beta_{L^*}$ as the $\mathscr{D}_{L^*}$-module obtained by gluing the $\mathscr{D}_{\mathbb{C}^* \times U_i}$-modules $(\widetilde{\psi}_{i,+} \mathcal{O}^\beta_{\mathbb{C}^* \times U_i})_{i \in I}$ along the isomorphisms $\mu_{ij} \colon \widetilde{\psi}_{ij,+} \mathcal{O}^\beta_{\mathbb{C}^* \times U_{ij}} \xrightarrow{\cong} \mathcal{O}^\beta_{\mathbb{C}^* \times U_{ij}}$ given by right-multiplication with $\alpha^m_{ij} = s^m_i / s^m_j$ with respect to the cyclic description given in Lemma 2.2.*

Three remarks are in order: First of all, this construction is independent of the chosen trivializing open cover on X and the chosen local non-vanishing sections $s_i \in \Gamma(U_i, \mathscr{F})$

corresponding to the choice of local trivializing isomorphisms ψ_{s_i} of F. This means that if we choose another open cover $(U_i')_{i\in I'}$ of X trivializing F and local non-vanishing sections $s_i' \in \Gamma(U_i', \mathscr{F})$, then the two $\mathscr{D}_{L^*}$-modules we obtain by gluing are isomorphic.

Secondly, if $k' \in \mathbb{Z}$ divides k and $k'\beta \in \mathbb{Z}$, then the construction of $\mathcal{O}^\beta_{L^*}$ from the k-th root F of L agrees with the construction from the k'-th root $F^{\otimes(k/k')}$ of L. In this sense, $\mathcal{O}^\beta_{L^*}$ does not depend on the choice of k as a denominator for β.

Thirdly, however, we must emphasize that the $\mathscr{D}_{L^*}$-module $\mathcal{O}^\beta_{L^*}$ *does* depend on the choice of the line bundle F as the k-th root of L in the Picard group: If the Picard group has k-torsion, then there exists another line bundle F' on X with $F \ncong F'$, but $F^{\otimes k} \cong (F')^{\otimes k}$. In this case, the two $\mathscr{D}_{L^*}$-modules $\mathcal{O}^\beta_{F^{\otimes k},*}$ and $\mathcal{O}^\beta_{(F')^{\otimes k},*}$ are in general not isomorphic.

Proof of Proposition 2.1. As just remarked, the isomorphism class of $\mathcal{O}^\beta_{L^*}$ does not depend on the choice of the trivializing open cover for $F \to X$ and the choice of local non-vanishing sections describing the trivializing isomorphisms. Thus, a given local non-vanishing section $s \in \Gamma(U, \mathscr{F})$ may be assumed to be part of the data used in the gluing. This shows that the construction in Definition 2.3 describes a $\mathscr{D}_{L^*}$-module with the desired property.

It remains to show uniqueness. Since the local description of $\mathcal{O}^\beta_{L^*}$ of F on a trivializing open cover is prescribed, the only part in the construction where we may have a choice was when defining the gluing data μ_{ij}. According to Lemma 2.2, this amounts to the choice of $h_{ij} \in \Gamma(U_{ij}, \mathcal{O}^\times_{U_{ij}})$ with

$$h_{ij}^{-1}\frac{\partial h_{ij}}{\partial x_\ell} = (\beta+1)\cdot \alpha_{ij}^{-k}\frac{\partial \alpha_{ij}^k}{\partial x_\ell} \qquad \text{for } \ell = 1, \dots, n.$$

Using $k\cdot(\beta+1) = m$, this is equivalent to

$$\frac{\partial(h_{ij}/\alpha_{ij}^m)}{\partial x_\ell} = 0 \qquad \text{for } \ell = 1, \dots, n.$$

Hence, $h_{ij} = c_{ij}\alpha_{ij}^m$ for some $c_{ij} \in \mathbb{C}^*$. The compatibility condition (5) then forces $c_{ij}c_{j\ell} = c_{i\ell}$ and $c_{ii} = 1$ for $i, j, \ell \in I$. This guarantees that we can write $c_{ij} = \tilde{c}_i/\tilde{c}_j$ for non-zero constants $\tilde{c}_i \in \mathbb{C}^*$. The $\mathscr{D}_{L^*}$-module obtained from gluing over U_{ij} via right-multiplication with α_{ij}^m is isomorphic to the one via right-multiplication with $c_{ij}\alpha_{ij}^m$. Indeed, there is an isomorphism locally given on U_i by multiplication with $\tilde{c}_i$ in the cyclic representations. □

Next, we observe that $\mathcal{O}^\beta_{L^*}$ only depends on the class of β modulo $\mathbb{Z}$:

Proposition 2.4. *Let $\beta, \beta' \in \mathbb{C}$ with $k\beta, k\beta' \in \mathbb{Z}$. Then*

$$\mathcal{O}^\beta_{L^*} \cong \mathcal{O}^{\beta'}_{L^*} \quad \Leftrightarrow \quad \beta - \beta' \in \mathbb{Z}.$$

Proof. An isomorphism $\xi\colon \mathcal{O}^\beta_{L^*} \xrightarrow{\cong} \mathcal{O}^{\beta'}_{L^*}$ is given by isomorphisms over the open cover

$$\xi_i\colon \mathcal{O}^\beta_{L^*}\big|_{\pi^{-1}(U_i)^*} \xrightarrow{\cong} \mathcal{O}^{\beta'}_{L^*}\big|_{\pi^{-1}(U_i)^*}$$

such that, for all $i, j \in I$, the restrictions of ξ_i and ξ_j to $\pi^{-1}(U_{ij})$ agree. By construction, there are isomorphisms

$$\chi_i \colon \mathcal{O}^{\beta}_{L^*}\big|_{\pi^{-1}(U_i)^*} \xrightarrow{\cong} \widetilde{\psi}_{i,+}\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_i}, \qquad \chi'_i \colon \mathcal{O}^{\beta'}_{L^*}\big|_{\pi^{-1}(U_i)^*} \xrightarrow{\cong} \widetilde{\psi}_{i,+}\mathcal{O}^{\beta'}_{\mathbb{C}^*\times U_i}.$$

Therefore, ξ_i must be of the form $(\chi'_i)^{-1} \circ \widetilde{\psi}_{i,+}(\varphi_i) \circ \chi_i$ for some $\varphi_i \colon \mathcal{O}^{\beta}_{\mathbb{C}^*\times U_i} \xrightarrow{\cong} \mathcal{O}^{\beta'}_{\mathbb{C}^*\times U_i}$. Using the description as a cyclic $\mathscr{D}$-module

$$\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_i} = \mathscr{D}_{\mathbb{C}^*\times U_i}/\mathscr{D}_{\mathbb{C}^*\times U_i}\big(\partial_t t + \beta,\ \partial_{x_1}, \ldots, \partial_{x_n}\big)$$

(with respect to a local coordinate system), one sees that $\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_i}$ and $\mathcal{O}^{\beta'}_{\mathbb{C}^*\times U_i}$ are only isomorphic in case $\beta - \beta' \in \mathbb{Z}$, and in that case φ_i may be chosen to be right-multiplication with $t^{\beta'-\beta}$. This defines isomorphisms ξ_i for β, β' satisfying $\beta - \beta' \in \mathbb{Z}$, which we assume from now on.

In order to show that ξ_i and ξ_j agree on $V := \pi^{-1}(U_{ij})^*$, we need to check that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\psi}_{ij,+}\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_{ij}} & \xrightarrow{\widetilde{\psi}_{ij,+}(\varphi_i|_{\mathbb{C}^*\times U_{ij}})} & \widetilde{\psi}_{ij,+}\mathcal{O}^{\beta'}_{\mathbb{C}^*\times U_{ij}} \\
\big\downarrow{\scriptstyle \mu_{ij}} & & \big\downarrow{\scriptstyle \mu'_{ij}} \\
\mathcal{O}^{\beta}_{\mathbb{C}^*\times U_{ij}} & \xrightarrow{\varphi_j|_{\mathbb{C}^*\times U_{ij}}} & \mathcal{O}^{\beta'}_{\mathbb{C}^*\times U_{ij}},
\end{array}$$

where the vertical arrows are $\widetilde{\psi}^+_j(\chi_{j|V} \circ (\chi_{i|V})^{-1})$ and $\widetilde{\psi}^+_j(\chi'_{j|V} \circ (\chi'_{i|V})^{-1})$, which are the gluing isomorphisms μ_{ij} and μ'_{ij} in the constructions of $\mathcal{O}^{\beta}_{L^*}$ and $\mathcal{O}^{\beta'}_{L^*}$, respectively. In the cyclic presentations of the involved $\mathscr{D}_{\mathbb{C}^*\times U_{ij}}$-modules from Lemma 2.2, they are given by right-multiplication with $\alpha_{ij}^{k\beta+1}$ and $\alpha_{ij}^{k\beta'+1}$, respectively. Meanwhile, $\varphi_i|_{\mathbb{C}^*\times U_{ij}} = \varphi_j|_{\mathbb{C}^*\times U_{ij}}$ is given by right-multiplication with $t^{\beta'-\beta}$. Applying the functor $\widetilde{\psi}_{ij,+}$ to this homomorphism, we see that the upper map in the above diagram is given by right-multiplication with $(\alpha_{ij}^{-k} t)^{\beta'-\beta}$. Since

$$\alpha_{ij}^{k(\beta'+1)} \cdot (\alpha_{ij}^{-k} t)^{\beta'-\beta} = t^{\beta'-\beta} \cdot \alpha_{ij}^{k(\beta+1)},$$

this shows that the diagram commutes, and therefore $\xi_i|_V = \xi_j|_V$. □

Remark 2.5. *The $\mathcal{O}_{L^*}$-module underlying the $\mathscr{D}_{L^*}$-module $\mathcal{O}^{\beta}_{L^*}$ for $\beta = \ell/k$ is $\pi_0^*\mathscr{F}^{\otimes \ell}$, the $\mathcal{O}$-module pull-back of $\mathscr{F}^{\otimes \ell}$ under $\pi_0 \colon L^* \xhookrightarrow{j_L} L \xrightarrow{\pi} X$, which is a torsion element of* Pic(L^*) *of order k. We will revisit the construction of $\mathcal{O}^{\beta}_{L^*}$ from this viewpoint in Section 6.3.*

There is a short exact sequence

$$0 \to \pi_0^*\mathscr{L} \to \Theta_{L^*} \to \pi_0^*\Theta_X \to 0$$

and $\pi_0^*\mathscr{L} = \mathcal{O}_X E$, where E is the vector field on L^* corresponding to the $\mathbb{C}^*$-action scaling the fibers, see Section 6.2. This short exact sequence splits locally over trivializations $\widetilde{\psi}_{s_i^k} \colon \mathbb{C}^* \times U_i \xrightarrow{\cong} \pi_0^{-1}(U_i)$ and the $\mathscr{D}_{L^*}$-module structure on $\mathcal{O}^{\ell/k}_{L^*} = \pi_0^*\mathscr{F}^{\otimes \ell}$ can be locally described by $E \cdot \pi_0^* s_i^{\ell} = \ell\, \pi_0^* s_i^{\ell}$ and $\pi_0^*\Theta_{U_i} \cdot \pi_0^* s_i^{\ell} = 0$.

2.2 Mixed Hodge module structure on $\mathcal{O}_{L^*}^{\beta}$

In this section, we show that the $\mathscr{D}_{L^*}$-module $\mathcal{O}_{L^*}^{\beta}$ constructed in the previous section underlies a complex pure Hodge module, and we consider its extension to a mixed Hodge module on L.

We continue using the notations from Section 2.1. In particular, we consider a line bundle $\pi\colon L \to X$ which is the k-th tensor power of a line bundle $\pi^F\colon F \to X$ for a fixed integer k, and call $L^* \subseteq L$ the complement of the zero section of L. Moreover, we fix an open cover $X = \bigcup_{i\in I} U_i$ with a choice of non-vanishing local sections $s_i \in \Gamma(U_i, \mathscr{F})$ inducing trivializations of F and L over U_i.

Proposition 2.6. *Let $\beta \in \mathbb{C}$ be such that $k\beta \in \mathbb{Z}$ and let $\ell \in I$. Let $U \subseteq X$ be a dense open subset over which the line bundle F trivializes. Then the $\mathscr{D}_{L^*}$-module $\mathcal{O}_{L^*}^{\beta}$ is the minimal extension of its restriction to $\pi^{-1}(U)^* \subseteq L^*$.*

Proof. We may assume that U is part of the open cover of X, i.e., $U = U_\ell$ for some $\ell \in I$. For all $i \in I$, denote by j_i the open embedding of $\pi^{-1}(U_i)^*$ into L^*. For $i, i' \in I$, let $j_{ii'}$ be the open embedding of $\pi^{-1}(U_{ii'})^*$ into $\pi^{-1}(U_{i'})^*$. We wish to show that the canonical homomorphism

$$\eta\colon \mathcal{O}_{L^*}^{\beta} \to (j_\ell)_{\dagger,+} j_\ell^+ \mathcal{O}_{L^*}^{\beta}$$

is an isomorphism. It suffices to prove this after restricting to $\pi^{-1}(U_i)^*$ for all $i \in I$, i.e., we need to show that $j_i^+(\eta)$ is an isomorphism. Note that we have natural isomorphisms

$$j_i^+(j_\ell)_{\dagger,+} j_\ell^+ \mathcal{O}_{L^*}^{\beta} \cong (j_{\ell i})_{\dagger,+} j_{i\ell}^+ j_\ell^+ \mathcal{O}_{L^*}^{\beta} \cong (j_{\ell i})_{\dagger,+} j_{\ell i}^+ j_i^+ \mathcal{O}_{L^*}^{\beta},$$

where the first isomorphism follows from base change and the second one from $j_\ell \circ j_{i\ell} = j_i \circ j_{\ell i}$.

By the construction of $\mathcal{O}_{L^*}^{\beta}$, there are isomorphisms $\chi_i\colon j_i^+ \mathcal{O}_{L^*}^{\beta} \xrightarrow{\cong} \widetilde{\psi}_{i,+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_i})$. Then the following diagram commutes:

$$\begin{array}{ccccc}
j_i^+ \mathcal{O}_{L^*} & \xrightarrow{j_i^+(\eta)} & j_i^+(j_\ell)_{\dagger,+} j_\ell^+ \mathcal{O}_{L^*}^{\beta} & \xrightarrow{\cong} & (j_{\ell i})_{\dagger,+} j_{\ell i}^+ j_i^+ \mathcal{O}_{L^*}^{\beta} \\
\chi_i \downarrow \cong & & & & (j_{\ell i})_{\dagger,+} j_{\ell i}^+(\chi_i) \downarrow \cong \\
\widetilde{\psi}_{i,+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_i}) & & \xrightarrow{\eta'} & & (j_{\ell i})_{\dagger,+} j_{\ell i}^+ \widetilde{\psi}_{i,+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_i}) \\
& \searrow^{\widetilde{\psi}_{i,+}(\tilde{\eta}')} & & & \downarrow \cong \\
& & & & \widetilde{\psi}_{i,+}(\tilde{j}_{\ell i})_{\dagger,+} \tilde{j}_{\ell i}^+(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_i}),
\end{array}$$

where η' and $\tilde{\eta}'$ are the canonical homomorphisms analogous to η and $\tilde{j}_{\ell i}$ is the open embedding of $\mathbb{C}^* \times U_{\ell i}$ into $\mathbb{C}^* \times U_i$.

It remains to be seen that $\tilde{\eta}'$ is an isomorphism. Let $\xi\colon \mathcal{O}_{U_i} \to j_{\dagger,+} j^+ \mathcal{O}_{U_i}$ denote the natural homomorphism associated to the open embedding $j\colon U_{\ell i} \hookrightarrow U_i$. Note that $\mathcal{O}_{U_i}$ is the minimal extension of $\mathcal{O}_{U_{\ell i}} = j^+ \mathcal{O}_{U_i}$ under j, i.e., ξ is an isomorphism. Thus, $\tilde{\eta}' = \mathrm{id}_{\mathbb{C}^*} \boxtimes \xi$ is an isomorphism, too, which concludes the proof. □

Corollary 2.7. *Let $\beta \in \mathbb{R}$ be such that $k\beta \in \mathbb{Z}$. Then the $\mathscr{D}_{L^*}$-module $\mathcal{O}_{L^*}^{\beta}$ underlies a (complex) constant pure Hodge module (i.e. a smooth element in* $\mathrm{HM}(L^*, \mathbb{C})$*) that we denote by ${}^H\underline{\mathbb{C}}_{L^*}^{\beta}$.*

Proof. We choose an open dense subset $U \subseteq X$ over which the line bundle F trivializes and pick a non-vanishing section $s \in \Gamma(U, \mathscr{F})$. By Proposition 2.1 and Proposition 2.6, we have

$$\mathcal{O}^{\beta}_{L^*} \cong j_{\dagger,+}\widetilde{\psi}_{s^k,+}(\mathcal{O}^{\beta}_{\mathbb{C}^*} \boxtimes \mathcal{O}_U),$$

where j is the open embedding of $\pi^{-1}(U)^*$ into L^*.

Using that the functors involved in this formula lift to the category of mixed Hodge modules, we obtain

$${}^H\underline{\mathbb{C}}^{\beta}_{L^*} = j_{!*}\widetilde{\psi}_{s^k,*}({}^H\underline{\mathbb{C}}^{\beta}_{\mathbb{C}^*} \boxtimes {}^H\underline{\mathbb{C}}_{U_\ell}).$$

Moreover, $j_{!*}$ conserves weights, and so does $\widetilde{\psi}_{s^k,*}$ since $\widetilde{\psi}_{s^k}$ is an isomorphism, therefore, we obtain that ${}^H\underline{\mathbb{C}}^{\beta}_{L^*} \in \mathrm{HM}(L^*, \mathbb{C})$, as claimed. □

Denote by $j_L \colon L^* \hookrightarrow L$ the canonical open embedding. Then the previous results yield the following.

Corollary 2.8. *The $\mathscr{D}_L$-modules $j_{L,+}\mathcal{O}^{\beta}_{L^*}$ resp. $j_{L,\dagger}\mathcal{O}^{\beta}_{L^*}$ underly the mixed Hodge modules $j_{L,*}{}^H\underline{\mathbb{C}}^{\beta}_{L^*}$ resp. $j_{L,!}{}^H\underline{\mathbb{C}}^{\beta}_{L^*}$ on L. If $\beta \notin \mathbb{Z}$, then*

$$j_{L,*}{}^H\underline{\mathbb{C}}^{\beta}_{L^*} \cong j_{L,!}{}^H\underline{\mathbb{C}}^{\beta}_{L^*} \cong j_{L,!*}{}^H\underline{\mathbb{C}}^{\beta}_{L^*},$$

which is pure of weight $\dim(X) + 1$.

Proof. The first statement is obvious from the previous Corollary 2.7. The second one follows from the well-known fact that $j_{\mathbb{C},+}\mathcal{O}^{\beta}_{\mathbb{C}^*} \cong j_{\mathbb{C},\dagger}\mathcal{O}^{\beta}_{\mathbb{C}^*} \cong j_{\mathbb{C},\dagger,+}\mathcal{O}^{\beta}_{\mathbb{C}^*}$ for $\beta \notin \mathbb{Z}$, using that for any $i \in I$, we have $j_i^+\mathcal{O}^{\beta}_{L^*} \cong \widetilde{\psi}_{i,+}(\mathcal{O}^{\beta}_{\mathbb{C}^*} \boxtimes \mathcal{O}_{U_i})$. □

3 Fourier–Laplace transforms

The purpose of this section is twofold: First we recall a few basic properties of general Fourier-Laplace transformations on (not necessarily trivial) vector bundles. We then apply these constructions to study a (complex of) $\mathscr{D}$-module(s) that generically computes cohomology groups of hyperplane sections of projective varieties. These results are used later in Section 7 for the special case of homogeneous spaces and their corresponding tautological systems.

3.1 Basics: Fourier–Laplace transforms on vector bundles

Definition 3.1. *Given a vector bundle $E \to X$ on a smooth projective variety X, we consider the canonical projections $p_1 \colon E \times_X E^\vee \to E$ and $p_2 \colon E \times_X E^\vee \to E^\vee$. Let $\alpha \colon E \times_X E^\vee \to \mathbb{C} \times X \to \mathbb{C}$ be the natural pairing and denote $\mathscr{K} := \alpha^+(\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t + 1))$. The **Fourier–Laplace transform** is the functor $\mathrm{FL}^E_X \colon D^b_{qc}(\mathscr{D}_E) \to D^b_{qc}(\mathscr{D}_{E^\vee})$ given as*

$$\mathrm{FL}^E_X(\mathscr{M}) := p_{2,+}(p_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} \mathscr{K}).$$

The following three basic properties follow directly from the projection formula and base change.

Lemma 3.2. *Let $\pi\colon E \to X$ be a vector bundle on a smooth variety and let $\pi^\vee\colon E^\vee \to X$ be its dual bundle. For $\mathscr{M} \in D^b_{qc}(\mathscr{D}_E)$ and $\mathscr{N} \in D^b_{qc}(\mathscr{D}_X)$, we have*

$$\mathrm{FL}^E_X(\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_E} \pi^+\mathscr{N}) \cong \mathrm{FL}^E_X(\mathscr{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_{E^\vee}} \pi^{\vee,+}\mathscr{N}.$$

Proof. We have

$$\begin{aligned}
\mathrm{FL}^E_X(\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_E} \pi^+\mathscr{N}) &= p_{2,+}(p_1^+(\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_E} \pi^+\mathscr{N}) \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} \mathscr{K})\\
&\cong p_{2,+}(p_1^+\pi^+\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} p_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} \mathscr{K})\\
&\cong p_{2,+}(p_2^+\pi^{\vee,+}\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} p_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} \mathscr{K})\\
&\overset{(\star)}{\cong} \pi^{\vee,+}\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{E^\vee}} p_{2,+}(p_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} \mathscr{K})\\
&\cong \mathrm{FL}^E_X(\mathscr{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_{E^\vee}} \pi^{\vee,+}\mathscr{N},
\end{aligned}$$

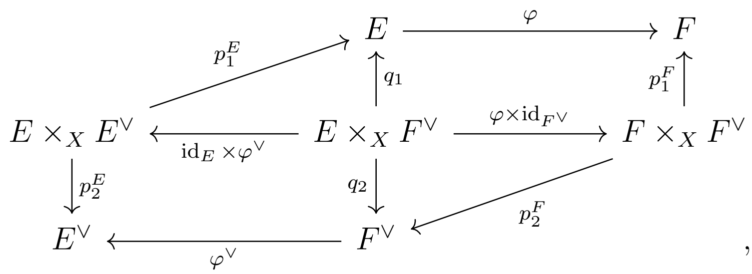
where $(\star)$ follows from the projection formula in the derived category of quasi-coherent $\mathscr{D}$-modules. □

Lemma 3.3. *Let $\varphi\colon E \to F$ be a morphism of vector bundles over X and denote by $\varphi^\vee\colon F^\vee \to E^\vee$ the induced morphism of dual vector bundles. Then*

$$\mathrm{FL}^F_X \circ \varphi_+ \cong \varphi^{\vee,+} \circ \mathrm{FL}^E_X$$

as functors $D^b_{qc}(\mathscr{D}_E) \to D^b_{qc}(\mathscr{D}_{F^\vee})$.

Proof. We denote $\mathscr{K}^E := (\alpha^E)^+(\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t + 1))$ and $\mathscr{K}^F := (\alpha^F)^+(\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t + 1))$, where $\alpha^E\colon E \times_X E^\vee \to \mathbb{C}$ and $\alpha^F\colon F \times_X F^\vee \to \mathbb{C}$ are the natural pairings. Moreover, denote by $q_1\colon E \times_X F^\vee \to E$ and $q_2\colon E \times_X F^\vee \to F^\vee$ the projections onto the first and second factor. Consider the commutative diagram



,

whose squares are cartesian. For every $\mathscr{M} \in D^b_{qc}(\mathscr{D}_E)$, we have:

$$\begin{aligned}
&\mathrm{FL}^F_X(\varphi_+\mathscr{M})\\
&= p^F_{2,+}(p_1^{F,+}\varphi_+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{F\times F^\vee}} \mathscr{K}^F)\\
&\cong p^F_{2,+}\big((\varphi \times \mathrm{id}_{F^\vee})_+ q_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{F\times F^\vee}} \mathscr{K}^F\big) && \text{(base change)}\\
&\cong p^F_{2,+}(\varphi \times \mathrm{id}_{F^\vee})_+\big(q_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times F^\vee}} (\varphi \times \mathrm{id}_{F^\vee})^+\mathscr{K}^F\big) && \text{(projection formula)}\\
&\cong q_{2,+}\big(q_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times F^\vee}} (\varphi \times \mathrm{id}_{F^\vee})^+\mathscr{K}^F\big) && (q_2 = p_2^F \circ (\varphi \times \mathrm{id}_{F^\vee}))\\
&\cong q_{2,+}\big(q_1^+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times F^\vee}} (\mathrm{id}_E \times \varphi^\vee)^+\mathscr{K}^E\big) && (\alpha^E \circ (\mathrm{id}_E \times \varphi^\vee) = \alpha^F \circ (\varphi \times \mathrm{id}_{F^\vee}))\\
&\cong q_{2,+}\big((\mathrm{id}_E \times \varphi^\vee)^+ p_1^{E,+}\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times F^\vee}} (\mathrm{id}_E \times \varphi^\vee)^+\mathscr{K}^E\big) && (q_1 = p_1^E \circ (\mathrm{id}_E \times \varphi^\vee))\\
&\cong \varphi^\vee_+ p_2^{E,+}(p_1^{E,+}\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times E^\vee}} \mathscr{K}^E) && \text{(base change)}\\
&= \varphi^\vee_+ \mathrm{FL}^E_X(\mathscr{M}).
\end{aligned}$$

□

Lemma 3.4. *Consider a cartesian square*

$$\begin{array}{ccc} E & \xrightarrow{g} & F \\ \downarrow & \times & \downarrow \\ X & \longrightarrow & Y, \end{array}$$

where the vertical arrows are vector bundles over smooth varieties. Denote by $g^\vee \colon E^\vee \to F^\vee$ the corresponding morphism of dual vector bundles. Then

$$\mathrm{FL}_Y^F \circ g_+ \cong g_+^\vee \circ \mathrm{FL}_X^E \qquad \textit{and} \qquad \mathrm{FL}_X^E \circ g^+ \cong g^{\vee,+} \circ \mathrm{FL}_Y^F$$

as functors $D^b_{qc}(\mathscr{D}_E) \to D^b_{qc}(\mathscr{D}_{F^\vee})$ and $D^b_{qc}(\mathscr{D}_F) \to D^b_{qc}(\mathscr{D}_{E^\vee})$, respectively.

Proof. We use notations as in the proof of Lemma 3.3. Note that we have the following commutative diagram with cartesian squares:

$$\begin{array}{ccccc} E & \xleftarrow{p_1^E} & E \times_X E^\vee & \xrightarrow{p_2^E} & E^\vee \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g\times g^\vee} & & \downarrow{\scriptstyle g^\vee} \\ F & \xleftarrow{p_1^F} & F \times_Y F^\vee & \xrightarrow{p_2^F} & F^\vee. \end{array}$$

For every $\mathscr{M} \in D^b_{qc}(\mathscr{D}_E)$, we have:

$$\begin{aligned}
&\mathrm{FL}_Y^F(g_+\mathscr{M}) \\
&= p^F_{2,+}(p_1^{F,+} g_+\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{F\times_Y F^\vee}} \mathscr{K}^F) \\
&\cong p^F_{2,+}\big((g\times g^\vee)_+ p_1^{E,+}\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{F\times_Y F^\vee}} \mathscr{K}^F\big) && \text{(base change)} \\
&\cong p^F_{2,+}(g\times g^\vee)_+\big(p_1^{E,+}\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times_X E^\vee}} (g\times g^\vee)^+\mathscr{K}^F\big) && \text{(projection formula)} \\
&\cong g^\vee_+ p^E_{2,+}\big(p_1^{E,+}\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times_X E^\vee}} (g\times g^\vee)^+\mathscr{K}^F\big) && (p_2^F \circ (g\times g^\vee) = g^\vee \circ p_2^E) \\
&\cong g^\vee_+ p^E_{2,+}(p_1^{E,+}\mathscr{M} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times_X E^\vee}} \mathscr{K}^E) && (\alpha^E = \alpha^F \circ (g\times g^\vee)) \\
&= g^\vee_+ \,\mathrm{FL}_X^E(\mathscr{M}).
\end{aligned}$$

Similarly, for $\mathscr{N} \in D^b_{qc}(\mathscr{D}_F)$, we get:

$$\begin{aligned}
&\mathrm{FL}_X^E(g^+\mathscr{N}) \\
&= p^E_{2,+}(p_1^{E,+} g^+\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times_X E^\vee}} \mathscr{K}^E) \\
&\cong p^E_{2,+}\big((g\times g^\vee)^+ p_1^{F,+}\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times_X E^\vee}} \mathscr{K}^E\big) && (g\circ p_1^E = p_1^F \circ (g\times g^\vee)) \\
&\cong p^E_{2,+}\big((g\times g^\vee)^+ p_1^{F,+}\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{E\times_X E^\vee}} (g\times g^\vee)^+\mathscr{K}^F\big) && (\alpha^E = \alpha^F \circ (g\times g^\vee)) \\
&\cong g^{\vee,+} p^F_{2,+}\big(p_1^{F,+}\mathscr{N} \otimes^{\mathbb{L}}_{\mathcal{O}_{F\times_Y F^\vee}} \mathscr{K}^F\big) && \text{(base change)} \\
&= g^{\vee,+}\,\mathrm{FL}_Y^F(\mathscr{N}).
\end{aligned}$$

□

In the following, we wish to relate Fourier–Laplace transforms on vector bundles with classical Fourier–Laplace transforms on a finite-dimensional vector space (which is the special case of a vector bundle over a point). For this, we consider the following situation: Let $\pi\colon E \to X$ be a vector bundle on a smooth variety and denote by $\mathcal{E}$ its sheaf of sections, i.e., $E = \mathrm{Tot}(\mathcal{E}) := \mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}^\bullet \mathcal{E}^\vee$. Let $W \subseteq \Gamma(X, \mathcal{E})$ be a non-zero finite-dimensional vector space of global sections of E and let $V := W^\vee$ be its dual vector space. There are natural bundle morphisms $ev\colon W \times X \to E$ and $ev^\vee\colon E^\vee \to V \times X$, where $E^\vee$ denotes the dual vector bundle to E.

Proposition 3.5. *Let W be a finite-dimensional space of global sections of a vector bundle $E \to X$ on a smooth variety. Let V denote its dual vector space. If $a_V\colon V \times X \to V$ and $a_W\colon W \times X \to W$ denote the projections onto the first factors, we have*

$$\mathrm{FL}^V(a_{V,+} ev^\vee_+ \mathscr{M}) \cong a_{W,+} ev^+ \mathrm{FL}_X^{E^\vee}(\mathscr{M})$$

for all $\mathscr{M} \in D^b_{qc}(\mathscr{D}_{E^\vee})$.

Proof. The claim follows from Lemma 3.3 and Lemma 3.4 considering the diagram

$$\begin{array}{ccccc} E^\vee & \xrightarrow{ev^\vee} & V \times X & \xrightarrow{a_V} & V \\ & \searrow & \downarrow & \times & \downarrow \\ & & X & \longrightarrow & \mathrm{Spec}\ \mathbb{C}. \end{array}$$

□

3.2 Fourier–Laplace transform of extensions of $\mathcal{O}^\beta_{L^*}$

We now determine the Fourier–Laplace transform of the $\mathscr{D}_{L^*}$-modules $\mathcal{O}^\beta_{L^*}$ defined in Section 2, where L^* is the complement of the zero section of a line bundle $\pi\colon L \to X$. Recall that the definition of $\mathcal{O}^\beta_{L^*}$ depends on the choice of a line bundle F with $L = F^{\otimes k}$ for some $k \in \mathbb{Z}$ satisfying $k\beta \in \mathbb{Z}$.

Note that the dual line bundle $\pi^\vee\colon L^\vee \to X$ is the $(-k)$-th tensor power $F^{\otimes(-k)}$. In what follows, we will consider the $\mathscr{D}_{L^{\vee,*}}$-module $\mathcal{O}^{-\beta}_{L^{\vee,*}}$ whose definition we always base on the choice of F as a $(-k)$-th root of $L^\vee$ (or, equivalently, based on $F^\vee$ as a k-th root of $L^\vee$).

We denote by $j_L\colon L^* \hookrightarrow L$ and $j_{L^\vee}\colon L^{\vee,*} \hookrightarrow L^\vee$ the open embeddings from the complements of the zero section into L and $L^\vee$, respectively.

Proposition 3.6. *Let $\beta \in \mathbb{C}$ with $k\beta \in \mathbb{Z}$. Then*

$$\mathrm{FL}^L_X(j_{L,+}\mathcal{O}^\beta_{L^*}) \cong j_{L^\vee,\dagger}\mathcal{O}^{-\beta}_{L^{\vee,*}}.$$

Proof. Denote $\mathscr{M} := \mathrm{FL}^L_X(j_{L,+}\mathcal{O}^\beta_{L^*})$ and $\mathscr{N} := j_{L^\vee,\dagger}\mathcal{O}^{-\beta}_{L^{\vee,*}}$. We construct the isomorphism between $\mathscr{M}$ and $\mathscr{N}$ locally over the trivializing open cover. To this end, we need to produce isomorphisms $\xi_i\colon \mathscr{M}|_{\pi^{\vee,-1}(U_i)} \to \mathscr{N}|_{\pi^{\vee,-1}(U_i)}$ for all $i \in I$ such that the restrictions of ξ_i and ξ_j to $\pi^{\vee,-1}(U_{ij})$ agree for all $i, j \in I$.

By construction, the restriction of $\mathcal{O}^\beta_{L^*}$ to $\pi^{-1}(U_i)^*$ is isomorphic to $\widetilde{\psi}_{i,+}(\mathcal{O}^\beta_{\mathbb{C}^*} \boxtimes \mathcal{O}_{U_i})$, where $\widetilde{\psi}_i = \widetilde{\psi}_{s_i^k}$ is the restriction to the complement of the zero section of the trivializing isomorphism $\psi_i = \psi_{s_i^k} \colon \mathbb{C} \times U_i \xrightarrow{\cong} \pi^{-1}(U_i)$ corresponding to the nonvanishing section $s_i^k \in \Gamma(U_i, \mathscr{L})$. Analogously, denote by

$$\widetilde{\psi}_i^\vee := \widetilde{\psi}_{s_i^{-k}} \colon \mathbb{C}^* \times U_i \xrightarrow{\cong} \pi^{\vee,-1}(U_i)^* \qquad \text{and} \qquad \psi_i^\vee := \psi_{s_i^{-k}} \colon \mathbb{C} \times U_i \xrightarrow{\cong} \pi^{\vee,-1}(U_i)$$

the corresponding isomorphisms for the dual line bundle $\pi^\vee \colon L^\vee \to X$. Then, by construction, the restricton of $\mathcal{O}_{L^\vee,*}^{-\beta}$ to $\pi^{\vee,-1}(U_i)^*$ is isomorphic to $\widetilde{\psi}_{i,+}(\mathcal{O}_{\mathbb{C}^*}^{-\beta} \boxtimes \mathcal{O}_{U_i})$.

We denote by $j_\mathbb{C}$ and $j_{\mathbb{C}\times U_i}$ the open embeddings $\mathbb{C}^*$ into $\mathbb{C}$ and of $\mathbb{C}^* \times U_i$ into $\mathbb{C} \times U_i$, respectively. Then

$$\begin{aligned} \mathscr{M}|_{\pi^{\vee,-1}(U_i)} &\cong \mathrm{FL}_{U_i}^{\pi^{-1}(U_i)} \left(\psi_{i,+} j_{\mathbb{C}\times U_i,+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_i})\right) \\ &\cong \psi_{i,+}^\vee \, \mathrm{FL}_{U_i}^{\mathbb{C}\times U_i} \left(j_{\mathbb{C}\times U_i,+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_i})\right) \\ &\cong \psi_{i,+}^\vee \big(\mathrm{FL}^{\mathbb{C}}(j_{\mathbb{C},+}\mathcal{O}_{\mathbb{C}^*}^{\beta}) \boxtimes \mathcal{O}_{U_i}\big), \end{aligned}$$

where the first isomorphism follows from Lemma 3.4, the second one from Lemma 3.3 and the final one from combining Lemma 3.2 and Lemma 3.4. On the other hand, it follows from base change that

$$\mathscr{N}|_{\pi^{\vee,-1}(U_i)} \cong \psi_{i,+}^\vee j_{\mathbb{C}\times U_i,\dagger}(\mathcal{O}_{\mathbb{C}^*}^{-\beta} \boxtimes \mathcal{O}_{U_i}) \cong \psi_{i,+}^\vee (j_{\mathbb{C},\dagger}\mathcal{O}_{\mathbb{C}^*}^{-\beta} \boxtimes \mathcal{O}_{U_i}).$$

There is an isomorphism $\eta \colon \mathrm{FL}^{\mathbb{C}}(j_{\mathbb{C},+}\mathcal{O}_{\mathbb{C}^*}^{\beta}) \xrightarrow{\cong} j_{\mathbb{C},\dagger}\mathcal{O}_{\mathbb{C}^*}^{-\beta}$. It gives rise to

$$\xi_i := \psi_{i,+}^\vee(\eta \boxtimes \mathrm{id}_{U_i}) \colon \mathscr{M}|_{\pi^{\vee,-1}(U_i)} \xrightarrow{\cong} \mathscr{N}|_{\pi^{\vee,-1}(U_i)}.$$

We need to check that ξ_i and ξ_j agree when restricted to $\pi^{\vee,-1}(U_{ij})$. In terms of the transition maps

$$\psi_{ij} = \psi_{s_i^k/s_j^k} \colon \mathbb{C} \times U_{ij} \xrightarrow{\cong} \mathbb{C} \times U_{ij} \quad \text{and} \quad \psi_{ij}^\vee = \psi_{s_i^{-k}/s_j^{-k}} \colon \mathbb{C} \times U_{ij} \xrightarrow{\cong} \mathbb{C} \times U_{ij}$$

of the line bundles L and $L^\vee$ as in (3), or their restrictions $\widetilde{\psi}_{ij}$ and $\widetilde{\psi}_{ij}^\vee$ to $\mathbb{C}^* \times U_{ij}$, this is equivalent to showing that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{FL}_{U_{ij}}^{\mathbb{C}\times U_{ij}} \left(j_{\mathbb{C}\times U_{ij},+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_{ij}})\right) & \xrightarrow[\cong]{\eta\boxtimes\mathrm{id}_{U_{ij}}} & j_{\mathbb{C}\times U_{ij},\dagger}(\mathcal{O}_{\mathbb{C}^*}^{-\beta} \boxtimes \mathcal{O}_{U_{ij}}) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{FL}_{U_{ij}}^{\mathbb{C}\times U_{ij}} \left(j_{\mathbb{C}\times U_{ij},+}\widetilde{\psi}_{ij}^{+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_{ij}})\right) & & j_{\mathbb{C}\times U_{ij},\dagger}\widetilde{\psi}_{ij}^{\vee,+}(\mathcal{O}_{\mathbb{C}^*}^{-\beta} \boxtimes \mathcal{O}_{U_{ij}}) \\
\wr\Vert & & \wr\Vert \\
\psi_{ij}^{\vee,+}\left(\mathrm{FL}_{U_{ij}}^{\mathbb{C}\times U_{ij}} \left(j_{\mathbb{C}\times U_{ij},+}(\mathcal{O}_{\mathbb{C}^*}^{\beta} \boxtimes \mathcal{O}_{U_{ij}})\right)\right) & \xrightarrow[\psi_{ij}^{\vee,+}(\eta\boxtimes\mathrm{id}_{U_{ij}})]{\cong} & \psi_{ij}^{\vee,+} j_{\mathbb{C}\times U_{ij},\dagger}(\mathcal{O}_{\mathbb{C}^*}^{-\beta} \boxtimes \mathcal{O}_{U_{ij}}).
\end{array}$$

The vertical arrows are induced from the isomorphisms

$$\widetilde{\psi}_{ij}^{+}(\mu_{ij}) \colon \mathcal{O}_{\mathbb{C}^*}^{\beta}\boxtimes\mathcal{O}_{U_{ij}} \xrightarrow{\cong} \widetilde{\psi}_{ij}^{+}(\mathcal{O}_{\mathbb{C}^*}^{\beta}\boxtimes\mathcal{O}_{U_{ij}}) \quad \text{and} \quad \widetilde{\psi}_{ij}^{\vee,+}(\mu_{ij}^\vee) \colon \mathcal{O}_{\mathbb{C}^*}^{-\beta}\boxtimes\mathcal{O}_{U_{ij}} \xrightarrow{\cong} \widetilde{\psi}_{ij}^{\vee,+}(\mathcal{O}_{\mathbb{C}^*}^{-\beta}\boxtimes\mathcal{O}_{U_{ij}}),$$

from (4) in the construction of $\mathcal{O}_{L^k,*}^{\beta}$ and $\mathcal{O}_{L^{-k},*}^{-\beta}$, respectively.

First, we treat the case $\beta \notin \mathbb{Z}$. Fixing a local coordinate system $(t, x_1, \ldots, x_n)$ on $V := \mathbb{C} \times U_{ij}$, we have homomorphisms between cyclic $\mathscr{D}$-modules, which we may now write out explicitly. Since $\beta \notin \mathbb{Z}$, we have $j_{\mathbb{C},+}\mathcal{O}^{\beta}_{\mathbb{C}^*} \cong j_{\mathbb{C},\dagger}\mathcal{O}^{\beta}_{\mathbb{C}^*} = \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t + \beta)$. Notice that $\mathrm{FL}^{\mathbb{C}}(\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t + \beta)) = \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - \beta - 1) \xrightarrow{\cong} \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - \beta)$, where the last map is given by right-multiplication with t, which is an isomorphism for $\beta \neq 0$. Under these identifications, the isomorphism η from above is simply given by the composition

$$\eta\colon\ \mathrm{FL}^{\mathbb{C}}(j_{\mathbb{C},+}\mathcal{O}^{\beta}_{\mathbb{C}^*}) \cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - \beta) \xrightarrow{\mathrm{id}} j_{\mathbb{C},\dagger}\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - \beta) \cong \mathcal{O}^{-\beta}_{\mathbb{C}^*}.$$

Using this as well as Lemma 2.2 and recalling the notation $\alpha_{ij} = s_i/s_j \in \Gamma(U_{ij}, \mathcal{O}^{\times}_{U_{ij}})$, the diagram from above becomes:

$$\begin{array}{ccc}
\mathscr{D}_V/\mathscr{D}_V\big(\partial_t t - \beta, \partial_{x_\ell} \mid \ell\big) & \xrightarrow{\qquad\mathrm{id}\qquad} & \mathscr{D}_V/\mathscr{D}_V\big(\partial_t t - \beta, \partial_{x_\ell} \mid \ell\big) \\
\big\downarrow{\scriptstyle \cdot\alpha_{ij}^{k(\beta+1)}} & & \big\downarrow{\scriptstyle \cdot\alpha_{ij}^{(-k)\cdot(-\beta+1)}} \\
\mathscr{D}_V/\mathscr{D}_V\big(\partial_t t - \beta, \partial_{x_\ell} + k(\beta+1)\alpha_{ij}^{-1}\frac{\partial\alpha_{ij}}{\partial x_\ell} \mid \ell\big) & & \mathscr{D}_V/\mathscr{D}_V\big(\partial_t t - \beta, \partial_{x_\ell} + k(\beta-1)\alpha_{ij}^{-1}\frac{\partial\alpha_{ij}}{\partial x_\ell} \mid \ell\big) \\
\big\downarrow{\scriptstyle \cdot\alpha_{ij}^{-k}} & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\mathscr{D}_V/\mathscr{D}_V\big(\partial_t t - \beta, \partial_{x_\ell} + k\beta\alpha_{ij}^{-1}\frac{\partial\alpha_{ij}}{\partial x_\ell} \mid \ell\big) & \xrightarrow{\alpha_{ij}^{-k}} & \mathscr{D}_V/\mathscr{D}_V\big(\partial_t t - \beta, \partial_{x_\ell} + k(\beta-1)\alpha_{ij}^{-1}\frac{\partial\alpha_{ij}}{\partial x_\ell} \mid \ell\big),
\end{array}$$

which commutes. On the other hand, if $\beta \in \mathbb{Z}$, then we have $j_{\mathbb{C},+}\mathcal{O}^{\beta}_{\mathbb{C}^*} \cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t)$, but $j_{\mathbb{C},\dagger}\mathcal{O}^{\beta}_{\mathbb{C}^*} \cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - 1)$. (Notice that for $\beta \in \mathbb{Z}$, the modules $\mathcal{O}^{\beta}_{\mathbb{C}^*}$ are all isomorphic to both $\mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t t)$ and $\mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_t t - 1)$.) Then clearly

$$\mathrm{FL}^{\mathbb{C}}(j_{\mathbb{C},+}\mathcal{O}^{\beta}_{\mathbb{C}^*}) \cong \mathrm{FL}^{\mathbb{C}}(\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t)) \cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - 1),$$

and under these identifications, the isomorphism η is again given by the identity map on $\mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(\partial_t t - 1)$. Hence, we obtain the commutativity of the corresponding diagram by simply putting β to one in the above diagram. □

Corollary 3.7. *Let $k \in \mathbb{Z}$ and let $\beta \in \mathbb{R}$ with $k\beta \in \mathbb{Z}$. Then the Fourier–Laplace transform on L of the $\mathscr{D}_L$-module $j_{L,+}\mathcal{O}^{\beta}_{L^*}$ can be equipped with the structure of a complex mixed Hodge module which is pure of weight* $\dim(X) + 1$ *if $\beta \notin \mathbb{Z}$.*

Proof. We have just seen in the previous Proposition 3.6 that

$$\mathrm{FL}^{L}_{X}(j_{L,+}\mathcal{O}^{\beta}_{L^*}) \cong j_{L^\vee,\dagger}\mathcal{O}^{-\beta}_{L^\vee,*}.$$

On the other hand, we know by Corollary 2.8 that $j_{L^\vee,\dagger}\mathcal{O}^{-\beta}_{L^\vee,*}$ underlies the the object

$$j_{L^\vee,!}\,{}^{H}\underline{\mathbb{C}}^{-\beta}_{L^\vee,*} \in \mathrm{MHM}(L^\vee, \mathbb{C}),$$

and that it is pure if $\beta \notin \mathbb{Z}$. □

3.3 Twisted cohomology of hyperplane sections

In this subsection, we describe a complex of $\mathcal{D}$-modules that generically computes certain twisted cohomologies of hyperplane sections of our variety X (resp. the complement of those). We show that they underlie objects of the derived category of mixed Hodge modules. In the more specific situation studied later in Section 7, when X arises a homogenous space, these $\mathcal{D}$-modules will appear as tautological systems.

With the notations from before, we fix a non-zero finite-dimensional subspace W of $\Gamma(X, \mathscr{L})$. Let $V := W^\vee$ denote its dual vector space. The linear system W on X defines a rational map $g\colon X \dashrightarrow \mathbb{P}V$. The natural evaluation morphism

$$ev\colon W \times X \to L, \tag{7}$$

is a morphism of vector bundles over X and it induces a dual bundle morphism

$$ev^\vee\colon L^\vee \to V \times X.$$

The following diagram commutes:

$$\begin{array}{ccc} L^\vee & \xrightarrow{ev^\vee} & V \times X \\ {\scriptstyle \pi^\vee}\downarrow & & \downarrow \\ X & \xrightarrow{g\times \mathrm{id}_X} & \mathbb{P}V \times X \end{array}$$

If the linear system W is base-point-free, then $g\colon X \to \mathbb{P}V$ is a morphism and $ev^\vee$ restricts to a morphism

$$\widetilde{ev}^\vee\colon L^{\vee,*} \to (V \setminus 0) \times X$$

of complements of zero sections. In this case, we have the following commutative diagram:

$$\begin{array}{ccccc} L^\vee & \xrightarrow{ev^\vee} & V \times X & \xrightarrow{a_V} & V \\ {\scriptstyle j_{L^\vee}}\uparrow & & {\scriptstyle j_V \times \mathrm{id}_X}\uparrow & \times & {\scriptstyle j_V}\uparrow \\ L^{\vee,*} & \xrightarrow{\widetilde{ev}^\vee} & (V\setminus 0) \times X & \longrightarrow & V \setminus 0 \\ \downarrow & & \downarrow & \times & \downarrow \\ X & \xrightarrow{g\times \mathrm{id}_X} & \mathbb{P}V \times X & \longrightarrow & \mathbb{P}V \end{array}$$

If, moreover, the linear system W separates points and tangent directions (in particular, $\mathscr{L}$ is very ample in this case), then $g\colon X \to \mathbb{P}V$ is a locally closed embedding. In this case, $L^{\vee,*}$ is isomorphic to $\hat{X} \setminus 0$, where $\hat{X} \subseteq V$ is the affine cone over $g(X) \subseteq \mathbb{P}V$, and $L^\vee$ is the blow-up of $\hat{X}$ in the origin: $L^\vee \cong \mathrm{Bl}_{\{0\}} \hat{X}$.

Proposition 3.8. *Assume L to be very ample and let $W \subseteq H^0(X, \mathscr{L})$ be a finite-dimensional linear system defining a locally closed embedding $g\colon X \hookrightarrow \mathbb{P}V$, where $V := W^\vee$. Let $\hat{\imath}\colon L^{\vee,*} \cong \hat{X} \setminus 0 \hookrightarrow V$ denote the locally closed embedding of the punctured affine cone over X into V. Then, for all $\beta \in \mathbb{C}$ with $k\beta \in \mathbb{Z}$, the $\mathscr{D}_W$-module*

$$\mathrm{FL}^V(\hat{\imath}_+ \mathcal{O}^\beta_{L^{\vee,*}})$$

underlies an element of $D^b\mathrm{MHM}(W, \mathbb{C})$ that we denote by ${}^H\!\mathcal{M}^\beta_L$.

For $\beta \notin \mathbb{Z}$, the cohomology modules $\mathcal{H}^i({}^H\!\mathcal{M}^\beta_L)$ are pure Hodge modules of weight $\dim(X) + \dim(W) + i$.

Proof. We combine Proposition 3.5 and Proposition 3.6 to get a purely functorial description of this $\mathscr{D}_W$-module not involving Fourier–Laplace transforms, namely

$$\begin{aligned}
\mathrm{FL}^V(\hat{\iota}_+\mathcal{O}^\beta_{L^\vee,*}) &\cong \mathrm{FL}^V(a_{V,+}ev^\vee_+ j_{L^\vee,+}\mathcal{O}^\beta_{L^\vee,*}) \\
&\cong a_{W,+}ev^+\,\mathrm{FL}^{L^\vee}_X(j_{L^\vee,+}\mathcal{O}^\beta_{L^\vee,*})) \\
&\cong a_{W,+}ev^+ j_{L,\dagger}\mathcal{O}^{-\beta}_{L^*} \\
&\cong a_{W,+}ev^\dagger j_{L,\dagger}\mathcal{O}^{-\beta}_{L^*} \\
&= a_{W,+}\, ev^\dagger[\dim L - \dim(W\times X)]\, j_{L,\dagger}\mathcal{O}^{-\beta}_{L^*}[\dim W - 1].
\end{aligned}$$

where the fourth isomorphism follows from the fact that ev is smooth.

Recall from Corollary 2.7 that $\mathcal{O}^{-\beta}_{L^*}$ underlies the complex pure Hodge module ${}^H\underline{\mathbb{C}}^{-\beta}_{L^*}$, therefore, we obtain that

$${}^H\mathcal{M}^\beta_L := a_{W,*}ev^* j_{L,!}\,{}^H\underline{\mathbb{C}}^{-\beta}_{L^*}[\dim W - 1] \in D^b\mathrm{MHM}(W,\mathbb{C}).$$

If $\beta \notin \mathbb{Z}$, we have seen in Corollary 3.7 that $j_{L,!}\,{}^H\underline{\mathbb{C}}^{-\beta}_{L^*}$ is pure (of weight $\dim(X)+1$). Since the morphism ev is smooth, and since a_W is projective, the second assertion thus follows from [Sai88, Théorème 1]. □

We will discuss next a natural interpretation of the smooth part of the complex of mixed Hodge modules ${}^H\mathcal{M}^\beta_L$. For this purpose, fix some $\lambda \in W$ which lies outside of the singular locus of the complex ${}^H\mathcal{M}^\beta_L$, i.e. such that the inclusion $i_\lambda\colon \{\lambda\} \hookrightarrow W$ is non-characteristic for ${}^H\mathcal{M}^\beta_L$ in the sense of [Sai88, Section 3.5.1.].

Then, by definition, we have $\lambda \in \Gamma(X,\mathscr{L})$, and interpreting this global section as a morphism $\lambda\colon X \hookrightarrow L$, we can consider the image $L_\lambda := \mathrm{im}(\lambda) \subset L$. We identify the zero section of the projection $\pi\colon L \twoheadrightarrow X$ inside L with X and recall that $L^* := L\backslash X$ denotes the complement of the zero section. We denote by $H_\lambda := L_\lambda \cap X \subset X$ the zero locus of the section λ and by $U_\lambda := X\backslash H_\lambda$ its complement in X.

Notice that there is the family of zero loci of sections of L, given by $\mathcal{Y} := ev^{-1}(0) \to W$, $(s,\lambda) \mapsto \lambda$, and the fibre of this map over a point $\lambda \in W$ is exactly the hypersurface H_λ. Writing $\mathcal{U} := (W\times X)\backslash\mathcal{Y} = \bigcup_{\lambda\in W} U_\lambda$, the evaluation morphism ev from Formula (7) then restricts to a morphism

$$ev_{|\mathcal{U}} : \mathcal{U} \longrightarrow L^*.$$

Recall that we defined the constant (complex) pure Hodge module ${}^H\underline{\mathbb{C}}^\beta_{L^*}$ in Corollary 2.7. We then put

$${}^H\underline{\mathbb{C}}^\beta_{\mathcal{U}} := ev^*_{|\mathcal{U}}\,{}^H\underline{\mathbb{C}}^\beta_{L^*}[\dim W - 1] \in \mathrm{HM}(\mathcal{U},\mathbb{C}).$$

Moreover, for $\lambda \in W$ as above, consider the restriction $\lambda_{|U_\lambda}\colon U_\lambda \hookrightarrow L^*$. We put, for $\beta \in \mathbb{Q}$,

$${}^H\underline{\mathbb{C}}^\beta_\lambda := \lambda^*_{|U_\lambda}\,{}^H\underline{\mathbb{C}}^\beta_{L^*}[-1] \in \mathrm{HM}(U_\lambda,\mathbb{C})$$

Proposition 3.9. *We continue with the setup of Proposition 3.8 and additionally assume that X is projective. Let $\beta \in \mathbb{C}$ with $k\beta \in \mathbb{Z}$. Then the following statements hold true:*

(i) Let $a_\mathcal{U}\colon \mathcal{U} \to W$ be the restriction of the projection $a_W\colon W\times X \to W$. Then we have an isomorphism

$$a_{\mathcal{U},!}\,{}^H\underline{\mathbb{C}}^{-\beta}_{\mathcal{U}} \cong {}^H\mathcal{M}^\beta_L$$

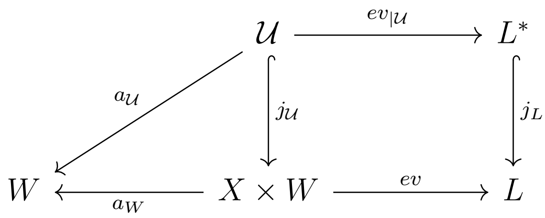
in $D^b\mathrm{MHM}(W,\mathbb{C})$.

(ii) For any $m \in \mathbb{N}$, and any $\lambda \in W$ outside the singular locus of ${}^H\mathcal{M}_L^\beta$ we have an isomorphism of (complex) mixed Hodge structures

$$H^m(i_\lambda^* \, {}^H\mathcal{M}_L^\beta[-\dim W]) \cong H_c^{\dim(X)+m}(U_\lambda, \, {}^H\underline{\mathbb{C}}_\lambda^{-\beta}).$$

Proof. In the course of the proof, we will make repeatedly use of the base change property for algebraic mixed Hodge modules, as stated in [Sai90, Section 4.4.3].

(i) This is almost immediate by considering the following cartesian diagram



which yields

$$\begin{aligned} {}^H\mathcal{M}_L^\beta &\overset{(*)}{\cong} a_{W,!}\, ev^*\, j_{L,!}\, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] \\ &\overset{(**)}{\cong} a_{W,!}\, j_{\mathcal{U},!}\, ev_{|\mathcal{U}}^* \, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] \\ &= a_{\mathcal{U},!}\, ev_{|\mathcal{U}}^* \, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] \\ &= a_{\mathcal{U},!}\, {}^H\underline{\mathbb{C}}_{\mathcal{U}}^{-\beta}, \end{aligned}$$

where the isomorphism $(*)$ holds because a_W is proper (since X is projective) and ev is smooth, and where $(**)$ follows by base change.

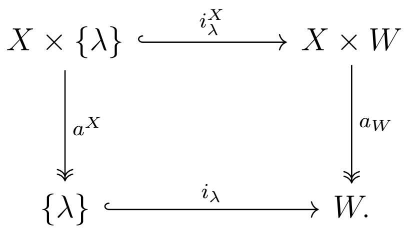
(ii) Write

$${}^H\mathcal{N}_L^\beta := ev^* j_{L,!}\, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] \in \mathrm{MHM}(X \times W, \mathbb{C}),$$

then by the proof of the previous Proposition 3.8 we have that

$${}^H\mathcal{M}_L^\beta \cong a_{W,!}\, {}^H\mathcal{N}_L^\beta.$$

Now consider the cartesian diagram



Then

$$\begin{aligned} i_\lambda^* \, {}^H\mathcal{M}_L^\beta &\cong i_\lambda^* a_{W,!}\, {}^H\mathcal{N}_L^\beta \\ &\cong a_!^X i_\lambda^{X,*}\, {}^H\mathcal{N}_L^\beta = a_!^X\, i_\lambda^{X,*} ev^* j_{L,!}\, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] && \text{(base change)} \\ &\cong a_!^X\, \lambda^* j_{L,!}\, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] && (ev \circ i_\lambda^X = \lambda) \\ &\cong a_*^X\, \lambda^* j_{L,!}\, {}^H\underline{\mathbb{C}}_{L^*}^{-\beta}[\dim W - 1] && (a^X \text{ proper}). \end{aligned}$$

Now we consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & L \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j_L} \\ U_\lambda & \xrightarrow{\lambda_{|U_\lambda}} & L^*, \end{array}$$

then base change yields $\lambda^* j_{L,!} \cong j_! \lambda^*_{|U_\lambda}$, so that we get an isomorphism of objects in $D^b\mathrm{MHM}(\{\lambda\}, \mathbb{C})$ (which we identify with the derived category of complex mixed Hodge structures).

$$\begin{aligned} i^*_\lambda {}^H\mathcal{M}^\beta_L &\cong a^X_* \, j_! \, \lambda^*_{|U_\lambda} {}^H\underline{\mathbb{C}}^{-\beta}_{L^*}[\dim W - 1] \cong a^X_* \, j_! \, {}^H\underline{\mathbb{C}}^{-\beta}_\lambda[\dim W] \\ &\cong a^X_! \, j_! \, {}^H\underline{\mathbb{C}}^{-\beta}_\lambda[\dim W] = a^{U_\lambda}_! \, {}^H\underline{\mathbb{C}}^{-\beta}_\lambda[\dim W], \end{aligned}$$

where $a^{U_\lambda} : U_\lambda \twoheadrightarrow \{\lambda\}$ and where we have used $a^X_* = a^X_!$ since X is projective. We apply $H^m(-)$ to both sides to obtain an isomorphism of complex mixed Hodge structures

$$H^m(i^*_\lambda {}^H\mathcal{M}^\beta_L[-\dim W]) \cong H^m(a^{U_\lambda}_! \, {}^H\underline{\mathbb{C}}^{-\beta}_\lambda).$$

□

In the subsequent sections of this article, we will investigate to which extent tautological systems for homogeneous spaces X are examples of the $\mathscr{D}$-modules underlying ${}^H\mathcal{M}^\beta_L$ for particular line bundles L and values β.

4 Equivariant constructions and tautological systems

In this section, we consider varieties with group actions and introduce particular $\mathscr{D}$-modules that arise naturally from global vector fields induced by the group action.

Notations about vector fields: For a smooth variety X, we denote its tangent sheaf by Θ_X, which coincides with the sheaf $\mathscr{D}er(\mathcal{O}_X)$ of derivations on X, i.e., $\Gamma(U, \Theta_X) = \mathrm{Der}(\mathcal{O}_U)$ for $U \subseteq X$. A vector field on X is a global section of Θ_X. A morphism $f : X \to Y$ between smooth varieties induces a push-forward homomorphism of $\mathcal{O}_X$-modules $\mathrm{d}f : \Theta_X \to f^*\Theta_Y$. Under the natural isomorphism

$$\begin{aligned} f^*\Theta_Y &\cong f^*\mathscr{H}om_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_Y) \cong \mathscr{H}om_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{f^{-1}\mathcal{O}_Y}, \mathcal{O}_X) \\ &\cong \mathscr{H}om_{f^{-1}\mathcal{O}_Y}(\Omega_{f^{-1}\mathcal{O}_Y}, \mathcal{O}_X) \cong \mathscr{D}er(f^{-1}\mathcal{O}_Y, \mathcal{O}_X), \end{aligned}$$

the push-forward homomorphism $\mathrm{d}f$ is given by pre-composing derivations on (an open subset of) X with the homomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ induced by f. We identify the fiber of Θ_X at a point $p \in X$ with the tangent space T_pX. The push-forward homomorphism $\mathrm{d}f$ induces on the level of fibers the differential map $\mathrm{d}f_p : T_pX \to T_{f(p)}Y$ between tangent spaces. For a tangent vector $v \in T_pX$, we will simply write $\mathrm{d}f(v)$ instead of $\mathrm{d}f_p(v)$ whenever convenient.

4.1 Vector fields from group actions

In this section we consider the action of a connected linear algebraic group G' on a variety Y. The main case of interest arises when we are given an action of an algebraic group G on a variety X, and an equivariant line bundle $\mathscr{L}$ on X. Denoting G' the group $\mathbb{C}^* \times G$, we let G' act (in various ways depending on the choice of a character) on Y, which we take to be the total space L (or the complement L^* of the zero section) of $\mathscr{L}$.

We begin with recalling some facts concerning actions of algebraic groups on smooth varieties.

Lemma 4.1. *Let G' be an algebraic group acting on a smooth variety Y. Then there is a unique Lie algebra homomorphism*

$$Z_Y : \mathfrak{g}' \to \Gamma(Y, \Theta_Y)$$

associating to every element ξ of the Lie algebra $\mathfrak{g}'$ of G' a vector field $Z_Y(\xi)$ on Y with the following point-wise description: At a point $y \in Y$, the tangent vector of the vector field $Z_Y(\xi)$ is given by $-\,\mathrm{d}\varphi^y(\xi)$, where $\varphi^y : G' \to Y$, $g \mapsto g \cdot y$, and ξ is understood as a tangent vector to G' at the point $1 \in G'$.

Proof. If $\theta_1 \in \Gamma(G', \Theta_{G'})$ is a *right-invariant* vector field on G' and $\theta_2 \in \Gamma(Y, \Theta_Y)$ is a G'*-invariant* vector field on Y, then $\theta_1 \times \theta_2 \in \Gamma(G' \times Y, \Theta_{G' \times Y})$ is a G'-invariant vector field on $G' \times Y$ with respect to the action $g \cdot (h, y) := (hg^{-1}, gy)$ on $G' \times Y$. This implies that the push-forward $\mathrm{d}\varphi(\theta_1 \times \theta_2) \in \Gamma(G' \times Y, \varphi^* \Theta_Y)$ under the action morphism $\varphi : G' \times Y \to Y$ lifts uniquely to a vector field on Y. We apply this to θ_1 being the right-invariant vector field on G' associated to $\xi \in \mathfrak{g}'$ and $\theta_2 = 0$, and define $-Z_Y(\xi)$ to be the resulting vector field on Y. It follows from the construction and the chosen sign convention that this defines a Lie algebra homomorphism Z_Y.

It remains to show the point-wise description for $Z_Y(\xi)$. The right-invariant vector field θ_1 associated to $\xi \in T_1 G'$ is at a point $g \in G'$ given by the tangent vector $\mathrm{d}R^g(\xi) \in T_g G'$, where R_g is the automorphism of G' given by right-multiplication with g. The push-forward of the vector field $\theta_1 \times 0$ under the action $\varphi : G' \times Y \to Y$ associates to a point $(g, y) \in G' \times Y$ the tangent vector $\mathrm{d}\varphi(\mathrm{d}R^g(\xi), 0) = \mathrm{d}\varphi^y(\xi)$. Therefore, its unique lift $-Z_Y(\xi)$ is the vector field on Y whose tangent vector at a point $y \in Y$ is $\mathrm{d}\varphi^y(\xi)$. □

In the complex analytic category, the vector field $Z_Y(\xi)$ may be defined as the derivation

$$Z_Y(\xi)(f) = \frac{d}{dt} f\big(\exp(t\xi)^{-1} \cdot (-)\big)\big|_{t=0}.$$

If the G'-variety Y considered is clear from the context, we will drop the index and just write $Z(\xi)$. In the literature, the vector field $Z(\xi)$ is sometimes denoted by L_ξ, see e.g. [Hot98, II.2].

Example 4.2. *Consider the action of G' on itself by left-multiplication (i.e., $Y = G'$). Then $-Z_{G'}(\xi)$ is the right-invariant vector field associated to $\xi \in \mathfrak{g}'$. If, for example, $G' = (\mathbb{C}^*)^d$ and $\xi \in \mathbb{C}^d = \mathfrak{g}'$, then*

$$Z_{(\mathbb{C}^*)^d}(\xi) = -\sum_{i=1}^{d} \xi_i \, t_i \partial_{t_i},$$

where $(t_1, \ldots, t_d)$ are the standard coordinates on $(\mathbb{C}^*)^d$.

For group actions on finite-dimensional vector spaces, we also have the following description:

Lemma 4.3. *Let $\rho\colon G' \to \mathrm{GL}(V)$ be a finite-dimensional rational representation of an algebraic group G'. The induced left action of G' on $\mathbb{C}[V] = \bigoplus_{d\geq 0} \mathrm{Sym}^d V^\vee$ describes a morphism of algebraic groups $G' \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}[V])$ whose induced Lie algebra homomorphism $\mathfrak{g}' \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[V])$ makes the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Lie}(G') & \xrightarrow{\qquad\qquad} & \mathrm{End}_{\mathbb{C}}(\mathbb{C}[V]) \\ & {\scriptstyle Z_V}\searrow \qquad \nearrow & \\ & \mathrm{Der}(\mathbb{C}[V]) & \end{array}.$$

Explicitly, if we fix coordinates $x_1, \ldots, x_n$ on V and consider the associated Lie algebra representation $\mathrm{d}\rho\colon \mathfrak{g}' \to \mathfrak{gl}(V) = \mathfrak{gl}(n, \mathbb{C}) = \mathbb{C}^{n\times n}$, then

$$Z_V(\xi) = -\sum_{i,j=1}^{n} \mathrm{d}\rho(\xi)_{ji}\, x_i \partial_{x_j}$$

for all $\xi \in \mathrm{Lie}(G')$.

Proof. Let $e_1, \ldots, e_n$ be a basis of the vector space V and let $x_1, \ldots, x_n \in \Gamma(V, \mathcal{O}_V)$ be the corresponding coordinate functions on V, corresponding to the dual basis $e_1^*, \ldots, e_n^*$ of $V^\vee$. Let $\xi \in \mathfrak{g}'$, i.e., a tangent vector to G' at the point $1 \in G'$, which we may identify with a morphism $\gamma\colon \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \to G'$. The composition of γ with $\rho\colon G' \to \mathrm{GL}(V)$ is the $\mathbb{C}[\varepsilon]/\varepsilon^2$-valued point $\mathrm{Id} + \varepsilon\, \mathrm{d}\rho(\xi)$ of $\mathrm{GL}(V) = \mathrm{GL}(\mathbb{C}^n) \subseteq \mathbb{C}^{n\times n}$.

In particular, for $v \in V = \mathbb{C}^n$ and the morphism $\varphi^v\colon G' \to V$ mapping $g \mapsto g \cdot v$, we see that $\mathrm{d}\varphi^v(\xi)$ is the tangent vector to $V = \mathbb{C}^n$ at v given by $\mathrm{d}\rho(\xi)(v) \in \mathbb{C}^n$. Therefore, by Lemma 4.1, the vector field $-Z_V(\xi)$ is the unique vector field on $\mathbb{C}^n$ associating to $v \in \mathbb{C}^n$ the tangent vector whose j-th coordinate is $\sum_{i=1}^n \mathrm{d}\rho(\xi)_{ji} v_i$. This corresponds to the derivation

$$-Z_V(\xi) = \sum_{i,j=1}^{n} \mathrm{d}\rho(\xi)_{ji}\, x_i \partial_{x_j}.$$

On the dual representation $\rho^\vee\colon G' \to \mathrm{GL}(V^\vee)$, note that γ defines the $\mathbb{C}[\varepsilon]/\varepsilon^2$-valued point $\mathrm{Id} - \varepsilon\, \mathrm{d}\rho(\xi)^T$ of $\mathrm{GL}(V^\vee) = \mathrm{GL}(\mathbb{C}^n) \subseteq \mathbb{C}^{n\times n}$, mapping e_j^* to $e_j^* - \varepsilon \sum_{i=1}^n \mathrm{d}\rho(\xi)_{ji} e_i^*$. This induces the $\mathbb{C}[\varepsilon]/\varepsilon^2$-valued point of $\mathrm{GL}_{\mathbb{C}}(\mathbb{C}[V])$ which maps

$$\begin{aligned} x_j &\mapsto x_j - \varepsilon \sum_{i=1}^{n} \mathrm{d}\rho(\xi)_{ji} x_i \quad \text{and, more generally,} \\ x^\alpha &\mapsto x^\alpha - \varepsilon \sum_{i=1}^{n} \mathrm{d}\rho(\xi)_{ji}\, \alpha_j \frac{x_i}{x_j} x^\alpha = x^\alpha - \varepsilon \sum_{i=1}^{n} \mathrm{d}\rho(\xi)_{ji}\, x_i \frac{\partial x^\alpha}{\partial x_j} \quad \text{for } \alpha \in \mathbb{N}^n. \end{aligned}$$

In particular, the Lie algebra homomorphism $\mathfrak{g}' \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[V])$ maps ξ to the derivation $Z_V(\xi) \in \mathrm{Der}(\mathbb{C}[V])$. □

Example 4.4. *Let $G' = (\mathbb{C}^*)^d$ be a d-dimensional torus acting linearly on an n-dimensional vector space V. We identify V with $\mathbb{C}^n$ by picking a basis that diagonalizes the action, i.e., $t = (t_1, \dots, t_d) \in (\mathbb{C}^*)^d$ acts on $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ by*

$$t \cdot x = (t^{\alpha_1} x_1, \dots, t^{\alpha_n} x_n) \qquad \text{with } \alpha_1, \dots, \alpha_n \in \mathbb{Z}^d.$$

If $\xi \in \mathbb{Z}^d = \mathrm{Lie}\big((\mathbb{C}^*)^d\big)$ is the i-th standard basis vector e_i, we get the vector field

$$Z_V(e_i) = -\sum_{j=1}^{n} (\alpha_j)_i x_j \partial_{x_j}$$

on V. These are the vector fields showing up in GKZ-systems associated to the given torus action.

Lemma 4.5. *Let G' be an algebraic group acting transitively on a smooth variety Y. Fix $y \in Y$ and consider the morphism $\varphi^y\colon G' \to Y$, $g \mapsto g \cdot y$. Let $\xi \in \mathfrak{g}'$. Then $Z_Y(\xi)$ is the unique vector field on Y lifting the push-forward under φ^y of the right-invariant vector field on G' associated to $-\xi$.*

Proof. Let $\theta \in \Gamma(G', \Theta_{G'})$ be the right-invariant vector field associated to $-\xi \in \mathfrak{g}' = T_1 G'$. Point-wise, it associates to a point $g \in G'$ the tangent vector $\mathrm{d}R_g(-\xi)$, where $R_g\colon G' \to G'$ is given by right-multiplication with g. Its push-forward under φ^y associates to $g \in G'$ the tangent vector $\mathrm{d}\varphi^y(\mathrm{d}R_g(-\xi)) = -\,\mathrm{d}\varphi^{g\cdot y}(\xi)$ to Y at $g \cdot y$. By Lemma 4.1, $Z_Y(\xi)$ is a vector field on Y lifting $\mathrm{d}\varphi^y(\theta) \in \Gamma(G', \varphi^{y,*}\Theta_Y)$. On the other hand, since G' acts transitively on Y, the morphism φ^y is surjective, hence the lift of $\mathrm{d}\varphi^y(\theta)$ is unique. □

Example 4.6. *Let $X \subseteq \mathbb{P}^k$ be the rational normal curve of degree k, i.e., the image of*

$$\mathbb{P}^1 \xrightarrow{|\mathcal{O}(k)|} \mathbb{P}^k, \qquad [x_0 : x_1] \mapsto \big[\tbinom{k}{i}\, x_0^{k-i} x_1^i \mid i = 0, \dots, k\big],$$

and let $Y := \hat{X} \setminus 0$ be the punctured affine cone over X in $V := \mathbb{C}^{k+1}$. The group $\mathrm{SL}(2)$ acts on on $V = H^0(\mathbb{P}^1, \mathcal{O}(k))^\vee = \mathrm{Sym}^k(\mathbb{C}^2)$, the k-th symmetric power of the standard $\mathrm{SL}(2)$-representation, and we extend this to an action of $G' := \mathrm{SL}(2) \times \mathbb{C}^*$ by letting the $\mathbb{C}^*$-factor act by scaling on V. The Lie algebra $\mathfrak{g}'$ is generated by $E_{12}, E_{21}, E_{11} - E_{22} \in \mathfrak{sl}(2)$ and the generator $\mathbf{e}$ of $\mathrm{Lie}(\mathbb{C}^*) \cong \mathbb{C}$. The induced vector fields on V are

$$Z_V(E_{12}) = -\sum_{i=1}^{k} i\, z_i \partial_{z_{i-1}}, \qquad Z_V(E_{21}) = -\sum_{i=1}^{k} (k-i+1)\, z_{i-1} \partial_{z_i},$$
$$Z_V(E_{11} - E_{22}) = -\sum_{i=0}^{k} (k-2i)\, z_i \partial_{z_i}, \qquad Z_V(\mathbf{e}) = -\sum_{i=0}^{k} z_i \partial_{z_i},$$

where $z_0, \dots, z_k$ denote the coordinates on $V = \mathbb{C}^{k+1}$. Note that the minus signs appear because we differentiate the contragredient *action on the coordinate ring of V.*

On the G'-invariant subset Y, these vector fields restrict to the vector fields $Z_Y(\xi)$. In local charts, these can be expressed as follows: We may cover Y by the two open subsets U_0 and U_1 given by the non-vanishing of $x_0^k \in V^\vee$ and $x_1^k \in V^\vee$, respectively. Identifying

$$U_0 \cong \mathbb{C}^* \times \mathbb{C}, \qquad \lambda \cdot (1, ks, \tbinom{k}{2} s^2, \dots, ks^{k-1}, s^k) \;\leftarrow\!\shortmid\; (\lambda, s),$$
$$U_1 \cong \mathbb{C}^* \times \mathbb{C}, \qquad \mu \cdot (t^k, kt^{k-1}, \tbinom{k}{2} t^{k-2}, \dots, kt, 1) \;\leftarrow\!\shortmid\; (\mu, t),$$

the vector fields induced from the G'-action on Y are:

$$\begin{aligned} Z_Y(E_{12})|_{U_0} &= -ks\lambda\partial_\lambda + s^2\partial_s, & Z_Y(E_{12})|_{U_1} &= -\partial_t, \\ Z_Y(E_{21})|_{U_0} &= -\partial_s, & Z_Y(E_{21})|_{U_1} &= -kt\mu\partial_\mu + t^2\partial_t, \\ Z_Y(E_{11}-E_{22})|_{U_0} &= -k\lambda\partial_\lambda + 2s\partial_s, & Z_Y(E_{11}-E_{22})|_{U_1} &= k\mu\partial_\mu - 2t\partial_t, \\ Z_Y(\mathbf{e})|_{U_0} &= -\lambda\partial_\lambda, & Z_Y(\mathbf{e})|_{U_1} &= -\mu\partial_\mu. \end{aligned}$$

Note that these local expressions coincide on the intersection $U_0 \cap U_1$ under the gluing $\mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\cong} \mathbb{C}^* \times \mathbb{C}^*$, $(\lambda, s) \mapsto (\lambda s^k, s^{-1}) = (\mu, t)$.

4.2 $\mathscr{D}$-modules from group actions

Using the vector fields defined in the previous section, we introduce the following $\mathscr{D}$-modules:

Definition 4.7. *Let G' be an algebraic group acting on a smooth variety Y and let $\beta\colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism. Then we define the left $\mathscr{D}_Y$-module*

$$\mathscr{N}_Y^\beta := \omega_Y^\vee \otimes_{\mathcal{O}_Y} \mathscr{D}_Y/(Z_Y(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_Y.$$

Example 4.8. *Let $G' = T = (\mathbb{C}^*)^d$ be a d-dimensional torus acting on itself. We identify Lie algebra homomorphisms $\beta\colon \mathbb{C}^d = \mathfrak{g}' \to \mathbb{C}$ with vectors $\beta \in \mathbb{C}^d$. Then*

$$\mathscr{N}_T^\beta = \omega_T^\vee \otimes_{\mathcal{O}_T} \mathscr{D}_T/(-t_i\partial_{t_i} - \beta_i \mid i = 1, \dots, d)\mathscr{D}_T \cong \mathscr{D}_T/\mathscr{D}_T(\partial_{t_i} t_i - \beta_i \mid i = 1, \dots, d).$$

This $\mathscr{D}_T$-module was called $\mathcal{O}_T^{-\beta}$ in [RS20].

Example 4.9. *We reconsider the action of $G' = \mathrm{SL}(2) \times \mathbb{C}^*$ on the punctured affine cone Y over the rational normal curve of degree k from Example 4.6 and use the notations from before. Every Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathbb{C}$ is given by $\beta|_{\mathfrak{sl}(2)} \equiv 0$ and $\beta(\mathbf{e}) = \beta_0 \in \mathbb{C}$. By the computations in Example 4.6, in the local chart $U_0 \cong \mathbb{C}^* \times \mathbb{C} \subseteq Y$, the $\mathscr{D}_Y$-module $\mathscr{N}_Y^\beta$ can be expressed as*

$$\begin{aligned} \mathscr{N}_Y^\beta|_{U_0} &\cong \omega_{U_0}^\vee \otimes_{\mathcal{O}_{U_0}} \mathscr{D}_{U_0}/(-ks\lambda\partial_\lambda + s^2\partial_s,\ -\partial_s,\ -k\lambda\partial_\lambda + 2s\partial_s,\ -\lambda\partial_\lambda - \beta_0)\mathscr{D}_{U_0} \\ &\cong \mathscr{D}_{U_0}/\mathscr{D}_{U_0}(ks\partial_\lambda\lambda - \partial_s s^2,\ \partial_s,\ k\partial_\lambda\lambda - 2\partial_s s,\ \partial_\lambda\lambda - \beta_0) \\ &\cong \mathscr{D}_{U_0}/\mathscr{D}_{U_0}(ks\lambda\partial_\lambda - s^2\partial_s + (k-2)s,\ \partial_s,\ k\lambda\partial_\lambda - 2s\partial_s + (k-2),\ \lambda\partial_\lambda + 1 - \beta_0) \\ &\cong \mathscr{D}_{U_0}/\mathscr{D}_{U_0}(\partial_s,\ \lambda\partial_\lambda + 1 - \beta_0,\ k(-1+\beta_0) + (k-2)) \\ &\cong \begin{cases} \mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(\partial_\lambda\lambda - \beta_0) \boxtimes \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}} \cdot \partial_s & \text{if } \beta_0 = 2/k, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and similarly for the other local chart U_1 of Y. In particular, for one specific value for $\beta(\mathbf{e})$, we obtain a non-zero $\mathscr{D}_Y$-module that will be of interest to us.

Note that in contrast, if we define the cyclic left *module*

$$\tilde{\mathscr{N}}_Y^\beta := \mathscr{D}_Y/\mathscr{D}_Y(Z_Y(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}'),$$

then, in this example, we get

$$\tilde{\mathscr{N}}_Y^{\beta}\big|_{U_0} \cong \mathscr{D}_{U_0}/\mathscr{D}_{U_0}(-ks\lambda\partial_\lambda + s^2\partial_s,\ -\partial_s,\ -k\lambda\partial_\lambda + 2s\partial_s,\ -\lambda\partial_\lambda - \beta_0)$$
$$\cong \mathscr{D}_{U_0}/\mathscr{D}_{U_0}(\partial_s,\ \lambda\partial_\lambda + \beta_0,\ k\beta_0) \cong \begin{cases} \mathcal{O}_{U_0} & \text{if } \beta_0 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 2$, we have $\mathscr{N}_Y^{(\beta_0=1)} = \tilde{\mathscr{N}}_Y^{(\beta_0=0)}$, but in general they do not agree with each other. In fact, one can show that although $\mathscr{N}_Y^{\beta}$ is locally *a cyclic left $\mathscr{D}_Y$-module, it does not admit a* global *description as a cyclic left $\mathscr{D}_Y$-module for $k \geq 3$.*

The main reason we wish to consider the $\mathscr{D}_Y$-module $\mathscr{N}_Y^{\beta}$ defined via the right-left-transformation of a cyclic right-module is the following behavior under equivariant closed embeddings:

Proposition 4.10. *Let G' be an algebraic group and let $i\colon Y_1 \hookrightarrow Y_2$ be a G'-equivariant closed embedding between smooth G'-varieties Y_1, Y_2. Then, for all Lie algebra homomorphisms $\beta\colon \mathfrak{g}' \to \mathbb{C}$, we have*

$$i_+\mathscr{N}_{Y_1}^{\beta} \cong \omega_{Y_2}^{\vee} \otimes_{\mathcal{O}_{Y_2}} \mathscr{D}_{Y_2}/\big(\mathcal{I} + (Z_{Y_2}(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\big)\mathscr{D}_{Y_2}$$

where $\mathcal{I} \subseteq \mathcal{O}_{Y_2}$ is the ideal sheaf of Y_1 in Y_2.

Proof. Since $i\colon Y_1 \hookrightarrow Y_2$ is a closed embedding, the functor i_* is exact and the transfer module $\mathscr{D}_{Y_1\to Y_2}$ is a flat $\mathscr{D}_{Y_1}$-module. Therefore, the direct image of $\mathscr{N}_{Y_1}^{\beta}$ under i is given by

$$i_+\mathscr{N}_{Y_1}^{\beta} \cong \omega_{Y_2}^{\vee} \otimes_{\mathcal{O}_{Y_2}} i_*\big(\mathscr{D}_{Y_1}/(Z_{Y_1}(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_{Y_1} \otimes_{\mathscr{D}_{Y_1}} \mathscr{D}_{Y_1\to Y_2}\big).$$

Hence, the claim is that

$$\mathscr{D}_{Y_1}/(Z_{Y_1}(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_{Y_1} \otimes_{\mathscr{D}_{Y_1}} \mathscr{D}_{Y_1\to Y_2}$$
$$\overset{!}{\cong} i^{-1}\big(\mathscr{D}_{Y_2}/((Z_{Y_2}(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_{Y_2} + \mathcal{I}\mathscr{D}_{Y_2})\big).$$

as right $i^{-1}\mathscr{D}_{Y_2}$-modules. Note that $\mathscr{D}_{Y_1\to Y_2} \cong i^{-1}(\mathscr{D}_{Y_2}/\mathcal{I}\mathscr{D}_{Y_2})$ as right $i^{-1}\mathscr{D}_{Y_2}$-modules, since i is a closed embedding. Under the left $\mathscr{D}_{Y_1}$-module structure on $\mathscr{D}_{Y_1\to Y_2}$, vector fields on Y_1 act via the push-forward homomorphism

$$\mathrm{d}i\colon \Theta_{Y_1} \to i^*\Theta_{Y_2} = \mathcal{O}_{Y_1} \otimes_{i^{-1}\mathcal{O}_{Y_2}} i^{-1}\Theta_{Y_2} \cong i^{-1}(\mathcal{O}_{Y_2}/\mathcal{I} \otimes_{\mathcal{O}_{Y_2}} \Theta_{Y_2}).$$

We note that the push-forward of the vector field $Z_{Y_1}(\xi)$ on Y_1 agrees with the restriction of the vector field $Z_{Y_2}(\xi)$ on Y_2 to Y_1, i.e., $\mathrm{d}i(Z_{Y_1}(\xi)) = 1 \otimes Z_{Y_2}(\xi)$. Indeed, this follows from the construction of $Z_{Y_1}(\xi)$ and $Z_{Y_2}(\xi)$, using the commutativity of

$$\begin{array}{ccc} G' \times Y_1 & \xrightarrow{\varphi_1} & Y_1 \\ \big\downarrow{\scriptstyle \mathrm{id}_{G'} \times i} & & \big\downarrow{\scriptstyle i} \\ G' \times Y_2 & \xrightarrow{\varphi_2} & Y_2, \end{array}$$

where φ_1, φ_2 are the morphisms given by the G'-actions.

This shows that the vector field $Z_{Y_1}(\xi) \in \mathrm{Der}(\mathcal{O}_{Y_1})$ acts on the right $i^{-1}\mathscr{D}_{Y_2}$-module $\mathscr{D}_{Y_1\to Y_2} \cong i^{-1}(\mathscr{D}_{Y_2}/\mathcal{I}\mathscr{D}_{Y_2})$ by left-multiplication with $Z_{Y_2}(\xi)$. This implies the claimed description of $\mathscr{D}_{Y_1}/(Z_{Y_1}(\xi) - \beta(\xi) \mid \xi)\mathscr{D}_{Y_1} \otimes_{\mathscr{D}_{Y_1}} \mathscr{D}_{Y_1\to Y_2}$ as a cyclic right $i^{-1}\mathscr{D}_{Y_2}$-module, concluding the proof. □

The $\mathscr{D}$-modules in Proposition 4.10 look similar to the β-twistedly equivariant $\mathscr{D}$-modules considered in [Hot98, II.2], yet they are different: Instead of considering a cyclic *left* module obtained by quotienting out a G'-stable ideal and the vector fields induced by the group action (twisted with β), we instead consider the *right* module constructed in the same way and apply a right-left transformation to obtain a left $\mathscr{D}$-module. The behavior under direct images of closed embeddings in Proposition 4.10 is the reason why for our purposes we work with the definition via right modules in Definition 4.7.

We next consider the situation where Y is an orbit of a rational representation ρ of our group G' in a given vector space V. Recall from our basic Definition 1.1 that under this hypothesis, we can define, for any Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathbb{C}$, the $\mathscr{D}_V$-module $\hat{\tau}(\rho, \overline{Y}, \beta)$ (as well as its Fourier-Laplace transform $\tau(\rho, \overline{Y}, \beta)$ which was called tautological system in Definition 1.1). The next result tells us about a technically easy but imporant relation of this $\hat{\tau}(\rho, \overline{Y}, \beta)$ to the $\mathscr{D}_Y$-module $\mathscr{N}_Y^\beta$ considered above.

Corollary 4.11. *Let $\rho\colon G' \to \mathrm{GL}(V)$ be a finite-dimensional rational representation of an algebraic group and let $\beta\colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism. Let $Y \subseteq V$ be a G'-orbit, let $\overline{Y}$ be its closure and let $\partial Y := \overline{Y} \setminus Y$. Then*

$$j^+\hat{\tau}(\rho, \overline{Y}, \beta) \cong i_+\mathscr{N}_Y^\beta,$$

where $Y = \overline{Y} \setminus \partial Y \overset{i}{\hookrightarrow} V \setminus \partial Y \overset{j}{\hookrightarrow} V$.

In particular, if $\hat{\tau}(\rho, \overline{Y}, \beta)$ is localized at ∂Y (meaning $j_+j^+\hat{\tau}(\rho, \overline{Y}, \beta) \cong \hat{\tau}(\rho, \overline{Y}, \beta)$), then it is the direct image of $\mathscr{N}_Y^\beta$ under the locally closed embedding $Y \hookrightarrow V$.

Proof. We apply Proposition 4.10 to $Y_1 := Y$, $Y_2 := V \setminus \partial Y$ and the closed embedding $i\colon Y_1 \hookrightarrow Y_2$ to see that

$$i_+\mathscr{N}_Y^\beta \cong \omega_V^\vee \otimes_{\mathcal{O}_V} \mathscr{D}_V / \big(I + \{Z_V(\xi) - \beta(\xi)\}\big)\mathscr{D}_V,$$

since V is affine. Choosing coordinates $x_1, \ldots, x_n$ on V, we may express the vector field $Z_V(\xi)$ as the derivation $-\sum_{i,j=1}^n \mathrm{d}\rho(\xi)_{ji}\, x_i\partial_{x_j}$ by Lemma 4.3. The right-left transformation $\omega_V^\vee \otimes_{\mathcal{O}_V} (\cdot)$ is then explicitly given by transposing operators:

$$i_+\mathscr{N}_Y^\beta \cong \mathscr{D}_V / \mathscr{D}_V\big(I + \{Z_V(\xi)^T - \beta(\xi)\}\big).$$

An explicit computation of the transposed vector fields yields:

$$Z_V(\xi)^T = \sum_{i,j=1}^n \mathrm{d}\rho(\xi)_{ji}\, \partial_{x_j} x_i = \sum_{i,j=1}^n \mathrm{d}\rho(\xi)_{ji}\, x_i \partial_{x_j} + \sum_{i=1}^n \mathrm{d}\rho(\xi)_{ii} = -Z_Y(\xi) + \mathrm{trace}(\mathrm{d}\rho(\xi)),$$

hence $i_+\mathscr{N}_Y^\beta \cong j^+\hat{\tau}(\rho, \overline{Y}, \beta)$. ∎

Example 4.12 (GKZ-systems). *Consider a torus representation $\rho\colon (\mathbb{C}^*)^n \to \mathrm{GL}(n, \mathbb{C})$ given by $\rho(t_1, \ldots, t_d) = \mathrm{diag}(t^{\alpha_1}, \ldots, t^{\alpha_n})$ with $\alpha_i \in \mathbb{Z}^d$. Let $\overline{Y} \subseteq \mathbb{C}^n$ be the orbit closure of the point $(1, \ldots, 1) \in \mathbb{C}^n$; this is a (not necessarily normal) affine toric variety. The $\mathscr{D}_{\mathbb{C}^n}$-module $\hat{\tau}(\rho, \overline{Y}, \beta)$ is the Fourier–Laplace transform of the GKZ-system $\mathcal{M}_A(-\beta)$, where A is the $d \times n$-matrix whose i-th column is α_i and $\beta\colon \mathrm{Lie}\big((\mathbb{C}^*)^d\big) = \mathbb{Z}^d \to \mathbb{C}$ is identified with the vector $(\beta(e_i))_{i=1,\ldots,d} \in \mathbb{C}^d$.*

In this case, Corollary 4.11 applied to $Y = \overline{Y} \cap (\mathbb{C}^*)^n$ says that $\mathscr{M}_A(-\beta)$ is the direct image of $\mathscr{N}^{\beta}_{(\mathbb{C}^*)^d} = \mathcal{O}^{-\beta}_{(\mathbb{C}^*)^d}$ under the locally closed embedding $(\mathbb{C}^*)^d \cong Y \hookrightarrow \mathbb{C}^n$, whenever $\mathscr{M}_A(-\beta)$ is localized at the intersection of Y with the union of coordinate hyperplanes of $\mathbb{C}^n$. This was observed in [SW09a], where an explicit combinatorial characterization of the localization property in terms of A and β was proved using Euler–Koszul complexes.

Example 4.13. *Reconsider from Example 4.6 the punctured affine cone Y over the rational normal curve of degree k. This may be identified with the complement of the zero section in the line bundle $L = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(k)) \to \mathbb{P}^1$. The calculation in Example 4.9 shows that $\mathscr{N}^{\beta}_Y = \mathcal{O}^{-\beta(\mathbf{e})}_{L^*}$ if $\beta(\mathbf{e}) = 2/k$ and $\mathscr{N}^{\beta}_Y = 0$ otherwise. Corollary 4.11 shows that the restriction of the FL-transformed tautological system*

$$\begin{aligned}\hat{\tau}(\rho, \overline{Y}, \beta) = \mathscr{D}_V/\mathscr{D}_V \cdot \Bigg\{ \binom{k}{i_2}\binom{k}{j_2} z_{i_1} z_{j_1} - \binom{k}{i_1}\binom{k}{j_1} z_{i_2} z_{j_2} \mid i_1 + j_1 = i_2 + j_2 \Bigg\} \\ \cup \Bigg\{ -\sum_{i=1}^{k} i\, z_i \partial_{z_{i-1}},\ -\sum_{i=1}^{k} (k-i+1)\, z_{i-1}\partial_{z_i}, \\ -\sum_{i=0}^{k} (k-2i)\, z_i \partial_{z_i},\ -\sum_{i=0}^{k} z_i\partial_{z_i} - (k+1) + \beta(\mathbf{e}) \Bigg\}\end{aligned}$$

to the complement of the origin in V is

$$\hat{\tau}(\rho, \overline{Y}, \beta)\big|_{V \setminus 0} = \begin{cases} i_+ \mathcal{O}^{-\beta(\mathbf{e})}_{L^*} & \text{if } \beta(\mathbf{e}) = 2/k, \\ 0 & \text{otherwise.} \end{cases}$$

5 Non-vanishing criteria

The definition of a tautological system does not always describe a non-zero $\mathscr{D}$-module. In fact, for tautological systems arising from projective homogeneous spaces, this fails in a striking way, as we will see in Section 7: in that setup, tautological systems $\tau(\rho, \overline{Y}, \beta)$ will only be non-zero for very particular representations ρ and specific choices of β. In those cases however, tautological systems are particularly interesting.

The non-vanishing of tautological systems is by Corollary 4.11 closely tied to the non-vanishing of the $\mathscr{D}$-modules $\mathscr{N}^{\beta}_Y$. The aim of this section is to study criteria for $\mathscr{N}^{\beta}_Y$ to be (non-)zero.

5.1 $\mathcal{A}_Y$-modules

We fix notations for the entirety of Section 5. Let G' be a connected linear algebraic group acting on a smooth connected algebraic variety Y. Denote by $\mathfrak{g}'$ the Lie algebra of G' and by $\mathcal{U}(\mathfrak{g}')$ its universal enveloping algebra. Every element ξ of $\mathfrak{g}'$ induces a vector field $Z(\xi) \in \Gamma(Y, \Theta_Y)$ by Lemma 4.1, and this map extends to a homomorphism of $\mathcal{O}_Y$-modules

$$Z \colon \mathcal{O}_Y \otimes_{\mathbb{C}} \mathfrak{g}' \to \Theta_Y$$

via $Z(f \otimes \xi) = fZ(\xi)$ for $f \in \mathcal{O}_Y$, $\xi \in \mathfrak{g}'$.

Definition 5.1. *Given the G'-variety Y, we define*

$$\mathcal{A}_Y := \mathcal{O}_Y \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g}'),$$

which has the structure of an associative $\mathbb{C}$-algebra with multiplication given by

$$(f_1 \otimes \xi_1) \cdot (f_2 \otimes \xi_2) = f_1 f_2 \otimes \xi_1 \xi_2 + f_1 Z(\xi_1)(f_2) \otimes \xi_2.$$

The $\mathcal{O}_Y$-module homomorphism Z extends to a homomorphism of associative $\mathbb{C}$-algebras

$$\tilde{Z} \colon \mathcal{A}_Y \to \mathscr{D}_Y.$$

For any left $\mathcal{A}_Y$-module $\mathcal{M}$, we may consider the left $\mathscr{D}_Y$-module obtained by scalar extension

$$\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathcal{M}.$$

On the other hand, note that the homomorphism $\tilde{Z}$ induces a forgetful functor from the category of left $\mathscr{D}_Y$-modules to the category of left $\mathcal{A}_Y$-modules.

The associative algebra $\mathcal{A}_Y$ is the universal enveloping algebra of the Lie algebroid $(\mathcal{O}_Y \otimes_{\mathbb{C}} \mathfrak{g}', Z)$ on Y, see [BB93, 1.8.4.Example]. This is the reason why, in many ways, modules over $\mathcal{A}_Y$ behave similarly to modules over the algebra $\mathscr{D}_Y$ (which can be viewed as the universal enveloping algebra of the Lie algebroid Θ_Y). For example, the tensor product of two left $\mathcal{A}_Y$-modules over $\mathcal{O}_Y$ is again naturally a left $\mathcal{A}_Y$-module, while the tensor product of a left and a right $\mathcal{A}_Y$-module over $\mathcal{O}_Y$ naturally becomes a right $\mathcal{A}_Y$-module. Applying basic results on modules over universal enveloping algebras of Lie algebroids [CMNM05, Appendice] to $Z \colon \mathcal{A}_Y \to \mathscr{D}_Y$, we obtain the following elementary properties:

Lemma 5.2 ([CMNM05, Théorème A.6 and Corollaire A.2])**.** *Let $\mathcal{M}$ be a left $\mathcal{A}_Y$-module. Let $\mathcal{N}$ (resp. $\mathcal{N}'$) be a left (resp. right) $\mathscr{D}_Y$-module. Then there are natural isomorphisms*

(i) $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}) \cong (\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{N}$ *as left $\mathscr{D}_Y$-modules,*

(ii) $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}') \otimes_{\mathcal{A}_Y} \mathscr{D}_Y \cong (\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{N}'$ *as right $\mathscr{D}_Y$-modules.*

Here, on the left hand sides, $\mathcal{N}$ and $\mathcal{N}'$ are considered as $\mathcal{A}_Y$-modules via $\tilde{Z} \colon \mathcal{A}_Y \to \mathscr{D}_Y$.

Equivariant line bundles as $\mathcal{A}_Y$-modules: If $E \to Y$ is a G'-equivariant line bundle and we denote by $\mathscr{E}$ its sheaf of sections, then for every open subset $U \subseteq Y$, the Lie algebra $\mathfrak{g}'$ acts on $\Gamma(U, \mathscr{E})$. This makes $\mathscr{E}$ a left $\mathcal{A}_Y$-module. We will be particularly interested in the left $\mathscr{D}_Y$-module $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$ arising from this.

Remark 5.3. *If $U \subseteq Y$ is an open subset not invariant under G', then G' does not act on U. Yet, we still get $Z \colon \mathcal{O}_U \otimes_{\mathbb{C}} \mathfrak{g}' \to \Theta_U$, allowing us to define $\mathcal{A}_U$. While $\mathscr{E}|_U$ is not G'-equivariant, it still is a left $\mathcal{A}_U$-module, and we may consider $\mathscr{D}_U \otimes_{\mathcal{A}_U} \mathscr{E}|_U$. This suggests a generalized viewpoint, where we replace the G'-action on Y by a $\mathfrak{g}'$-action on $\mathcal{O}_Y$, and replace G'-equivariant line bundles with line bundles carrying a left $\mathcal{A}_Y$-module structure.*

5.2 Transitive group actions

Consider now the case that G' acts transitively on Y. Then the $\mathcal{O}_Y$-module homomorphism $Z\colon \mathcal{O}_Y \otimes_{\mathbb{C}} \mathfrak{g}' \to \Theta_Y$ is surjective, hence the same is true for $\tilde{Z}\colon \mathcal{A}_Y \to \mathscr{D}_Y$, so

$$\mathscr{D}_Y \cong \mathcal{A}_Y / \ker \tilde{Z}.$$

We observe that the kernel of $\tilde{Z}$ (which is a two-sided ideal in $\mathcal{A}_Y$) is generated as a left ideal in $\mathcal{A}_Y$ by the kernel of Z:

Lemma 5.4. *If G' acts transitively on Y, then*

$$\ker\big(\tilde{Z}\colon \mathcal{A}_Y \to \mathscr{D}_Y\big) = \mathcal{A}_Y \cdot \ker\big(Z\colon \mathcal{O}_Y \otimes \mathfrak{g}' \to \Theta_Y\big).$$

Proof. We check the claim locally. For this, let $p \in Y$ be an arbitrary point and let $U \subseteq Y$ be an open neighborhood of p admitting a local coordinate system $(x_1, \ldots, x_n)$, so that $\Theta_U = \bigoplus_{i=1}^n \mathcal{O}_U \partial_{x_i}$. We claim that by further shrinking the open set U, we may choose an appropriate $\mathcal{O}_U$-basis $\theta_1, \ldots, \theta_m$ of the free $\mathcal{O}_U$-module $\mathcal{O}_U \otimes \mathfrak{g}'$ such that the surjective homomorphism of $\mathcal{O}_U$-modules

$$Z|_U \colon \mathcal{O}_U \otimes \mathfrak{g}' \to \Theta_U$$

is given by

$$\theta_i \mapsto \begin{cases} \partial_{x_i} & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

Indeed, $Z|_U$ is a surjective homomorphism of free $\mathcal{O}_U$-modules of finite rank and we may represent it by an $n \times m$-matrix A (with $m \geq n$) by choosing *any* $\mathcal{O}_U$-basis of $\mathcal{O}_U \otimes \mathfrak{g}'$. By surjectivity of Z, some $n \times n$-minor of A does not vanish at the point p. After permuting the chosen $\mathcal{O}_U$-basis of $\mathcal{O}_U \otimes \mathfrak{g}'$, we may assume that the non-vanishing set $V \subseteq U$ of the minor given by the first n columns is an open neighborhood of p. Writing

$$A = (A_1 \mid A_2) \qquad \text{with } A_1 \in \mathrm{Mat}(n \times n, \mathcal{O}_U), A_2 \in \mathrm{Mat}(n \times (m-n), \mathcal{O}_U),$$

we have $A_1 \in \mathrm{GL}(n, \mathcal{O}_V)$. Changing the $\mathcal{O}_U$-basis on $(\mathcal{O}_U \otimes \mathfrak{g}')|_V = \mathcal{O}_V \otimes \mathfrak{g}'$ corresponds to right-multiplying A with an element of $\mathrm{GL}(m, \mathcal{O}_V)$. Then

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix} \cdot \begin{pmatrix} A_1^{-1} & -A_1^{-1}A_2 \\ 0 & \mathrm{Id}_{m-n} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_n & 0 \end{pmatrix}$$

shows that a choice of $\theta_1, \ldots, \theta_m$ as desired exists.

Now, every section of $\mathcal{A}_U$ can be expressed as a sum of elements of the form $f\theta_1^{a_1}\theta_2^{a_2}\ldots\theta_m^{a_m}$ with $f \in \mathcal{O}_U$, $a_1, \ldots, a_m \in \mathbb{N}$, each of which gets mapped under $\tilde{Z}$ to

$$f\theta_1^{a_1}\theta_2^{a_2}\ldots\theta_m^{a_m} \mapsto \begin{cases} f\partial_{x_1}^{a_1}\partial_{x_2}^{a_2}\ldots\partial_{x_n}^{a_n} & \text{if } a_{n+1} = \cdots = a_m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we can see that every section of $\mathcal{A}_U$ getting mapped to zero under $\tilde{Z}|_U$ is an element of

$$\mathcal{A}_U \cdot \{\theta_{n+1}, \ldots, \theta_m\} = \mathcal{A}_U \cdot \ker(Z|_U). \qquad \square$$

Lemma 5.4 is in fact a special case of a more general fact about Lie algebroids: If $\varphi\colon \mathscr{F}_1 \twoheadrightarrow \mathscr{F}_2$ is a surjective homomorphism of two locally free Lie algebroids of finite rank on the same variety Y, then the kernel of the induced homomorphism of universal enveloping algebras $\tilde{\varphi}\colon \mathcal{U}(\mathscr{F}_1) \twoheadrightarrow \mathcal{U}(\mathscr{F}_2)$ is generated by $\ker\varphi$ as a left $\mathcal{U}(\mathscr{F}_1)$-ideal. A similar proof to the above carries over.

5.3 Torsion line bundles

In this section, we see that equivariant torsion line bundles give rise to non-zero $\mathscr{D}$-modules, when the group action on the variety is transitive.

Proposition 5.5. *Assume G' acts transitively on Y. Let $\mathscr{E}$ be a G'-equivariant line bundle on Y. Then the following are equivalent:*

(i) $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E} \neq 0$,

(ii) $\mathscr{E} \to \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$ *is an isomorphism of left $\mathcal{A}_Y$-modules,*

(iii) $\mathscr{E}^{\otimes k} \to \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}^{\otimes k}$ *is an isomorphism of left $\mathcal{A}_Y$-modules for some $k \in \mathbb{Z}_{>0}$,*

(iv) $\mathscr{E}^{\otimes k} \to \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}^{\otimes k}$ *is an isomorphism of left $\mathcal{A}_Y$-modules for all $k \in \mathbb{Z}_{>0}$.*

Proof. First, we show that the first two items are equivalent: By transitivity of the group action, $\tilde{Z}\colon \mathcal{A}_Y \to \mathscr{D}_Y$ is surjective, hence the natural homomorphism of $\mathcal{A}_Y$-modules $\mathscr{E} \to \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$ is also surjective. Since the support of $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$ is a G'-equivariant subset of Y, by transitivity we must either have $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E} = 0$ or $\operatorname{Supp}(D_Y \otimes_{\mathcal{A}_Y} \mathscr{E}) = Y$. Since $\mathscr{E}$ is a line bundle on Y, the only quotient of the $\mathcal{O}_Y$-module $\mathscr{E}$ with support equal to Y is $\mathscr{E}$ itself. This shows (i) $\Leftrightarrow$ (ii).

The implication (iv) $\Rightarrow$ (iii) is trivial. To show the implication (ii) $\Rightarrow$ (iv), we assume for contradiction that there is some $k \geq 2$ for which the claim does not hold and assume k to be minimal. Applying Lemma 5.2.(i) to $\mathcal{M} := \mathscr{E}^{\otimes(k-1)}$ and $\mathcal{N} := \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$ gives:

$$\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}^{\otimes k} \cong (\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}^{\otimes(k-1)}) \otimes_{\mathcal{O}_Y} (\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}) \cong \mathscr{E}^{\otimes(k-1)} \otimes_{\mathcal{O}_Y} \mathscr{E} = \mathscr{E}^{\otimes k}$$

as left $\mathcal{A}_Y$-modules (by minimality of k). This is a contradiction to the choice of k.

It remains to show the implication (iii) $\Rightarrow$ (ii). Consider the two-sided ideal

$$\mathcal{I} := \ker\Big(\tilde{Z}\colon \mathcal{A}_Y \twoheadrightarrow \mathscr{D}_Y\Big)$$

of $\mathcal{A}_Y$. Note that the natural homomorphism $\mathscr{E} \to \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$ of left $\mathcal{A}_Y$-modules is an isomorphism if and only if $\mathcal{I}$ annihilates $\mathscr{E}$. Using Lemma 5.4, it suffices to prove that $\mathscr{E}$ is annihilated by $\ker(Z)$. Let $s \in \Gamma(U, \mathscr{E})$ be a non-zero local section of $\mathscr{E}$ and let $P \in \Gamma(U, \ker(Z)) \subseteq \mathcal{O}_U \otimes \mathfrak{g}'$. By assumption (ii), we have $\mathscr{E}^{\otimes k} \cong \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}^{\otimes k}$ as left $\mathcal{A}_Y$-modules for some $k \geq 1$, meaning that $\mathscr{E}^{\otimes k}$ is annihilated by $\mathcal{I}$. In particular, the local section $s^k \in \Gamma(U, \mathscr{E}^{\otimes k})$ is annihilated by P, so $P \cdot s^k = 0$. On the other hand, we have

$$P \cdot s^k = ks^{k-1}(P \cdot s).$$

Since Y is an irreducible variety, we deduce that $P \cdot s = 0$. This concludes the proof. $\square$

Corollary 5.6. *Assume G' acts transitively on Y. Let $\mathscr{E}$ be a torsion element of the equivariant Picard group $\mathrm{Pic}^{G'}(Y)$, i.e., $\mathscr{E}^{\otimes k} \cong \mathcal{O}_Y$ as equivariant line bundles for some $k \in \mathbb{Z}_{>0}$. Then the natural homomorphism*

$$\mathscr{E} \to \mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}$$

of left $\mathcal{A}_Y$-modules is an isomorphism.

Proof. By Proposition 5.5, it suffices to consider the case that $\mathscr{E} = \mathcal{O}_Y$ as equivariant line bundles. The Lie algebra $\mathfrak{g}'$ acts trivially on the 1-section of $\mathcal{O}_Y$, hence

$$\mathscr{E} \cong \mathcal{A}_Y / \mathcal{A}_Y(\xi \mid \xi \in \mathfrak{g}')$$

as left $\mathcal{A}_Y$-modules. Tensoring with $\mathscr{D}_Y$ over $\mathcal{A}_Y$ gives

$$\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E} \cong \mathscr{D}_Y / \mathscr{D}_Y(Z(\xi) \mid \xi \in \mathfrak{g}') = \mathscr{D}_Y / \mathscr{D}_Y \Theta_Y \cong \mathcal{O}_Y \cong \mathscr{E}.$$

Here, we use that the vector fields $Z(\xi)$ for $\xi \in \mathfrak{g}'$ generate the tangent bundle Θ_Y, as the action of G' on Y is transitive. □

Remark 5.7. *Note that the proof of Proposition 5.5 works more generally for any line bundle $\mathscr{E}$ with a left $\mathcal{A}_Y$-module structure, not necessarily arising from G'-equivariant structure on $\mathscr{E}$.*

Corollary 5.6 shows in particular that $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E} \neq 0$ for G'-equivariant torsion line bundles. Under certain assumptions on Y, the converse is also true:

Proposition 5.8. *Let G' act transitively on Y and assume that there is an open cover $Y = \bigcup_{i \in I} U_i$ such that for each $i \in I$, there is a subgroup N_i of G acting freely and transitively on U_i. Then*

$$\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E} \neq 0 \quad \Leftrightarrow \quad \mathscr{E} \cong \mathcal{O}_Y \text{ as } G'\text{-equivariant line bundles.}$$

We remark that under the assumptions on Y in Proposition 5.8, there are no non-trivial equivariant torsion line bundles on Y.

Proof. One implication is given by Corollary 5.6. For the converse, we assume that $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E} \neq 0$. Since $\mathscr{E}$ is G'-equivariant, the support of this $\mathscr{D}_Y$-module is a non-empty G'-invariant subset of Y, hence (by transitivity of the group action)

$$\mathrm{Supp}(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E}) = Y. \tag{8}$$

In particular, the restriction to U_i is a non-zero $\mathscr{D}_{U_i}$-module for each $i \in I$.

Denote by E^* the complement of the zero section of $E = \mathrm{Tot}(\mathscr{E}) \xrightarrow{\pi} Y$. For $i \in I$, the choice of a point $w_i \in E^*$ such that $p_i := \pi(w_i) \in U_i$ determines a local section $s_i \in \Gamma(U_i, \mathscr{E})$ geometrically given by

$$\begin{array}{rccccc} s_i\colon & U_i & \xrightarrow{\cong} & N_i & \to & \pi^{-1}(U_i) \\ & g \cdot p_i & \leftarrow\!\shortmid & g & \mapsto & g \cdot w_i. \end{array}$$

Here, we use that $N_i \to U_i$, $g \mapsto g \cdot p_i$ is an isomorphism, since N_i is assumed to act freely and transitively on U_i. Since E^* is invariant under the action of G' on E, the local section s_i does not vanish on U_i, hence $\mathscr{E}|_{U_i} = \mathcal{O}_{U_i} s_i$.

By definition, s_i is an N_i-invariant section of $\mathscr{E}|_{U_i}$, hence $\xi \cdot s_i = 0$ holds for all $\xi \in \mathrm{Lie}(N_i) =: \mathfrak{n}_i$. Since N_i acts transitively on U_i, the $\mathcal{O}_{U_i}$-module homomorphism $\mathcal{O}_{U_i} \otimes \mathfrak{n}_i \to \Theta_{U_i}$ is surjective, so from the above we may deduce that Θ_{U_i} annihilates the cyclic $\mathscr{D}_{U_i}$-module $(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E})|_{U_i}$ generated by $1 \otimes s_i$.

Take any $\xi \in \mathfrak{g}'$. Then $\xi \cdot s_i = f \cdot s_i$ for some $f \in \Gamma(U_i, \mathcal{O}_{U_i})$. But then f annihilates $(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \mathscr{E})|_{U_i}$, as $f \cdot (1 \otimes s_i) = 1 \otimes (\xi \cdot s_i) = Z_Y(\xi)|_{U_i} \cdot (1 \otimes s_i) = 0$. Because of (8), this forces

$$\xi \cdot s_i = 0 \qquad \text{for all } \xi \in \mathfrak{g}'.$$

On $U_{ij} := U_i \cap U_j$ for $i, j \in I$, the non-vanishing local sections s_i and s_j only differ by an invertible function:

$$s_i|_{U_{ij}} = \alpha_{ij} s_j|_{U_{ij}}, \qquad \alpha_{ij} \in \Gamma(U_{ij}, \mathcal{O}_{U_{ij}}^\times).$$

Since

$$0 = \xi \cdot s_i|_{U_{ij}} = \xi \cdot (\alpha_{ij} s_j|_{U_{ij}}) = Z_Y(\xi)|_{U_{ij}}(\alpha_{ij}) s_j|_{U_{ij}} + \alpha_{ij}(\xi \cdot s_j|_{U_{ij}}) = Z_Y(\xi)|_{U_{ij}}(\alpha_{ij}) s_j|_{U_{ij}},$$

we see that $\alpha_{ij} = 0$ is annihilated by all vector fields on U_{ij} (since Θ_Y is globally generated by the image of $Z\colon \mathcal{O}_Y \otimes \mathfrak{g}' \to \Theta_Y$). Therefore, $\alpha_{ij} \in \mathbb{C}^*$.

We may now fix some $k \in I$ and define non-vanishing sections

$$\tilde{s}_i := \alpha_{ki}^{-1} s_i \in \Gamma(U_i, \mathscr{E}) \qquad \text{for all } i \in I$$

which are still annihilated by the action of $\mathfrak{g}'$. Then $\tilde{s}_i$ and $\tilde{s}_j$ agree on U_{ij} for all $i, j \in I$, so they glue to a global non-vanishing section $\tilde{s} \in \Gamma(Y, \mathscr{E})$ annihilated by $\mathfrak{g}'$. This section defines an isomorphism $\mathscr{E} \cong \mathcal{O}_Y$ of left $\mathcal{A}_Y$-modules and hence of G'-equivariant line bundles. □

5.4 Equivariant line bundles from characters

We now return to the general situation, where we do not assume G' to act transitively on Y.

Definition 5.9. *Let $\rho\colon G' \to \mathbb{C}^*$ be a character. We define a G'-equivariant line bundle $\mathcal{O}_Y\{\rho\}$ on Y by equipping the trivial line bundle $\mathcal{O}_Y$ with a G'-equivariant structure such that the action of G' on $\mathrm{Tot}(\mathcal{O}_Y\{\rho\}) = \mathbb{C} \times Y$ is given by $g \cdot (\lambda, y) = (\rho(g)\lambda,\ g \cdot y)$.*

For any G'-equivariant line bundle $\mathscr{E}$, consider the G'-equivariant line bundle

$$\mathscr{E}\{\rho\} := \mathscr{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\rho\},$$

which has the same underlying $\mathcal{O}_Y$-module, but a different equivariant structure.

Note that

$$\mathrm{Hom}(G', \mathbb{C}^*) \to \mathrm{Pic}^{G'}(Y), \qquad \rho \mapsto \mathcal{O}_Y\{\rho\}$$

is a group homomorphism, i.e.,

(i) $\mathscr{O}_Y\{\rho_1\rho_2\} \cong \mathscr{O}_Y\{\rho_1\} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y\{\rho_2\}$,

(ii) $\mathscr{O}_Y\{\rho^{-1}\} \cong \mathscr{O}_Y\{\rho\}^\vee$,

(iii) $\mathscr{O}_Y\{1\} \cong \mathscr{O}_Y$.

Remark 5.10. *For a given equivariant line bundle $\mathscr{E}$ whose G'-action is given on $E := \operatorname{Tot}(\mathscr{E})$ as $\varphi\colon G' \times E \to E$, the G'-action on $\operatorname{Tot}(\mathscr{E}\{\rho\}) = E$ is given by*

$$G' \times E \to E, \qquad (g, e) \mapsto \mu\big(\rho(g), \varphi(g, e)\big),$$

where $\mu\colon \mathbb{C}^* \times E \to E$ denotes the natural $\mathbb{C}^*$-action on E by scaling fibers.

We have seen before that every G'-equivariant line bundle on Y is a left $\mathcal{A}_Y$-module, so for every character $\rho\colon G' \to \mathbb{C}^*$, we get the left $\mathcal{A}_Y$-module

$$\mathscr{O}_Y\{\rho\} \cong \mathcal{A}_Y/\mathcal{A}_Y(\xi - \mathrm{d}\rho(\xi) \mid \xi \in \mathfrak{g}'),$$

where $\mathrm{d}\rho\colon \mathfrak{g}' \to \mathbb{C}$ is the Lie algebra homomorphism induced by ρ.

Note that the left $\mathcal{A}_Y$-module structure on a G'-equivariant line bundle $\mathscr{E}$ results just from the infinitesimal action of $\mathfrak{g}'$ on local sections of $\mathscr{E}$. Therefore, it is also natural to more generally define the left $\mathcal{A}_Y$-module

$$\mathscr{O}_Y\{\beta\} := \mathcal{A}_Y/\mathcal{A}_Y(\xi - \beta(\xi) \mid \xi \in \mathfrak{g}')$$

for any Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathbb{C}$. This may in general not be a G'-equivariant line bundle, yet it still is a left $\mathcal{A}_Y$-module. Note that $\mathscr{O}_Y\{\rho\} \cong \mathscr{O}_Y\{\mathrm{d}\rho\}$ as $\mathcal{A}_Y$-modules for $\rho\colon G' \to \mathbb{C}^*$ inducing $\mathrm{d}\rho\colon \mathfrak{g}' \to \mathbb{C}$. Similarly to before, given a left $\mathcal{A}_Y$-module $\mathscr{E}$, we denote by $\mathscr{E}\{\beta\}$ the left $\mathcal{A}_Y$-module $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y\{\beta\}$.

5.5 Left-right transforms on $\mathcal{A}_Y$-modules

The line bundle $\alpha_Y := \bigwedge^{\dim G'}(\mathscr{O}_Y \otimes_{\mathbb{C}} \mathfrak{g}')^\vee$ on Y has the structure of a right $\mathcal{A}_Y$-module which is given by the negated Lie derivative: A Lie algebra element $\xi \in \mathfrak{g}'$ acts on an alternating form ω by mapping it to the alternating form $\omega \cdot \xi$ given by

$$(\omega \cdot \xi)(\theta_1, \ldots, \theta_m) = -Z(\xi)(\omega(\theta_1, \ldots, \theta_m)) + \sum_{i=1}^{m} \omega(\theta_1, \ldots, [\xi, \theta_i], \ldots, \theta_m)$$

for any $\theta_1, \ldots, \theta_m \in \mathscr{O}_Y \otimes_{\mathbb{C}} \mathfrak{g}'$. This defines transformations between left and right $\mathcal{A}_Y$-modules giving rise to an equivalence of categories

$$\begin{array}{ccc} \operatorname{Mod}(\mathcal{A}_Y) & \xrightarrow{\quad\cong\quad} & \operatorname{Mod}(\mathcal{A}_Y^{\mathrm{op}}), \\ \mathcal{M} & \longmapsto & \alpha_Y \otimes_{\mathscr{O}_Y} \mathcal{M}, \\ \alpha_Y^\vee \otimes_{\mathscr{O}_Y} \mathcal{M}' & \longleftarrow & \mathcal{M}'. \end{array}$$

Remark 5.11. *If $\xi_1, \ldots, \xi_m$ form a $\mathbb{C}$-basis of $\mathfrak{g}'$, then*

$$\alpha_Y = \mathscr{O}_Y\, \xi_1^* \wedge \cdots \wedge \xi_m^*$$

The right action on α_Y is given by

$$(f\,\xi_1^* \wedge \cdots \wedge \xi_m^*) \cdot \xi = \big(\mathrm{trace}(\mathrm{ad}(\xi)) - Z(\xi)(f)\big)\,\xi_1^* \wedge \cdots \wedge \xi_m^* \qquad \text{for } \xi \in \mathfrak{g}'.$$

In general, if $\mathcal{M}$ is a left $\mathcal{A}_Y$-module, then the right $\mathcal{A}_Y$-module structure on $\alpha_Y \otimes_{\mathcal{O}_Y} \mathcal{M}$ is given by

$$(\xi_1^* \wedge \cdots \wedge \xi_m^* \otimes s) \cdot \xi = \xi_1^* \wedge \cdots \wedge \xi_m^* \otimes \big(\mathrm{trace}(\mathrm{ad}(\xi)) - \xi\big) \cdot s \quad \text{for } \xi \in \mathfrak{g}', s \in \mathcal{M}.$$

The canonical bundle ω_Y on Y is a right $\mathscr{D}_Y$-module and hence, via the homomorphism $\tilde{Z}\colon \mathcal{A}_Y \to \mathscr{D}_Y$, it also has the structure of a right $\mathcal{A}_Y$-module. On the other hand, the action of G' on Y extends naturally to an action on the tangent bundle on Y, so $\omega_Y = \bigwedge^{\dim Y} \Theta_Y^\vee$ is naturally a G'-equivariant line bundle, which induces a left $\mathcal{A}_Y$-module structure. The next lemma states that these left and right module structures on ω_Y relate to each other via the transformation above:

Lemma 5.12. *Let $\delta := \mathrm{trace} \circ \mathrm{ad}\colon \mathfrak{g}' \to \mathbb{C}$ and let $\xi_1, \dots, \xi_m$ form a $\mathbb{C}$-basis of $\mathfrak{g}'$. There is an isomorphism of right $\mathcal{A}_Y$-modules*

$$\begin{aligned} \omega_Y &\xrightarrow{\cong} \alpha_Y \otimes_{\mathcal{O}_Y} \omega_Y\{\delta\} \\ s &\mapsto \xi_1^* \wedge \cdots \wedge \xi_m^* \otimes s, \end{aligned}$$

where on the left hand side, ω_Y is endowed with its right $\mathcal{A}_Y$-module structure induced from $\tilde{Z}\colon \mathcal{A}_Y \to \mathscr{D}_Y$, and on the right hand side, we consider ω_Y with its left $\mathcal{A}_Y$-module structure by viewing it as a G'-equivariant line bundle.

Proof. Denote $a := \xi_1^* \wedge \cdots \wedge \xi_m^* \in \Gamma(Y, \alpha_Y)$ and recall that $\xi_1, \dots, \xi_m$ are a $\mathbb{C}$-basis of $\mathfrak{g}'$. Since a is a non-vanishing global section of the line bundle α_Y, the homomorphism $\omega_Y \to \alpha_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\delta\} \otimes_{\mathcal{O}_Y} \omega_Y$, $s \mapsto a \otimes 1 \otimes s$ is an isomorphism of $\mathcal{O}_Y$-modules, hence it suffices to show:

$$(a \otimes 1 \otimes s) \cdot \xi \stackrel{!}{=} a \otimes 1 \otimes (s \cdot \xi)$$

for $s \in \omega_Y$, $\xi \in \mathfrak{g}'$.

The right $\mathcal{A}_Y$-module structure on ω_Y (inherited from the right $\mathscr{D}_Y$-module structure) is given by

$$(s \cdot \xi)(\theta_1, \dots, \theta_m) = -Z(\xi)(s(\theta_1, \dots, \theta_n)) + \sum_{i=1}^m s(\theta_1, \dots, [Z(\xi), \theta_i], \dots, \theta_n)$$

for $\xi \in \mathfrak{g}'$, $s \in \omega_Y$, $\theta_1, \dots, \theta_n \in \Theta_Y$. On the other hand, the right $\mathcal{A}_Y$-module α_Y satisfies $a \cdot \xi = \delta(\xi)a$, so the right $\mathcal{A}_Y$-module structure on $\alpha_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\delta\}$ satisfies

$$(a \otimes 1) \cdot \xi = 0.$$

The left $\mathcal{A}_Y$-module structure on ω_Y results from the left $\mathcal{A}_Y$-module structure on the G'-equivariant vector bundle Θ_Y given by

$$\xi \cdot \theta = [Z(\xi), \theta] \qquad \text{for all } \xi \in \mathfrak{g}'$$

The induced left $\mathcal{A}_Y$-module structure on $\bigwedge^n \Theta_Y$ is given by

$$\xi \cdot (\theta_1 \wedge \cdots \wedge \theta_n) = \sum_{i=1}^{n} \theta_1 \wedge \cdots \wedge [Z(\xi), \theta_i] \wedge \cdots \wedge \theta_n$$

for $\xi \in \mathfrak{g}'$. Passing to the dual line bundle ω_Y, we get

$$\begin{aligned}(\xi \cdot s)(\theta_1, \dots, \theta_n) &= Z(\xi)(s(\theta_1, \dots, \theta_n)) - \sum_{i=1}^{n} s(\theta_1, \dots, [Z(\xi), \theta_i], \dots, \theta_n) \\ &= -(s \cdot \xi)(\theta_1, \dots, \theta_n) \qquad \text{for all } \xi \in \mathfrak{g}'.\end{aligned}$$

The right $\mathcal{A}_Y$-module structure on $\alpha_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\delta\} \otimes_{\mathcal{O}_Y} \omega_Y$ resulting from this satisfies

$$(a \otimes 1 \otimes s) \cdot \xi = ((a \otimes 1) \cdot \xi) \otimes s - (a \otimes 1) \otimes (\xi \cdot s) = a \otimes 1 \otimes (s \cdot \xi)$$

for $\xi \in \mathfrak{g}'$. □

5.6 Cyclic right $\mathscr{D}_Y$-modules from group actions

Recall from Definition 4.7 that for any Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathbb{C}$ on a smooth connected G'-variety Y, we defined the left $\mathscr{D}_Y$-module

$$\mathscr{N}_Y^\beta := \omega_Y^\vee \otimes_{\mathcal{O}_Y} \mathscr{D}_Y/(Z(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_Y.$$

The following result shows that this $\mathscr{D}_Y$-module arises from a $\mathfrak{g}'$-module structure on the anticanonical bundle on Y:

Proposition 5.13. *Let $\beta\colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism. Considering ω_Y with its natural G'-equivariant structure, there is an isomorphism of left $\mathscr{D}_Y$-modules*

$$\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\omega_Y\{\beta\})^\vee \cong \mathscr{N}_Y^\beta.$$

Proof. Equivalently, we may show that there is an isomorphism between the corresponding right $\mathscr{D}_Y$-modules

$$\omega_Y \otimes_{\mathcal{O}_Y} \left(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\omega_Y\{\beta\})^\vee\right) \overset{!}{\cong} \mathscr{D}_Y/(Z(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_Y.$$

By Lemma 5.2.(ii), we have an isomorphism of right $\mathscr{D}_Y$-modules

$$\omega_Y \otimes_{\mathcal{O}_Y} \left(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\omega_Y\{\beta\})^\vee\right) \cong (\omega_Y \otimes_{\mathcal{O}_Y} (\omega_Y\{\beta\})^\vee) \otimes_{\mathcal{A}_Y} \mathscr{D}_Y,$$

where, on the right hand side, the first occurrence of ω_Y is equipped with the right $\mathcal{A}_Y$-module structure inherited from its right $\mathscr{D}_Y$-module structure. Combining this with Lemma 5.12, we obtain:

$$\begin{aligned}\omega_Y \otimes_{\mathcal{O}_Y} \left(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\omega_Y\{\beta\})^\vee\right) &\cong (\alpha_Y \otimes_{\mathcal{O}_Y} \omega_Y\{\delta\} \otimes_{\mathcal{O}_Y} (\omega_Y\{\beta\})^\vee) \otimes_{\mathcal{A}_Y} \mathscr{D}_Y \\ &\cong (\alpha_Y \otimes_{\mathcal{O}_Y} \omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\delta - \beta\}) \otimes_{\mathcal{A}_Y} \mathscr{D}_Y\end{aligned}$$

where $\delta := \operatorname{trace} \circ \operatorname{ad} \colon \mathfrak{g}' \to \mathbb{C}$ and ω_Y is now considered as a left $\mathcal{A}_Y$-module via its natural structure as a G'-equivariant line bundle. Since $\omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee \cong \mathcal{O}_Y$ as G'-equivariant line bundles (and therefore also as left $\mathcal{A}_Y$-module), we conclude:

$$\omega_Y \otimes_{\mathcal{O}_Y} \left(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} \omega_Y^\vee\{\beta\}\right) \cong \left(\alpha_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\delta - \beta\}\right) \otimes_{\mathcal{A}_Y} \mathscr{D}_Y.$$

Recall that $\mathcal{O}_Y\{\delta - \beta\} \cong \mathcal{A}_Y / \mathcal{A}_Y(\xi - (\delta - \beta)(\xi) \mid \xi \in \mathfrak{g}')$, so by Remark 5.11, we have

$$\alpha_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\{\delta - \beta\} \cong \mathcal{A}_Y / (\xi - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathcal{A}_Y$$

as right $\mathcal{A}_Y$-modules. Tensoring with $\mathscr{D}_Y$ over $\mathcal{A}_Y$ by means of the homomorphism $\tilde{Z} \colon \mathcal{A}_Y \to \mathscr{D}_Y$ yields the claimed result. □

5.7 Sufficient criterion for non-zero tautological systems

Proposition 5.14. *Let $\rho \colon G' \to \mathrm{GL}(V)$ be a finite-dimensional rational representation. Let $Y \subseteq V$ be a G'-orbit and let $\overline{Y}$ be its closure. Let $\beta \colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism. If $(\omega_Y\{\beta\})^{\otimes k} \cong \mathcal{O}_Y$ for some $k \in \mathbb{Z}$ as left $\mathcal{A}_Y$-modules, then $\hat{\tau}(\rho, \overline{Y}, \beta) \neq 0$.*

Proof. By Corollary 4.11, we have $i_+\mathscr{N}_Y^\beta \cong \hat{\tau}(\rho, \overline{Y}, \beta)\big|_U$, where i denotes the closed embedding of Y into $U := V \setminus \partial Y$ for $\partial Y := \overline{Y} \setminus Y$. With Proposition 5.13, we conclude that

$$\hat{\tau}(\rho, \overline{Y}, \beta)\big|_U \cong i_+\big(\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\omega_Y\{\beta\})^\vee\big)$$

as left $\mathscr{D}_{L^*}$-modules. To show that the right hand side is non-zero, it suffices to see that $\mathscr{D}_Y \otimes_{\mathcal{A}_Y} (\omega_Y\{\beta\})^\vee \neq 0$. But this follows from Corollary 5.6 respectively Remark 5.7, because we assumed that $(\omega_Y\{\beta\})^{\otimes k} \cong \mathcal{O}_Y$ as left $\mathcal{A}_Y$-modules. □

6 Total spaces of line bundles

In this section, we turn our attention to group actions on total spaces of equivariant line bundles.

6.1 Equivariant vector fields on line bundles

Let G' be a connected linear algebraic group acting on a smooth variety X. The following Lemma will allow us to analyze the vector fields $Z_L(\xi)$ on an equivariant line bundle L using local trivializations.

Lemma 6.1. *Let $\pi \colon L \to X$ be a G'-equivariant line bundle on X with sheaf of section $\mathscr{L}$. Let $U \subseteq X$ be a trivializing open subset and fix a non-vanishing local section $s \in \Gamma(U, \mathscr{L})$, so $\mathscr{L}|_U = \mathcal{O}_U\, s$. The $\mathfrak{g}'$-action on $\Gamma(U, \mathscr{L})$ may be expressed as*

$$\forall \xi \in \mathfrak{g}' : \quad \xi \cdot s = \gamma(\xi) s \qquad \text{for some } \gamma(\xi) \in \Gamma(U, \mathcal{O}_U).$$

Under the to s corresponding trivialization $\psi_s \colon \mathbb{C} \times U \xrightarrow{\cong} \pi^{-1}(U)$, $(\lambda, p) \mapsto \lambda\, s(p)$, the vector field $Z_L(\xi)$ on L for $\xi \in \mathfrak{g}'$ can then be expressed as:

$$\mathrm{d}\psi_s^{-1}\Big(Z_L(\xi)|_{\pi^{-1}(U)}\Big) = pr_2^*(\gamma(\xi)) \cdot pr_1^*(-t\partial_t) + pr_2^*(Z_Y(\xi)|_U),$$

where on the right hand side, t denotes a coordinate on $\mathbb{C}$ and we use the identification $\Theta_{\mathbb{C}\times U} \cong pr_1^*\Theta_{\mathbb{C}} \oplus pr_2^*\Theta_U$ with pr_1, pr_2 being the projections of $\mathbb{C}\times U$ onto its factors.

Proof. First, we make the $\mathfrak{g}'$-action on sections of $\mathscr{L}$ explicit on the level of points. For this, we view the local section $s \in \Gamma(U, \mathscr{L})$ as a morphism $s\colon U \to \pi^{-1}(U)$. Then the section $\xi \cdot s$ is the map $U \to \pi^{-1}(U)$ given on points as follows: Consider the open subset

$$\widetilde{U} := \{(g,q) \in G' \times U \mid g^{-1}\cdot q \in U\}$$

of $G'\times U$ and, for every $\lambda \in \mathbb{C}$, let

$$\alpha^\lambda\colon \widetilde{U} \to \pi^{-1}(U), \qquad (g,q) \mapsto g\cdot(\lambda s)(g^{-1}\cdot q);$$

for now, we will only be interested in $\alpha := \alpha^1$. For each point $p \in U$, note that the set $\widetilde{U}_p = \{g \in G' \mid g^{-1}p \in U\}$ is an open neighborhood of 1 in G'. We view $\mathfrak{g}'$ as the tangent space $T_1\widetilde{U}_p$. For $\alpha(\cdot,p)\colon \widetilde{U}_p \to \pi^{-1}(U)$, $g \mapsto g\cdot s(g^{-1}\cdot p)$, we have

$$\mathrm{d}(\alpha(\cdot,p))_1(\xi) \in \ker(T_{s(p)}L \to T_pX) \cong \mathcal{T}_{L|X,s(p)} \cong (\pi^*\mathscr{L})_{s(p)} \cong \mathscr{L}_p \cong L_p.$$

This element of the fiber $L_p \subseteq E$ is $(\xi\cdot s)(p) = \gamma(\xi)(p)\, s(p)$.

Now, we fix $\xi \in \mathfrak{g}'$ and show the claimed equality of vector fields point-wise. For this, let $(\lambda,p) \in \mathbb{C}\times U$. By definition, $Z_L(\xi)$ is the vector field whose tangent vector at $y := \psi_s(\lambda,p) = \lambda\, s(p)$ is $\mathrm{d}\bar{\varphi}^y(\xi)$, where $\bar{\varphi}^y\colon G' \to L$ maps g to $g^{-1}\cdot y$. Restricting $\bar{\varphi}^y$ to $\widetilde{U}_p$ gives a morphism $\widetilde{U}_p \to \pi^{-1}(U)$, whose composition with the isomorphism $\psi_s^{-1}\colon \pi^{-1}(U) \to \mathbb{C}\times U$ is given by $a^\lambda(\cdot,p)\colon \widetilde{U}_p \to \mathbb{C}\times U$, where

$$a^\lambda\colon V \to \mathbb{C}\times U, \qquad (g,q) \mapsto \psi_s^{-1}(g^{-1}\cdot\psi_s(\lambda,q)).$$

Note that $a^\lambda = \psi_s^{-1}\circ\alpha^\lambda\circ\chi$, where χ is the automorphism of $\widetilde{U}$ given by $\chi(g,q) = (g^{-1}, g^{-1}\cdot q)$. Using $T_{(1,p)}\widetilde{U} \cong T_1G' \oplus T_pU$ and $T_{(\lambda,p)}(\mathbb{C}\times U) \cong \mathbb{C}\oplus T_pU$, we get

$$\begin{aligned}
\big(\mathrm{d}\psi_s^{-1}(Z_L(\xi))\big)_{(\lambda,p)} &= \mathrm{d}\psi_s^{-1}(Z_L(\xi)_y) = \mathrm{d}\big(a^\lambda(\cdot,p)\big)_1(\xi) = \mathrm{d}a^\lambda_{(1,p)}(\xi,0)\\
&= \mathrm{d}(\psi_s^{-1}\circ\alpha^\lambda\circ\chi)_{(1,p)}(\xi,0)\\
&= \mathrm{d}(\psi_s^{-1}\circ\alpha^\lambda)_{(1,p)}(-\xi,\, Z_X(\xi)_p)\\
&= \mathrm{d}\big(\psi_s^{-1}\circ\alpha^\lambda(\cdot,p)\big)_1(-\xi) + \mathrm{d}\big(\psi_s^{-1}\circ\alpha^\lambda(1,\cdot)\big)_p(Z_X(\xi)_p)\\
&= (-\lambda\gamma(\xi)(p),\, 0) + (0,\, Z_X(\xi)_p) = (-\lambda\gamma(\xi)(p),\, Z_X(\xi)_p),
\end{aligned}$$

which is the tangent vector of the vector field $(\gamma(\xi)\circ pr_2)\cdot pr_1^*(-t\partial_t) + pr_2^*(Z_X(\xi)|_U)$ at the point (λ,p), as claimed. □

6.2 Canonical sheaf on total spaces of line bundles

We continue to consider a smooth variety X with the action of a connected linear algebraic group G'. Let $\mathscr{L}$ be a G'-equivariant line bundle on X, denote the total space by L and consider $L^* \subseteq L$, the complement of its zero section. The morphisms to X are denoted $\pi^L\colon L \to X$ and $\pi^{L^*}\colon L^* \to X$.

Lemma 6.2. *There is an isomorphism*

$$\omega_L \cong \pi^{L,*}\omega_X \otimes_{\mathcal{O}_L} \pi^{L,*}\mathscr{L}^\vee$$

of G'-equivariant line bundles on L.

Proof. Let $\mathscr{F}$ be a G'-equivariant vector bundle on X and put $F := \mathrm{Tot}(\mathscr{F})$ with projection $\pi^F \colon F \to X$. The variety F is then equipped with a G'-action. We first claim that there is an isomorphism

$$\Theta_{F/X} \cong \pi^{F,*}\mathscr{F}.$$

of G'-equivariant vector bundles on F, and a corresponding isomorphism $\Omega^1_{F/X} \cong \pi^{F,*}\mathscr{F}^\vee$ of dual vector bundles. Namely, any section $s \in \Gamma(U, \mathcal{F})$ can be considered as an element $s \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}^\vee, \mathcal{O}_X))$, and it extends via the Leibniz rule as a section of $\Gamma(U, \mathcal{D}er_{\mathcal{O}_X}(\mathcal{S}ym_{\mathcal{O}_X}(\mathscr{F}^\vee))$. This yields a G'-equivariant morphism of $\mathcal{O}_X$-modules

$$\mathscr{F} \longrightarrow \mathcal{D}er_{\mathcal{O}_X}(\mathcal{S}ym_{\mathcal{O}_X}(\mathscr{F}^\vee)) = \mathcal{D}er_{\mathcal{O}_X}(\pi^F_*\mathcal{O}_F).$$

It is also injective, since for any $s \neq 0$, there is some section of $\mathscr{F}^\vee$ that is not killed by s, so that s is not the zero derivation in $\mathcal{D}er_{\mathcal{O}_X}(\pi^F_*\mathcal{O}_F)$. Since both $\mathscr{F}$ and $\mathcal{D}er_{\mathcal{O}_X}(\pi^F_*\mathcal{O}_F)$ are locally free of the same rank, it follows that the cokernel of the inclusion $\mathscr{F} \hookrightarrow \mathcal{D}er_{\mathcal{O}_X}(\pi^F_*\mathcal{O}_F)$, if not zero, must be a torsion sheaf on X, but this is impossible since this map is equivariant, and so is its cokernel. We conclude that there is an isomorphism $\mathscr{F} \cong \mathcal{D}er_{\mathcal{O}_X}(\pi^F_*\mathcal{O}_F)$ of G'-equivariant vector bundles on X. Applying the functor $\pi^{F,*}$ then yields an isomorphism

$$\pi^{F,*}\mathscr{F} \xrightarrow{\cong} \pi^{F,*}\mathcal{D}er_{\mathcal{O}_X}(\pi^F_*\mathcal{O}_F) \overset{(\star)}{\cong} \mathcal{D}er_{\pi^{F,-1}\mathcal{O}_X}(\mathcal{O}_F) \cong \Theta_{F/X},$$

of G'-equivariant bundles on F, as required. Notice that the isomorphism $(\star)$ in the above displayed formula holds since the map π^F is affine.

We apply this to the special case $F = L$, i.e., $\mathrm{rk}(\mathscr{F}) = 1$, to obtain the $\mathcal{O}_L$-isomorphism

$$\omega_{L/X} \cong \pi^{L,*}\mathscr{L}^\vee, \tag{9}$$

which again is G'-equivariant.

Consider the cotangent sequence

$$0 \longrightarrow \pi^{L,*}\Omega^1_X \longrightarrow \Omega^1_L \longrightarrow \omega_{L/X} \longrightarrow 0,$$

which, since $\pi^L : L \to X$ is G'-equivariant, is an exact sequence of G'-equivariant vector bundles on L. Applying $\bigwedge^{\dim(X)+1}_{\mathcal{O}_L}(-)$ to this sequence, we get the following isomorphism of G'-equivariant line bundles on L:

$$\omega_L \cong \pi^{L,*}\omega_X \otimes_{\mathcal{O}_L} \omega_{L/X}.$$

Plugging in the isomorphism from Equation (9) yields

$$\omega_L \cong \pi^{L,*}\omega_X \otimes_{\mathcal{O}_L} \pi^{L,*}\mathscr{L}^\vee,$$

as required.

□

Lemma 6.3. *We have* $\pi^{L^*,*}\mathscr{L} \cong \mathcal{O}_{L^*}$ *as* G'*-equivariant line bundles on* L^**.*

Proof. Note that the G'-equivariant structure on $\pi^{L^*,*}\mathscr{L}$ corresponds to the diagonal G'-action on $\mathrm{Tot}(\pi^{L^*,*}\mathscr{L}) = L^* \times_X L \xrightarrow{pr_1} L^*$. Note further that the map

$$s\colon L^* \xrightarrow{\Delta} L^* \times_X L^* \hookrightarrow L^* \times_X L$$

is a G'-invariant global section of the line bundle $\pi^{L^*,*}\mathscr{L}$ that vanishes nowhere on L^*. Then

$$\mathcal{O}_{L^*} \to \pi^{L^*,*}\mathscr{L}, \qquad 1 \mapsto s$$

is an isomorphism of G'-equivariant line bundles. □

Proposition 6.4. *If* $\mathscr{L}^{\otimes \ell} \cong (\omega_X\{\rho\})^{\otimes k}$ *as* G'*-equivariant line bundles for some* $k, \ell \in \mathbb{Z}$*,* $k \neq 0$ *and some character* $\rho\colon G' \to \mathbb{C}^*$*, then, as* G'*-equivariant line bundles on* L^**:*

$$(\omega_{L^*}\{\rho\})^{\otimes k} \cong \mathcal{O}_{L^*}.$$

Proof. By Lemma 6.2 and Lemma 6.3, we have

$$\omega_{L^*} \cong \omega_L|_{L^*} \cong \pi^{L^*,*}\omega_X \otimes_{\mathcal{O}_{L^*}} \pi^{L^*,*}\mathscr{L}^\vee \cong \pi^{L^*,*}\omega_X$$

as G'-equivariant line bundles on L^*. Therefore,

$$(\omega_{L^*}\{\rho\})^{\otimes k} \cong ((\pi^{L^*,*}\omega_X)\{\rho\})^{\otimes k} \cong \pi^{L^*,*}((\omega_X\{\rho\})^{\otimes k}) \cong \pi^{L^*,*}\mathscr{L}^{\otimes \ell} \cong \mathcal{O}_{L^*},$$

where the last isomorphism is again due to Lemma 6.3. □

Corollary 6.5. *Assume* $\mathscr{L}^{\otimes \ell} \cong (\omega_X\{\rho\})^{\otimes k}$ *as* G'*-equivariant line bundles for some* $k, \ell \in \mathbb{Z}$*,* $k \neq 0$ *and some character* $\rho\colon G' \to \mathbb{C}^*$*. If* G' *acts transitively on* L^**, then*

$$\mathscr{N}_{L^*}^{\mathrm{d}\rho} \cong (\omega_{L^*}\{\rho\})^\vee$$

as left $\mathcal{A}_{L^*}$*-modules. In particular,* $\mathscr{N}_{L^*}^{\mathrm{d}\rho} \neq 0$*.*

Proof. Applying Proposition 5.13 to $Y = L^*$ and $\beta = \mathrm{d}\rho$, we get

$$\mathscr{N}_{L^*}^{\mathrm{d}\rho} \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} (\omega_{L^*}\{\rho\})^\vee$$

as left $\mathscr{D}_{L^*}$-modules. By Proposition 6.4, the assumption $\mathscr{L}^{\otimes \ell} \cong (\omega_X\{\rho\})^{\otimes k}$ ensures that $(\omega_{L^*}\{\rho\})^\vee$ is a torsion element of the G'-equivariant Picard group on L^*. Then the claim follows from Corollary 5.6. □

Remark 6.6. *When we replace the isomorphisms of* G'*-equivariant line bundles by isomorphisms of left* $\mathcal{A}_X$*- (resp.* $\mathcal{A}_{L^*}$*-)modules in Proposition 6.4 and Corollary 6.5, the statements hold more generally for arbitrary Lie algebra homomorphisms* $\beta\colon \mathfrak{g}' \to \mathbb{C}$ *that may not arise from group characters* $\rho\colon G' \to \mathbb{C}^*$*. The proofs carry over verbatim.*

6.3 Constructing $\mathcal{O}_{L^*}^{\beta}$ from group actions

Let G be a connected linear algebraic group and let $G' := G \times \mathbb{C}^*$. We denote the Lie algebras involved by $\mathfrak{g}' := \mathrm{Lie}(G')$, $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathrm{Lie}(\mathbb{C}^*) = \mathbb{C}\mathbf{e}$, so

$$\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}\mathbf{e}.$$

Let X be a variety with a transitive G-action, which we extend to a transitive G'-action, where the $\mathbb{C}^*$-factor acts trivially.

Let $L_0 \to X$ be a G-equivariant line bundle on X with sheaf of sections $\mathscr{L}_0$. We view $\mathscr{L}_0$ as a G'-equivariant line bundle on X by letting the $\mathbb{C}^*$-factor of G' act trivially on both L_0 and X. For every $k \in \mathbb{Z}$, we consider the character $\rho_k \colon G' = G \times \mathbb{C}^* \to \mathbb{C}^*$ given by $(g,t) \mapsto t^k$. We define $\mathscr{L}_k := \mathscr{L}_0\{\rho_k\}$ and $L_k := \mathrm{Tot}(\mathscr{L}_k)$. In other words, $L_k \to X$ is the G'-equivariant line bundle on X whose action of G' is given by extending the given action of G on L_0 by letting the $\mathbb{C}^*$-factor of G' act via scaling the fibers of the line bundle with k-th powers. Note that all L_k for $k \in \mathbb{Z}$ are the same line bundle over X with different G'-actions.

Remark 6.7. *Let $\mathscr{L}$ and $\mathcal{M}$ are two G'-equivariant line bundles on X, denote by L the total space of $\mathscr{L}$ and let $\pi_0 \colon L^* \to X$ be the complement of the zero section. If $\mathscr{L}^{\otimes \ell} \cong \mathcal{M}^{\otimes k}$, then $\pi_0^* \mathcal{M}$ is a torsion element of the equivariant Picard group on L^* by Lemma 6.3. Hence, by Corollary 5.6, we already know that this defines a non-zero $\mathscr{D}_{L^*}$-module $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^*\mathcal{M}$. The aim of this section is to describe this non-zero $\mathscr{D}_{L^*}$-module more explicitly by showing that it agrees with the construction from Section 2.*

Proposition 6.8. *Let $\mathscr{L}_0, \mathcal{M}_0$ be two G-equivariant line bundles on X and consider the G'-equivariant line bundles $\mathscr{L} := \mathscr{L}_0\{\rho_k\}$, $\mathcal{M} := \mathcal{M}_0\{\rho_\ell\}$ for some $k, \ell \in \mathbb{Z}$, $k \neq 0$. Let $\pi \colon L \to X$ denote the total space of $\mathscr{L}$ with the induced G'-action. Assume that $\mathscr{L}$ admits an r-th root $\mathscr{F}$ in $\mathrm{Pic}^{G'}(X)$ for $r := k/\gcd(k,\ell)$. The choice of a non-vanishing local section $s \in \Gamma(U, \mathscr{F})$ determines an isomorphism $\psi_{s^r} \colon \mathbb{C} \times U \xrightarrow{\cong} \pi^{-1}(U)$ given by $(\lambda, p) \mapsto \lambda s(p)$. Then*

$$\psi_{s^r}^{+}\Big((\mathscr{D}_L \otimes_{\mathcal{A}_L} \pi^*\mathcal{M})\big|_{\pi^{-1}(U)} \Big) \cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(t\partial_t + \ell/k) \boxtimes (\mathscr{D}_X \otimes_{\mathcal{A}_X} \mathscr{E})|_U,$$

where $\mathscr{E} := \mathcal{M} \otimes_{\mathcal{O}_X} \mathscr{F}^{\otimes(-\ell/\gcd(k,\ell))}$.

Proof. Every $\xi \in \mathfrak{g}'$ acts on the section $s \in \Gamma(U, \mathscr{F})$ as

$$\xi \cdot s = \gamma_{\mathscr{F}}(\xi)\, s \qquad \text{for some } \gamma_{\mathscr{F}}(\xi) \in \Gamma(U, \mathcal{O}_U).$$

The non-vanishing section $s^r \in \Gamma(U, \mathscr{L})$ corresponds to the 1-section of $\mathbb{C} \times U$ under ψ_{s^r}. Then

$$\xi \cdot s^r = r\gamma_{\mathscr{F}}(\xi)\, s^r \qquad \text{for all } \xi \in \mathfrak{g}'.$$

Note in particular that $\mathbf{e} \cdot s^r = k\, s^r$ implies $\gamma_{\mathscr{F}}(\mathbf{e}) = \gcd(k,\ell)$. We may assume that $\mathcal{M}$ also trivializes over U and choose any non-vanishing local section $s' \in \Gamma(U, \mathcal{M})$. Then $\xi \in \mathfrak{g}'$ acts by

$$\xi \cdot s' = \gamma_{\mathcal{M}}(\xi)\, s' \qquad \text{for some } \gamma_{\mathcal{M}}(\xi) \in \Gamma(U, \mathcal{O}_U).$$

Note that $\gamma_{\mathcal{M}}(\mathbf{e}) = \ell$ by definition. The local section $s_{\mathscr{E}} := s' \otimes s^{\otimes(-\ell/\gcd(k,\ell))} \in \Gamma(U, \mathscr{E})$ of $\mathscr{E}$ is acted on by $\xi \in \mathfrak{g}'$ via

$$\xi \cdot s_{\mathscr{E}} = \gamma_{\mathscr{E}}(\xi) s_{\mathscr{E}} \qquad \text{with} \qquad \gamma_{\mathscr{E}}(\xi) := \gamma_{\mathcal{M}}(\xi) - \frac{\ell}{\gcd(k,\ell)} \gamma_{\mathscr{F}}(\xi).$$

The pull-back section $\pi^* s' \in \Gamma(\pi^{-1}(U), \pi^*\mathcal{M})$ does not vanish on $U' := \pi^{-1}(U)$ and we have

$$\xi \cdot (\pi^* s') = \pi^*(\gamma_{\mathcal{M}}(\xi))\, \pi^* s' \qquad \text{for all } \xi \in \mathfrak{g}'.$$

We conclude that

$$(\pi^*\mathcal{M})|_{U'} \cong \mathcal{A}_{U'}/\mathcal{A}_{U'} \cdot (\xi - \pi^*(\gamma_{\mathcal{M}}(\xi)) \mid \xi \in \mathfrak{g}').$$

as left $\mathcal{A}_{U'}$-modules, and therefore

$$(\mathscr{D}_L \otimes_{\mathcal{A}_L} \pi^*\mathcal{M})|_{U'} \cong \mathscr{D}_{U'}/\mathscr{D}_{U'} \cdot (Z_L(\xi)|_{U'} - \pi^*(\gamma_{\mathcal{M}}(\xi)) \mid \xi \in \mathfrak{g}').$$

Hence, if we denote by pr_1 and pr_2 the projections of $\mathbb{C} \times U$ onto the first and second factor, then

$$\psi_{s^r}^+((\mathscr{D}_L \otimes_{\mathcal{A}_L} \pi^*\mathcal{M})|_{U'}) \cong \mathscr{D}_{\mathbb{C}\times U}/\mathscr{D}_{\mathbb{C}\times U} \cdot (\mathrm{d}\psi_{s^r}^{-1}(Z_L(\xi)|_V) - pr_2^*(\gamma_{\mathcal{M}}(\xi)) \mid \xi \in \mathfrak{g}').$$

With Lemma 6.1 and using $\mathfrak{g}' = \mathbb{C}\mathbf{e} \oplus \mathfrak{g}$, we get

$$\begin{aligned}
&\psi_{s^r}^+((\mathscr{D}_L \otimes_{\mathcal{A}_L} \pi^*\mathcal{M})|_{U'}) \\
&\cong \mathscr{D}_{\mathbb{C}\times U}/\mathscr{D}_{\mathbb{C}\times U} \cdot \big(pr_2^*(Z_X(\xi)|_U) - r\, pr_2^*(\gamma_{\mathscr{F}}(\xi)) \cdot pr_1^*(t\partial_t) - pr_2^*(\gamma_{\mathcal{M}}(\xi)) \mid \xi \in \mathfrak{g}'\big) \\
&\cong \mathscr{D}_{\mathbb{C}\times U}/\Big(\mathscr{D}_{\mathbb{C}\times U} \cdot (-k\, pr_2^*(t\partial_t) - \ell) \\
&\qquad\qquad + \mathscr{D}_{\mathbb{C}\times U} \cdot \big(pr_2^*(Z_X(\xi)|_U) - r\, pr_2^*(\gamma_{\mathscr{F}}(\xi)) \cdot pr_1^*(t\partial_t) - pr_2^*(\gamma_{\mathcal{M}}(\xi)) \mid \xi \in \mathfrak{g}\big)\Big) \\
&\cong \mathscr{D}_{\mathbb{C}\times U}/\Big(\mathscr{D}_{\mathbb{C}\times U} \cdot (pr_2^*(t\partial_t) + \ell/k) \\
&\qquad\qquad + \mathscr{D}_{\mathbb{C}\times U} \cdot (pr_2^*(Z_X(\xi)|_U) + r \cdot pr_2^*(\gamma_{\mathscr{F}}(\xi)) \cdot \ell/k - pr_2^*(\gamma_{\mathcal{M}}(\xi)) \mid \xi \in \mathfrak{g})\Big) \\
&\cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(t\partial_t + \ell/k) \boxtimes \mathscr{D}_U/\mathscr{D}_U(Z_X(\xi)|_U - \gamma_{\mathscr{E}}(\xi) \mid \xi \in \mathfrak{g}) \\
&\cong \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(t\partial_t + \ell/k) \boxtimes (\mathscr{D}_X \otimes_{\mathcal{A}_X} \mathscr{E})|_U.
\end{aligned}$$

□

Corollary 6.9. *Assume that X admits an open cover $X = \bigcup_{i\in I} U_i$ such that for each $i \in I$, there is a subgroup N_i of G acting freely and transitively on U_i. Let $\mathscr{L}_0, \mathcal{M}_0$ be two G-equivariant line bundles on X and consider the G'-equivariant line bundles $\mathscr{L} := \mathscr{L}_0\{\rho_k\}$, $\mathcal{M} := \mathcal{M}_0\{\rho_\ell\}$ for some $k, \ell \in \mathbb{Z}$, $k \neq 0$. Let L denote the total space of $\mathscr{L}$ with the induced G'-action and let $\pi_0 \colon L^* \to X$ be the complement of the zero section. Then*

$$\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^*\mathcal{M} \cong \begin{cases} \mathcal{O}_{L^*}^{\ell/k} & \text{if } \mathscr{L}^{\otimes \ell} \cong \mathcal{M}^{\otimes k}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, we assume that $\mathscr{L}$ admits an r-th root $\mathscr{F}$ in $\mathrm{Pic}^{G'}(X)$, where $r := k/\gcd(k,\ell)$. Applying Proposition 6.8 and restricting to L^*, we get that $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^*\mathcal{M}$ is locally of the form

$$\mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(t\partial_t + \ell/k) \boxtimes (\mathscr{D}_X \otimes_{\mathcal{A}_X} \mathscr{E})|_U$$

for $\mathscr{E} := \mathcal{M} \otimes_{\mathcal{O}_X} \mathscr{F}^{\otimes(-\ell/\gcd(k,\ell))}$. In particular, this is non-zero if and only if $\mathscr{D}_X \otimes_{\mathcal{A}_X} \mathscr{E}$ is non-zero (since the support of the latter is a G-invariant subset of X). According to Proposition 5.8, this happens if and only if $\mathscr{E} \cong \mathcal{O}_X$, i.e., if $\mathcal{M} \cong \mathscr{F}^{\otimes(-\ell/\gcd(k,\ell))}$. This is equivalent to $\mathscr{L}^{\otimes \ell} \cong \mathcal{M}^{\otimes k}$ (using that there is no torsion in $\operatorname{Pic}^{G'}(X)$). In this case, $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^* \mathcal{M}$ is locally of the form $\mathscr{D}_{\mathbb{C}^*}/\mathscr{D}_{\mathbb{C}^*}(t\partial_t + \ell/k) \boxtimes \mathcal{O}_U$, so it is $\mathcal{O}_{L^*}^{-1+\ell/k} \cong \mathcal{O}_{L^*}^{\ell/k}$ by Proposition 2.1 and Proposition 2.4.

If $\mathscr{L}$ does not admit an r-th root in the equivariant Picard group, then $\mathscr{L}^{\otimes \ell} \ncong \mathcal{M}^{\otimes k}$, as otherwise $\mathscr{F} := \mathscr{L}^{\otimes a} \otimes_{\mathcal{O}_X} \mathcal{M}^{\otimes b}$ for $ak + b\ell = \gcd(k,\ell)$ would satisfy $\mathscr{F}^{\otimes r} \cong \mathscr{L}$. We must therefore show that $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^* \mathcal{M} = 0$. If we assume for contradiction that $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^* \mathcal{M} \neq 0$, then, by Proposition 5.5, the map $\pi_0^* \mathcal{M} \to \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^* \mathcal{M}$ is an isomorphism of left $\mathcal{A}_{L^*}$-modules and we have

$$\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} (\pi_0^* \mathcal{M}^{\otimes r}) \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} (\pi_0^* \mathcal{M})^{\otimes r} \neq 0.$$

However, the G'-equivariant line bundle $\mathcal{M}' := \mathcal{M}^{\otimes r}$ is $\mathcal{M}_0\{\rho_{\ell'}\}$ for $\ell' := r\ell$ and $k/\gcd(k,\ell') = 1$. Therefore, we may apply the already shown first part to $\mathscr{L}$ and $\mathcal{M}'$. From $\mathscr{D}_L \otimes_{\mathcal{A}_L} (\pi^* \mathcal{M}') \neq 0$, we may thus deduce $\mathscr{L}^{\otimes \ell'} \cong (\mathcal{M}')^{\otimes k}$, i.e., $\mathscr{L}^{\otimes r\ell} \cong \mathcal{M}^{\otimes rk}$. This contradicts $\mathscr{L}^{\otimes \ell} \ncong \mathcal{M}^{\otimes k}$, as $\operatorname{Pic}^{G'}(X)$ is torsion-free under the assumption on X. □

Note from the proof that the assumption on the existence of a suitable open cover was only used to show the vanishing $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^* \mathcal{M} = 0$ for $\mathscr{L}^{\otimes \ell} \ncong \mathcal{M}^{\otimes k}$. The description $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L^*}} \pi_0^* \mathcal{M} \cong \mathcal{O}_{L^*}^{\ell/k}$ in the other case holds without this assumption.

7 Tautological systems associated to homogeneous spaces

The purpose of this section is to gather all the previous results and to apply them to the special case where we are given a projective variety X with a transitive group action (one may weaken this condition slightly, see Lemma 7.1 below) together with a very ample equivariant line bundle. We obtain a representation on the space of sections, and we can therefore consider the corresponding tautological system. The non-vanishing results from Section 5 then apply. Moreover, we show in Section 7.2 a localization property of the corresponding Fourier transform $\hat{\tau}$. These results, combined with the discussion in Section 3 will finally give our main result (Theorem 7.8) stating that the tautological system $\tau(\rho, \hat{X}, \beta)$, if non-zero, underlies a complex pure Hodge module.

7.1 Fourier transform of tautological systems from homogeneous spaces

Let G be a connected algebraic group acting on a variety X satisfying:

$$\begin{array}{c}\text{Every point of } X \text{ admits an open neighborhood} \\ \text{on which a subgroup of } G \text{ acts freely and transitively.}\end{array} \qquad (\star)$$

Note that assumption $(\star)$ implies in particular that G acts transitively on X.

Lemma 7.1. *Let G be a connected reductive linear algebraic group acting transitively on a projective variety X. Then $(\star)$ is fulfilled.*

Proof. For any point $p \in X$, the stabilizer $P := \{g \in G \mid g \cdot p = p\}$ describes the variety as a quotient:

$$G/P \xrightarrow{\cong} X, \qquad gP \mapsto g \cdot p.$$

Since X is projective, the subgroup $P \subseteq G$ is parabolic (see Section 8.1). Let $N^- \subseteq G$ be the unipotent radical of the opposite parabolic subgroup to P in G. Then $N^- \cap P = 1$, which shows that N^- acts freely and transitively on the N^--orbit $N^- \cdot p$. On the other hand, we have $\mathrm{Lie}(N^-) \oplus \mathrm{Lie}(P) = \mathfrak{g}$ as $\mathbb{C}$-vector spaces, so $N^- \cdot p$ is of dimension $\dim G - \dim P = \dim X$, hence it is an open neighborhood of p in X. □

As in the previous section, let $L_0 \to X$ be a G-equivariant line bundle on X with sheaf of sections $\mathscr{L}_0$. We consider $G' := G \times \mathbb{C}^*$ and denote the Lie algebras involved by $\mathfrak{g}' := \mathrm{Lie}(G')$, $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathrm{Lie}(\mathbb{C}^*) = \mathbb{C}\mathbf{e}$, so

$$\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}\mathbf{e}.$$

We view $\mathscr{L}_0$ as a G'-equivariant line bundle on X by letting the $\mathbb{C}^*$-factor of G' act trivially on both L_0 and X. For every $k \in \mathbb{Z}$, we consider the character $\rho_k \colon G' = G \times \mathbb{C}^* \to \mathbb{C}^*$ given by $(g,t) \mapsto t^k$. We define $\mathscr{L}_k := \mathscr{L}_0\{\rho_k\}$ and $L_k := \mathrm{Tot}(\mathscr{L}_k)$. When we consider L_k or L_k^* as a variety irrespective of the group action, we just denote it by L or L^*, respectively.

Proposition 7.2. *Let $\beta \colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism with $\beta|_{\mathfrak{g}} \equiv 0$ and $\beta(\mathbf{e}) \in \mathbb{Q}$. If $\mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}$ as G-equivariant line bundles for some $k, \ell \in \mathbb{Z}$, $k \neq 0$ with $\beta(\mathbf{e}) = \ell/k$, then*

$$\mathscr{N}^{\beta}_{L^*_{-1}} \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L_k^*}} \omega^{\vee}_{L_k^*}\{\rho_\ell\} \cong \mathcal{O}_{L^*}^{\ell/k}$$

as left $\mathscr{D}_{L^*}$-modules, and this is isomorphic to $\omega^{\vee}_{L_k^*}\{\rho_\ell\}$ as left $\mathcal{A}_{L_k^*}$-module.

If, on the other hand, $\mathscr{L}_0^{\otimes \ell} \not\cong \omega_X^{\otimes(-k)}$ for $\beta(\mathbf{e}) = \ell/k$, then $\mathscr{N}^{\beta}_{L^*_{-1}} = 0$.

Proof. Applying Proposition 5.13 to $Y = L_k^*$ and $\beta = \mathrm{d}\rho_{-\ell}$, we know

$$\mathscr{N}^{-k\beta}_{L_k^*} \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L_k^*}} (\omega_{L_k^*}\{\rho_{-\ell}\})^{\vee} \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L_k^*}} \omega^{\vee}_{L_k^*}\{\rho_\ell\}.$$

Moreover, since G acts transitively on X and $k \neq 0$, the group G' acts transitively on L_k^*. Under the assumption $\mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}$, Corollary 6.5 shows

$$\mathscr{N}^{-k\beta}_{L_k^*} \cong \omega^{\vee}_{L_k^*}\{\rho_\ell\}$$

as left $\mathcal{A}_{L_k^*}$-modules. This is in particular non-zero, as the underlying $\mathcal{O}_{L^*}$-module is the anticanonical line bundle. In general, note that $\omega^{\vee}_{L_k^*}\{\rho_\ell\} \cong \pi^{L_k^*,*}(\omega^{\vee}_X\{\rho_\ell\})$ by Lemma 6.2 and Lemma 6.3. If $\mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}$, then

$$\mathscr{L}_k^{\otimes \ell} \cong (\mathscr{L}_0\{\rho_k\})^{\otimes \ell} \cong \mathscr{L}_0^{\otimes \ell}\{\rho_{k\ell}\} \cong \omega_X^{\otimes(-k)}\{\rho_{k\ell}\} \cong (\omega_X^{\vee}\{\rho_\ell\})^{\otimes k},$$

so Corollary 6.9 shows $\mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L_k^*}} \omega_{L_k^*}^\vee\{\rho_\ell\} \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L_k^*}} \pi^{L_k^*,*}(\omega_X^\vee\{\rho_\ell\}) \cong \mathcal{O}_{L^*}^{\ell/k}$. On the other hand, if $\mathscr{L}_0^{\otimes \ell} \not\cong \omega_X^{\otimes(-k)}$, then $\mathscr{L}_k^{\otimes \ell} \not\cong (\omega_X^\vee\{\rho_\ell\})^{\otimes k}$, in which case Corollary 6.9 shows $\mathcal{N}_{L_k^*}^{k\beta} \cong \mathscr{D}_{L^*} \otimes_{\mathcal{A}_{L_k^*}} \pi^{L_k^*,*}(\omega_X^\vee\{\rho_\ell\}) = 0$.

It remains to show that

$$\mathscr{N}^{\beta}_{L_{-1}^*} \cong \mathscr{N}^{-k\beta}_{L_k^*} \quad \text{for all } k \in \mathbb{Z},\, k \neq 0. \tag{10}$$

For this, it suffices to observe that the corresponding cyclic right $\mathscr{D}_{L^*}$-modules

$$\mathscr{D}_{L^*}/(Z_{L_{-1}^*}(\xi) - \beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_{L^*} \quad \text{and} \quad \mathscr{D}_{L^*}/(Z_{L_k^*}(\xi) + k\beta(\xi) \mid \xi \in \mathfrak{g}')\mathscr{D}_{L^*}$$

agree. This is easily deduced from the definition of the G'-action on L_{-1}^* and L_k^*: We have $Z_{L_k^*}(\xi) = Z_{L_{-1}^*}(\xi)$ for all $\xi \in \mathfrak{g}$ and $Z_{L_k^*}(\mathbf{e}) = -kZ_{L_{-1}^*}(\mathbf{e})$. This shows (10). □

Remark 7.3. *In Proposition 7.2, we assumed $\beta(\mathbf{e}) \in \mathbb{Q}$ and obtain a non-zero $\mathscr{D}_{L^*}$-module if and only if $\mathscr{L}_0$ is a corresponding rational power of the anticanonical bundle. If we lift the rationality assumption on $\beta(\mathbf{e})$, then re-examining the arguments that led to Proposition 7.2, we see that $\mathscr{N}^{\beta}_{L_{-1}^*} = 0$, whenever $\beta|_{\mathfrak{g}} \equiv 0$ and $\beta(\mathbf{e}) \in \mathbb{C} \setminus \mathbb{Q}$.*

In the following, we assume additionally that X is projective and $\mathscr{L}_0$ is very ample. The G'-equivariant structure on $\mathscr{L}_{-1}$ induces a representation $\rho\colon G' \to \mathrm{GL}(V)$ for $V := H^0(X, \mathscr{L}_{-1})^\vee$ and the complete linear system $|\mathscr{L}_{-1}|$ gives an equivariant closed embedding of X into $\mathbb{P}V$. Note that the $\mathbb{C}^*$-factor of $G' = G \times \mathbb{C}^*$ acts by simple scaling on V, while acting by *inverse* scaling on the fibers of $L_{-1} \to X$.

If $\hat{X} \subseteq V$ denotes the affine cone over X, we may identify $L^\vee$ with the blow-up $\mathrm{Bl}_{\{0\}} \hat{X}$, and this identification induces an isomorphism of $L^{\vee,*}$ with $\hat{X} \setminus 0$. Combined with the isomorphism $inv\colon L^{\vee,*} \xrightarrow{\cong} L^*$ given by inverting the $\mathbb{C}^*$-fibers over X, we get the G'-equivariant isomorphism $\hat{X} \setminus 0 \cong L_{-1}^*$. Denote the closed embedding of $\hat{X} \setminus 0$ into $V \setminus 0$ by i. Since clearly $\hat{X}$ is a G'-variety, we can consider, for any Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathbb{C}$, the $\mathscr{D}_V$-module $\hat{\tau}(\rho, \hat{X}, \beta)$ from Definition 1.1. Then we have the following result.

Theorem 7.4. *Let $\beta\colon \mathfrak{g}' \to \mathbb{C}$ satisfy $\beta|_{\mathfrak{g}} \equiv 0$. Then*

$$\hat{\tau}(\rho, \hat{X}, \beta)\big|_{V \setminus 0} \cong \begin{cases} i_+\mathcal{O}_{L^*}^{\ell/k} & \text{if } \beta(\mathbf{e}) = \ell/k \in \mathbb{Q} \text{ and } \mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows directly from the work above by combining Corollary 4.11 (applied to $Y = \hat{X} \setminus 0$) and Proposition 7.2 together with Remark 7.3. For convenience, we roughly summarize the main steps that led to this result:

- The restriction of $\hat{\tau}(\rho, \hat{X}, \beta)$ to $V \setminus 0$ is supported on $\hat{X} \setminus 0 \cong L_{-1}^*$ and can be described as $i_+\mathscr{N}^{\beta}_{\hat{X}\setminus 0}$ (Corollary 4.11). Here, $\mathscr{N}^{\beta}_{\hat{X}\setminus 0}$ arises from a cyclic right $\mathscr{D}$-module constructed from the vector fields induced by the group action (Definition 4.7).

- $\mathscr{N}^{\beta}_{\hat{X}\setminus 0}$ is alternatively described as $\mathscr{D}_{\hat{X}\setminus 0} \otimes_{\mathcal{A}_{\hat{X}\setminus 0}} (\omega_{\hat{X}\setminus 0}\{\beta\})^\vee$ (Proposition 5.13), where $\mathcal{A}_{\hat{X}\setminus 0} = \mathcal{O}_{\hat{X}\setminus 0} \otimes \mathcal{U}(\mathfrak{g}')$ and $(\omega_{\hat{X}\setminus 0}\{\beta\})^\vee$ is the anticanonical bundle with a $\mathfrak{g}'$-module structure determined by β.

- Identifying $\hat{X}\backslash 0$ with L^*_{-1}, we can argue that $\mathscr{D}_{\hat{X}\backslash 0}\otimes_{\mathcal{A}_{\hat{X}\backslash 0}}(\omega_{\hat{X}\backslash 0}\{\beta\})^\vee$ is non-zero if $\mathscr{L}$ is a ℓ/k-th rational power of $\omega_X^\vee$ and is equipped with a suitable equivariant structure (Corollary 6.5). The geometric reason is that in this case $(\omega_{\hat{X}\backslash 0}\{\beta\})^{\otimes k} \cong \mathcal{O}_{\hat{X}\backslash 0}$ (Corollary 5.6 and Proposition 6.4).

- A more precise analysis (independent of the previous item) interprets $\mathscr{N}^{\beta}_{\hat{X}\backslash 0}$ as $\mathscr{D}_{L^*}\otimes_{\mathcal{A}_{L^*}}\pi_0^*\mathcal{M}$ for $\mathcal{M}$ being the anticanonical bundle on X (with an appropriate equivariant structure) and such a construction yields either $\mathcal{O}_{L^*}^{\ell/k}$ or 0 depending on whether $\mathscr{L}$ is a ℓ/k-th rational power of $\mathcal{M}$ or not (Corollary 6.9). □

Remark 7.5. *If G is a semisimple linear algebraic group, then we have $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. This shows that $\beta|_{\mathfrak{g}}\equiv 0$ holds for every Lie algebra homomorphism $\beta\colon \mathfrak{g}'\to\mathbb{C}$, so this condition in Theorem 7.4 is always fulfilled in this case.*

7.2 Localization property of $\widehat{\tau}$

The purpose of this section is to prove a key property of the differential system $\hat{\tau}(\rho,\hat{X},\beta)$. We do this in a slightly more general setup. For this reason, we let for a moment V be any finite-dimensional vector space, and let us consider the Euler operator E on V (i.e. the differential of the scaling action). For $\lambda\in\mathbb{C}$, define $\mathbf{Eig}(V,\lambda)$ to be the full subcategory of $\mathrm{Mod}(\Gamma(V,\mathscr{D}_V))$ consisting of modules M satisfying

$$M=\bigoplus_{\mu\in\lambda+\mathbb{Z}} M_\mu, \tag{11}$$

where $M_\mu := \ker(E-\mu)\subseteq M$.

Proposition 7.6. *Let $\mathscr{M}\in\mathrm{Mod}_{qc}(\mathscr{D}_V)$. If $\Gamma(V,\mathscr{M})\in\mathbf{Eig}(V,\lambda)$ for some $\lambda\in\mathbb{C}\setminus\mathbb{Z}$, then*

$$\mathscr{M}\cong j_+j^+\mathscr{M},$$

where j denotes the open embedding of $V\setminus 0$ into V.

Proof. Let $i\colon\{0\}\hookrightarrow V$ be the closed embedding of the origin into V. Let $N:=\dim V$ and choose coordinates $x_1,\dots,x_N$ on V. The distinguished triangle

$$\mathrm{R}\Gamma_{\{0\}}(\mathscr{M})\to\mathscr{M}\to j_+j^+\mathscr{M}\xrightarrow{+1}$$

in $D^b_{qc}(\mathscr{D}_V)$ (see [HTT08, Prop. 1.7.1(i)]) shows that it suffices to prove $\mathrm{R}\Gamma_{\{0\}}(\mathscr{M})=0$ in order to conclude the claim. Since V is affine and $\mathscr{M}$ is quasi-coherent, we actually just need to show

$$H^i_{\mathfrak{m}}(M)=0\quad\text{for all } i,$$

where $M=\Gamma(V,\mathscr{M})$ and $\mathfrak{m}=(x_1,\dots,x_N)$.

Recall that $H^i_{\mathfrak{m}}(M)$ may be computed as the cohomology of the Čech complex

$$0\to M\to\bigoplus_i M_{x_i}\to\bigoplus_{i,j}M_{x_ix_j}\to\cdots\to M_{x_1\cdots x_N}\to 0.$$

A straightforward application of the definition of eigenvector implies (a) that each term in this complex is also in $\mathbf{Eig}(V, \lambda)$, and (b) that $\mathbf{Eig}(V, \lambda)$ is closed under taking subquotients. Hence, each $H^i_{\mathfrak{m}}(M)$ is in $\mathbf{Eig}(V, \lambda)$.

Since $H^i_{\mathfrak{m}}(M)$ is $\mathfrak{m}$-torsion, it remains to show that every $\mathfrak{m}$-torsion module in $\mathbf{Eig}(V, \lambda)$ is zero. Let M' be one such module, and assume there is a nonzero $n \in M'$. Without loss of generality, we may assume that $n \in M'_\mu$ for some $\mu \in \lambda + \mathbb{Z}$ and that $\mathfrak{m}n = 0$. Then

$$\mu n = E \cdot n = \sum_{i=0}^{N} x_i \partial_i \cdot n = \sum_{i=0}^{N} (\partial_i x_i - 1) \cdot n = -Nn.$$

Thus, because $n \neq 0$, α must be $-N$—in particular, α, and therefore also λ, must be an integer, which is false by assumption. Hence, $M' = 0$. □

Corollary 7.7. *Let $\rho \colon G' \to \mathrm{GL}(V)$ be a finite-dimensional rational representation of an algebraic group of the form $G' = G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts by scaling elements of V. Let $Y \subseteq V$ be a G'-orbit and let $\overline{Y}$ be its closure. Let $\beta \colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism with $\beta(\mathbf{e}) \in \mathbb{C} \setminus \mathbb{Z}$. Then the $\mathscr{D}_V$-module $\hat{\tau}(\rho, \overline{Y}, \beta)$ from Definition 1.1 satisfies Then*

$$\hat{\tau}(\rho, \overline{Y}, \beta) \cong j_+ j^+ \hat{\tau}(\rho, \overline{Y}, \beta),$$

where j denotes the open embedding of $V \setminus 0$ into V.

Proof. For brevity, we denote $\Gamma(V, \hat{\tau}(\rho, \overline{Y}, \beta))$ simply by $\hat{\tau}$. By Proposition 7.6, it suffices to prove that $\hat{\tau} \in \mathbf{Eig}(V, \beta(\mathbf{e}))$. To do this, let

$$P = \sum_{\alpha\gamma} c_{\alpha\gamma} x^\alpha \partial^\gamma$$

be a global section of $\mathscr{D}_V$. In $\hat{\tau}$, we have

$$\begin{aligned} EP &= \sum_{\alpha\gamma} c_{\alpha\gamma}(|\alpha| - |\gamma|) x^\alpha \partial^\gamma + \sum_{\alpha\gamma} c_{\alpha\gamma} x^\alpha \partial^\gamma E \\ &= \sum_{\alpha\gamma} c_{\alpha\gamma}(|\alpha| - |\gamma|) x^\alpha \partial^\gamma + \sum_{\alpha\gamma} \beta(\mathbf{e}) c_{\alpha\gamma} x^\alpha \partial^\gamma \\ &= \sum_{\alpha\gamma} (\beta(\mathbf{e}) + |\alpha| - |\gamma|) c_{\alpha\gamma} x^\alpha \partial^\gamma \\ &\in \bigoplus_{\mu \in \beta(\mathbf{e}) + \mathbb{Z}} \hat{\tau}_\mu. \end{aligned}$$

Thus, $\hat{\tau} \in \mathbf{Eig}(V, \beta(\mathbf{e}))$. □

7.3 Tautological systems as mixed Hodge modules

We now get back to the more specific situation as discussed in Theorem 7.4. Let us recall the relevant notation: We let X be a projective variety, and we consider an action of a connected algebraic group G on X that satisfies assumption $(\star)$ from the beginning of this section. We let $L_0 \to X$ be a very ample G-equivariant line bundle, and we consider the G'-equivariant line bundles $L_k \to X$ defined by a character $\rho_k \colon G' \to \mathbb{C}^*$, where

$G' = \mathbb{C}^* \times G$. Put $V := H^0(X, \mathscr{L}_{-1})^\vee$, and consider the closed embedding $X \hookrightarrow \mathbb{P}V$ defined by $|\mathscr{L}_{-1}|$. Finally, $\hat{X} \subset V$ is the affine cone of X in V, and we write $\iota\colon L^*_{-1} \cong \hat{X} \setminus 0 \to V$ for the locally closed embedding given as the composition of the closed embedding $i : \hat{X} \backslash 0 \hookrightarrow V \backslash 0$ with the canonical open embedding $j : V \backslash 0 \hookrightarrow V$. We let, as before, $\beta\colon \mathfrak{g}' \to \mathbb{C}$ be a Lie algebra homomorphism satisfying $\beta|_{\mathfrak{g}} \equiv 0$. Denote $W := V^\vee$.

Theorem 7.8. *Under the above hypotheses, assume that $\beta(\mathbf{e}) = \ell/k \in \mathbb{Q} \setminus \mathbb{Z}$. The tautological system $\tau(\rho, \hat{X}, \beta)$ is non-zero if and only if $\mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}$. In this case, we have an isomorphism*

$$\tau(\rho, \hat{X}, \beta) \cong \mathrm{FL}^V(\iota_+ \mathcal{O}_{L^*}^{\ell/k})$$

in $\mathrm{Mod}(\mathscr{D}_W)$*, and the $\mathscr{D}_W$-module $\tau(\rho, \hat{X}, \beta)$ underlies a complex pure Hodge module on W of weight* $\dim(X) + \dim(W)$.

Proof. Under the assumptions of the theorem, we have

$$j^+ \hat{\tau}(\rho, \hat{X}, \beta) \cong \begin{cases} i_+ \mathcal{O}_{L^*}^{\ell/k} & \text{if } \mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}, \\ 0 & \text{otherwise.} \end{cases}$$

by Theorem 7.4. Moreover, since $\ell/k \notin \mathbb{Z}$, we know from Corollary 7.7 that

$$\hat{\tau}(\rho, \hat{X}, \beta) \cong j_+ j^+ \hat{\tau}(\rho, \hat{X}, \beta).$$

Therefore, since $\iota = j \circ i$, we conclude that

$$\hat{\tau}(\rho, \hat{X}, \beta) \cong \begin{cases} \iota_+ \mathcal{O}_{L^*}^{\ell/k} & \text{if } \mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}, \\ 0 & \text{otherwise.} \end{cases}$$

As we have $\mathrm{FL}^V(\hat{\tau}(\rho, \hat{X}, \beta)) = \tau(\rho, \hat{X}, \beta)$ by Definition 1.1, we obtain

$$\tau(\rho, \hat{X}, \beta) \cong \begin{cases} \mathrm{FL}^V(\iota_+ \mathcal{O}_{L^*}^{\ell/k}) & \text{if } \mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}, \\ 0 & \text{otherwise,} \end{cases}$$

as required.

For the second statement, assume $\mathscr{L}_0^{\otimes \ell} \cong \omega_X^{\otimes(-k)}$ and recall from Proposition 3.8 that

$$\mathrm{FL}^V(\hat{\iota}_+ \mathcal{O}_{L^{\vee,*}}^{-\ell/k}) \cong a_{W,+} ev^\dagger j_{L,\dagger} \mathcal{O}_{L^*}^{\ell/k}$$

as elements in $D_h^b(\mathscr{D}_W)$, using the notations from Proposition 3.8. Here, $\hat{\iota}\colon L^{\vee,*} \hookrightarrow V$ is the composition of $\iota\colon L^* \hookrightarrow V$ with the isomorphism $inv\colon L^{\vee,*} \to L^*$ given by inverting the $\mathbb{C}^*$-fibers over X. Since we have $inv_+ \mathcal{O}_{L^{\vee,*}}^{-\ell/k} \cong \mathcal{O}_{L^*}^{\ell/k}$, we get

$$\mathrm{FL}^V(\iota_+ \mathcal{O}_{L^*}^{\ell/k}) \cong a_{W,+} ev^\dagger j_{L,\dagger} \mathcal{O}_{L^*}^{\ell/k}$$

in $D_h^b(\mathscr{D}_W)$. However, as we have just proved, this is actually a single degree module isomorphic to the tautological system $\tau(\rho, \hat{X}, \beta)$. Hence it follows from the second statement of Proposition 3.8 that this $\mathscr{D}_W$-module underlies the pure complex Hodge module

$${}^H\mathcal{M}_L^{\ell/k} = \mathcal{H}^0({}^H\mathcal{M}_L^{\ell/k}) = a_{W,*} ev^* j_{L,!} {}^H\underline{\mathbb{C}}_{L^*}^{\ell/k}[\dim W - 1]$$

which has weight $\dim(X) + \dim(W)$. ∎

As a consequence, we can show the following statement which can be considered as a generalization to the case of arbitrary line bundles of the proof in [HLZ16, Theorem 1.4] of the holonomic rank problem from [BHL$^+$14]. Recall from the discussion before Proposition 3.9 that $\mathcal{U} := (W \times X)\backslash ev^{-1}(0) \subset W \times X$ and that $a_{\mathcal{U}} \colon \mathcal{U} \to W$ denotes the restriction of the first projection $a_W : W \times X \to W$.

Corollary 7.9. *Under the assumptions of the previous Theorem 7.8, we have an isomorphism in* $\mathrm{Mod}_h(\mathscr{D}_W)$

$$\tau(\rho, \hat{X}, \beta) \cong a_{\mathcal{U},\dagger} ev^+_{|\mathcal{U}} \mathcal{O}^{\ell/k}_{L^*}.$$

As a consequence, the holonomic rank of the tautological system $\tau(\rho, \hat{X}, \beta)$ *is given as*

$$\dim H^{\dim(X)}_c(U_\lambda, \underline{\mathbb{C}}^{\ell/k}_\lambda),$$

for any value $\lambda \in W$ *that lies outside the singular locus of* $\tau(\rho, \hat{X}, \beta)$ *and where* U_λ *and* $\underline{\mathbb{C}}^{\ell/k}_\lambda$ *are as in Proposition 3.9.*

Proof. Using the previous Theorem 7.8, the first statement is exactly the $\mathscr{D}$-module version of Proposition 3.9, 1. Similarly, the second statement follows from Proposition 3.9.(ii). □

8 A formula for β using representation theory

We continue with the notations and assumptions of Theorem 7.4. Theorem 7.4 proves that $\hat{\tau}(\rho, \hat{X}, \beta)|_{V\backslash 0}$ is nonzero if and only if $\mathscr{L}_0$ is a rational power of the anticanonical bundle, and this rational power determines $\beta(\mathbf{e})$. Under the additional assumption that

$$G \text{ is semisimple,}$$

we can give an alternate formula for $\beta(\mathbf{e})$ in terms of the highest weight of the (necessarily irreducible by Borel–Weil, see e.g. [Ser54] or [Bot57, Cor. to Th. V]) G-representation $V := H^0(X, \mathscr{L}_{-1})^\vee$.

To state this formula, let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, let $\Phi^+ \subseteq \mathfrak{t}^\vee$ be a choice of positive roots, and set

$$\delta := \frac{1}{2} \sum_{\lambda \in \Phi^+} \lambda.$$

Theorem 8.1. *Let* μ *be the highest weight of the irreducible* G*-representation* $V := H^0(X, \mathscr{L}_{-1})^\vee$. *If* $\hat{\tau}(\rho, \hat{X}, \beta)|_{V\backslash 0}$ *is nonzero, then*

$$\beta(\mathbf{e}) = \frac{2\langle \delta, \mu \rangle}{|\mu|^2},$$

where $\langle -, - \rangle$ *is the inner product on* $\mathfrak{t}^\vee$ *dual to (the restriction to* $\mathfrak{t}$ *of) the Killing form, and* $|\mu| = \sqrt{\langle \mu, \mu \rangle}$.

We will give two proofs of Theorem 8.1.

8.1 Some notation

Lie groups/algebras

- $\mathfrak{g}' = \mathbb{C}\mathbf{e} \oplus \mathfrak{g}$ – the Lie algebra of G'
- T – a maximal torus of G
- B – a Borel subgroup of G containing T
- $\mathfrak{t}$ – the Lie algebra of T

Roots

- $\Phi(M, T)$ – the roots of an (affine) algebraic group M relative to a subtorus T. This is the set of characters $\lambda\colon T \to \mathbb{C}^*$ of T such that

$$\mathfrak{m}_\alpha := \{\xi \in \mathfrak{m} \mid \mathrm{Ad}(t)\xi = \lambda(t)\xi\} \neq 0,$$

  where $\mathfrak{m}$ is the Lie algebra of M. (Cf. [Hum75])

- $\Phi := \Phi(G, T)$ – the root system of G relative to T. As usual, we view this as a subset of $\mathfrak{t}^\vee$.
- $\Phi^+ := \Phi(B, T)$ – the choice of positive roots corresponding to B
- $\Delta \subseteq \Phi^+$ – the simple roots
- $\delta := \dfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ – the Weyl vector.
- $\mathsf{B}(-,-)$ – the Killing form on $\mathfrak{g}$
- $\langle -,- \rangle$ – the symmetric bilinear form on $\mathfrak{t}^\vee$ induced by the restriction to $\mathfrak{t}$ of the Killing form. Since $\mathfrak{g}$ is semisimple, $\langle -,- \rangle$ is nondegenerate.

Since $\mathfrak{g}$ is semisimple, there is a decomposition

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha} \right).$$

Each $\mathfrak{g}_\alpha$ is one-dimensional. In fact, one can choose a generator $E_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$ such that $[E_\alpha, E_{-\alpha}] =: H_\alpha \in \mathfrak{t}$, $\mathsf{B}(E_\alpha, E_{-\alpha}) = 1$, $[H_\alpha, E_\alpha] = 2E_\alpha$, and $[H_\alpha, E_{-\alpha}] = -2E_{-\alpha}$. Note that the H_α might not form a basis for $\mathfrak{t}$, as there may be too many of them.

A straightforward argument shows that for each $\alpha \in \Phi^+$, H_α is the unique element of $\mathfrak{t}$ for which $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{t}$. We can use this property to define H_λ for all $\lambda \in \mathfrak{t}^\vee$—then the nondegenerate bilinear form $\langle -,- \rangle$ is given by

$$\langle \lambda, \lambda' \rangle = \mathsf{B}(H_\lambda, H_{\lambda'}) = \lambda(H_{\lambda'}) = \lambda'(H_\lambda).$$

8.2 Representation-theoretic proof

This first proof of Theorem 8.1 will be delayed to Section 8.2.3. The Casimir element is defined in Section 8.2.1, and we compute its action on irreducible representations (Lemma 8.3). In Section 8.2.2, we discuss the bare bones of differential operators on affine varieties—the full power of such operators is not needed here, so we only touch on a very small bit of the theory.

8.2.1 The Casimir element

Definition 8.2. *The (second order)* Casimir element *is the element*

$$C := \sum_i A_i B_i \in \mathcal{U}(\mathfrak{g}),$$

where $\{A_i\}$ is any basis for $\mathfrak{g}$, and $\{B_i\}$ is the dual basis under the Killing form. In particular, if $\{H_i\}$ is an orthonormal basis of $\mathfrak{t}$ with respect to the Killing form, then

$$C = \sum_i H_i^2 + \sum_{\alpha\in\Phi^+} E_\alpha E_{-\alpha} + \sum_{\alpha\in\Phi^+} E_{-\alpha}E_\alpha.$$

A straightforward exercise shows that C is in the center of $\mathcal{U}(\mathfrak{g})$.

Lemma 8.3. *Let U be an irreducible representation of $\mathfrak{g}$ with lowest weight λ and lowest weight vector v_λ. Then*

$$C \cdot v_\lambda = \big(|\lambda|^2 - 2\langle\delta,\lambda\rangle\big)v_\lambda.$$

Proof. Then

$$\begin{aligned} C\cdot v_\lambda &= \left(\sum_i H_i^2 + \sum_{\alpha\in\Phi^+} E_\alpha E_{-\alpha} + \sum_{\alpha\in\Phi^+} E_{-\alpha}E_\alpha\right)\cdot v_\lambda \\ &= \left(\sum_i H_i^2 - \sum_{\alpha\in\Phi^+} H_\alpha + 2\sum_{\alpha\in\Phi^+} E_\alpha E_{-\alpha}\right)\cdot v_\lambda \\ &= \left(\sum_i H_i^2 - 2H_\delta\right)\cdot v_\lambda, \end{aligned}$$

since E_α kills v_λ. Now use that $\sum_i H_i^2 \cdot v_\lambda = \sum_i \lambda(H_i)v_i = |\lambda|^2 v_i$ and $H_\delta \cdot v_\lambda = \lambda(H_\delta)v_\lambda$. ☐

8.2.2 Differential operators on affine varieties

Set

$$R := \Gamma(V,\mathcal{O}_V), \quad D_V := \Gamma(V,\mathscr{D}_V), \quad \text{and} \quad S := R/I,$$

where I is the defining ideal of $\hat{X}$.

Define

$$\begin{aligned} A &:= \{P \in D_V \mid P(I) \subseteq I\} \\ J &:= \{P \in D_V \mid P(R) \subseteq I\} = \sum_{\alpha \in \mathbb{N}^n} I\partial^\alpha \\ D_S &:= \operatorname{im}(\psi), \end{aligned}$$

where $\psi\colon A \to \operatorname{End}_{\mathbb{C}}(S)$ via $\psi(P) \bullet \overline{f} = \overline{P \bullet f}$. Note that it is *not* immediately obvious that this D_S is the same as the ring of Grothendieck differential operators of S over $\mathbb{C}$—that it is is the content of the surjectivity part of [Mil99, 1.2. Prop.]. That said, for our application in the proof of Theorem 8.1, we will start with a particular element of A, and we will need to show that it is in fact in J. In other words, we need the following

Lemma 8.4 ([Mil99, part of 1.2. Prop.]). $\ker\psi = J$. *In particular,* $\ker\psi \subseteq ID_V$.

8.2.3 Proof of Theorem 8.1

Under the assumption that G is semi-simple, the definition of $\hat{\tau}(\rho, \hat{X}, \beta)$ simplifies to

$$\hat{\tau}(\rho, \hat{X}, \beta) = \mathscr{D}_V / \big(\mathscr{D}_V I + \mathscr{D}_V(Z(\xi) \mid \xi \in \mathfrak{g}) + \mathscr{D}_V(Z(\mathbf{e}) - \dim V + \beta(\mathbf{e}))\big).$$

Here, we denote by $Z(\xi)$ the vector field $Z_V(\xi)$ defined in Section 4.1 and we will also denote by Z the map $\mathcal{U}(\mathfrak{g}') \to D_V$ extending it.

Because X is G-invariant, the ideal I is $\mathfrak{g}'$-stable, i.e. $Z(\xi)(I) \subseteq I$ for all $\xi \in \mathfrak{g}'$. Hence, the map Z induces a $\mathfrak{g}'$-module structure on S for which the elements of $\mathfrak{g}'$ act via derivations. If S_d is the dth graded component of S, then

$$\xi \cdot S_d \subseteq S_d \quad \text{for all } \xi \in \mathfrak{g},$$

and

$$\mathbf{e} \cdot f = -df \quad \text{for all } f \in S_d. \tag{12}$$

Denote the induced map $\mathcal{U}(\mathfrak{g}') \to \operatorname{End}_{\mathbb{C}}(S)$ by Z_S.

Lemma 8.5. $Z_S(C) = Z_S(\mathbf{e})^2|\mu|^2 - 2Z_S(\mathbf{e})\langle\delta, \mu\rangle$.

Proof. By definition, $R_1 = V^\vee$. The construction of the embedding $X \hookrightarrow \mathbb{P}V$ implies that $R_1 = S_1$ also. Hence, if $x \in S_1 \cong V^\vee$ is a lowest weight vector, it has lowest weight $-\mu$ (recall that μ is the *highest* weight of V). A straightforward argument shows that for all $d \in \mathbb{N}$, the element x^d is a lowest weight vector of S_d with lowest weight $-d\mu$. Hence, by Lemma 8.3,

$$C \cdot x^d = (|-d\mu|^2 - 2\langle\delta, -d\mu\rangle)x^d.$$

Since C is in the center of the universal enveloping algebra, it acts on the irreducible $\mathfrak{g}$-representation S_d as a scalar, which then must be the factor on the right hand side. Now use that $\mathbf{e}$ acts on S_d as multiplication by $-d$ (eq. (12)). □

By definition, the operators $Z(C)$ and $Z(\mathbf{e})^2|\mu|^2 - 2Z(\mathbf{e})\langle\delta,\mu\rangle$ are contained in the subalgebra A from Section 8.2.2. By Lemma 8.5, their difference is in the kernel of the map ψ from Section 8.2.2. Hence, by Lemma 8.4, we know that

$$Z(C) - \big(Z(\mathbf{e})^2|\mu|^2 - 2Z(\mathbf{e})\langle\delta,\mu\rangle\big) \in ID_V.$$

Applying the standard D-module transpose $(-)^\top$ and identifying $Z(\mathbf{e})$ with minus the Euler differential operator[1] $-E$ gives

$$(Z(C))^\top - \big((E+\dim V)^2|\mu|^2 - 2(E+\dim V)\langle\delta,\mu\rangle\big) \in D_V I, \tag{13}$$

since $(-E)^\top = E + \dim V$ and I is homogeneous.

Lemma 8.6. *$(Z(C))^\top = Z(C)$.*

Proof. Because $\mathfrak{g}$ is semisimple, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Hence, $\mathfrak{g}$ acts on V via trace-free matrices. Let $\xi \in \mathfrak{g}$ act on V via a square matrix $A = [a_{ij}]$. Then $Z(\xi)$ is the derivation $-\sum_{i,j} a_{ji}x_i\partial_{x_j}$. The transpose of this operator is then $\sum a_{ji}\partial_{x_j}x_i = \sum a_{ji}x_i\partial_{x_j} + \mathrm{Tr}(A) = -Z(\xi)$. As C arises as evaluation on elements of $\mathfrak{g}$ of a homogeneous quadric, $(Z(C))^\top = Z(C)$. □

Combining Lemma 8.6 and eq. (13) yields

$$Z(C) - \big((E+\dim V)^2|\mu|^2 - 2(E+\dim V)\langle\delta,\mu\rangle\big) \in D_V I.$$

Taking cosets in $\hat{\tau}(\rho,\hat{X},\beta)$, we find

$$\begin{aligned} 0 = \overline{Z(C)} &= \overline{(E+\dim V)^2|\mu|^2 - 2(E+\dim V)\langle\delta,\mu\rangle} \\ &= \overline{(E+\dim V)\big((E+\dim V)|\mu|^2 - 2\langle\delta,\mu\rangle\big)}. \end{aligned}$$

On the other hand, the defining ideal of $\hat{\tau}(\rho,\hat{X},\beta)$ also contains

$$Z(\mathbf{e}) - \dim V + \beta(\mathbf{e}) = -E - \dim V + \beta(\mathbf{e})$$

So, we deduce

$$\beta(\mathbf{e}) \in \left\{0, \frac{2\langle\delta,\ \mu\rangle}{|\mu|^2}\right\}.$$

It remains to show that $\beta(\mathbf{e}) \neq 0$. But this follows from Theorem 7.4, because $\mathscr{L}_0^{\otimes 0} \cong \mathcal{O}_X$ is never isomorphic to a (nonzero) power of ω_X. This completes the proof of Theorem 8.1.

8.3 Direct proof

The second proof of Theorem 8.1 will be delayed to Section 8.3.3. In Section 8.3.1, we recall some facts about parabolic subgroups. In Section 8.3.2, we recall the correspondence between characters of P and equivariant line bundles on G/P, where P is parabolic, and identify the character corresponding to the canonical bundle $\omega_{G/P}$ (Lemma 8.9).

[1] The minus sign comes from the fact that the action of $\mathfrak{g}$ on the coordinate ring of V is the contragredient action.

8.3.1 Parabolic subgroups

Recall ([Hum75, §21.3]) that a closed subgroup $P \leq G$ is called *parabolic* if G/P is projective. We recall some facts about parabolic subgroups.

Lemma 8.7.

(a) *If X is a projective homogeneous G-space, then $X \cong G/P$ for some parabolic subgroup P of G containing B.*

(b) *There is an inclusion-preserving bijection between subsets $I \subseteq \Delta$ and parabolic subgroups P_I containing B.*[2]

(c) $\Phi(P_I, T) = \Phi^+ \cup (\Phi^- \cap \mathbb{Z}I)$.

Proof. (a) This is standard. It uses that every parabolic subgroup contains a Borel subgroup ([Hum75, Cor. 21.3.B.]), and that all Borel subgroups are conjugate ([Hum75, Th. 21.3]).

(b) [Hum75, Th. 29.3].

(c) [Hum75, Th. 30.1]. □

8.3.2 Equivariant line bundles on homogeneous spaces

Let P be a parabolic subgroup of G containing a maximal torus T. Recall that the characters $\lambda\colon T \to \mathbb{C}^*$ which are extendable to P correspond one-to-one with G-equivariant line bundles $L_{\lambda,P}$ on G/P; see, e.g., [HTT08, §9.11] (although the argument there is for $P = B$, the same argument works verbatim with N^- replaced by the unipotent radical of the parabolic subgroup of G opposite P). Note that there are two common conventions for this correspondence—we choose the convention for which P acts on the fiber of $L_{\lambda,P}$ at P as $b \cdot v = \lambda(b)v$;[3] In this case, the sheaf of sections $\mathscr{L}_{\lambda,P}$ of $L_{\lambda,P}$ is given by

$$\Gamma(U, \mathscr{L}_{\lambda,P}) = \{f \in \Gamma(q^{-1}(U), \mathscr{O}_G) \mid f(gb) = \lambda(b)^{-1} f(g) \text{ for all } g \in G, b \in P\}, \qquad (14)$$

where $q\colon G \to G/P$ is the quotient map. Since $L_{\lambda,P}$ is G-equivariant, there is a G-equivariant structure on $\mathscr{L}_{\lambda,P}$. Although we won't need to know this structure explicitly, it may help the reader to note that the induced action of G on $\Gamma(G/P, \mathscr{L}_{\lambda,P})$ is given by

$$(g \cdot f)(g') = f(g^{-1}g') \qquad (g, g' \in G,\ f \in \Gamma(G/P, \mathscr{L}_\lambda)).$$

There are many proofs of the following theorem throughout the literature. It is often stated and proved only for $P = B$. However, it was originally proven for all parabolic subgroups, e.g. in [Ser54] or [Bot57, Cor. to Th. V].

Theorem 8.8 (Borel–Weil)**.** *If $-\lambda$ is a dominant weight which is extendable to the parabolic subgroup P, then $\Gamma(G/P, \mathscr{L}_{\lambda,P})^\vee$ is the irreducible representation of G with highest weight $-\lambda$.*

[2]Although we won't need it, the actual definition of P_I can be found directly above Th. 29.2 in [Hum75].

[3]The other convention is $b \cdot v = \lambda(b)^{-1}v$.

Lemma 8.9. *Let $I \subseteq \Delta$ be a subset of the set of simple roots, and let P_I be the corresponding parabolic subgroup. Define*

$$\delta_I := \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus \mathbb{Z}I} \alpha.$$

Then

$$\omega_{G/P_I} \cong \mathscr{L}_{2\delta_I, P_I}.$$

Proof. The following argument is based on the argument given in the MathOverflow post [Sco].

According to [CG10, Lem. 1.4.9], $T^*(G/P_I)$ is the (unique) G-equivariant vector bundle on G/P_I whose fiber over P_I is the P_I-module

$$\mathfrak{p}_I^\perp := \{\xi \in \mathfrak{g} \mid \langle \xi, x \rangle = 0 \text{ for all } x \in \mathfrak{p}_I\},$$

where P_I acts via the coadjoint action. But, letting T be a maximal torus of G contained in P_I, we have a sequence of P_I-isomorphisms

$$\begin{aligned} \mathfrak{p}_I^\perp &\cong (\mathfrak{g}/\mathfrak{p}_I)^\vee \\ &\cong \left(\bigoplus_{\substack{\alpha \in \Phi \text{ not a root} \\ \text{of } P_I \text{ rel. to } T}} \mathfrak{g}_\alpha \right)^\vee \\ &\cong \left(\bigoplus_{\alpha \in -(\Phi^+ \setminus \mathbb{Z}I)} \mathfrak{g}_\alpha \right) \\ &\cong \bigoplus_{\alpha \in \Phi^+ \setminus \mathbb{Z}I} \mathfrak{g}_\alpha. \end{aligned}$$

Taking the determinant gives the P_I-equivariant line bundle whose fiber at P_I is the P_I-module

$$\bigotimes_{\alpha \in \Phi^+ \setminus \mathbb{Z}I} \mathfrak{g}_\alpha.$$

The action of P_I on this module is determined by the action of T, and the action of $t \in T$ is just multiplication by

$$\sum_{\alpha \in \Phi^+ \setminus \mathbb{Z}I} \alpha(t) = 2\delta_I(t).$$

Thus, $\omega_{G/P_I} \cong \mathscr{L}_{2\delta_I, P_I}$. ☐

8.3.3 Proof of Theorem 8.1

We need one more technical lemma before beginning the proof.

Lemma 8.10. *Let $I \subseteq \Delta$ be a subset of the simple roots. Then $\langle \delta, \delta_I \rangle = \langle \delta_I, \delta_I \rangle$.*

Proof. Set $\delta_I' \coloneqq \delta - \delta_I = \frac{1}{2}\sum_{\alpha \in \Phi^+ \cap \mathbb{Z}I} \alpha$. We want to show that $\langle \delta_I', \delta_I \rangle = 0$. To begin with, let Φ_I be $\Phi \cap \mathbb{R}I$ viewed as a subset of the vector space $\mathbb{R}I$. It is immediately clear that Φ_I is a root system (in $\mathbb{R}I$), and that I forms a base of Φ_I. Hence, δ_I' is the Weyl vector δ_{Φ_I} of Φ_I (with respect to this base). Moreover, the inner product of two elements of $\mathbb{R}I$ is the same as in the ambient vector space $\mathfrak{t}^\vee$ of the root system Φ. So, the coroots of Φ_I are the coroots $H_\alpha \coloneqq 2\alpha/\langle \alpha, \alpha \rangle$ of Φ for $\alpha \in \Phi_I$. Therefore, by [Hal15, Prop. 8.38],

$$\langle \delta_I', H_\alpha \rangle = \langle \delta_{\Phi_I}, H_\alpha \rangle = 1$$

for all $\alpha \in I$. Hence, for all $\alpha \in I$, we have

$$\begin{aligned} \frac{2\langle \alpha, \delta_I \rangle}{\langle \alpha, \alpha \rangle} &= \langle H_\alpha, \delta_I \rangle \\ &= \langle H_\alpha, \delta - \delta_I' \rangle \\ &= \langle H_\alpha, \delta \rangle - \langle H_\alpha, \delta_I' \rangle \\ &= 1 - 1 = 0, \end{aligned}$$

where the final equality again uses [Hal15, Prop. 8.38]. Therefore, $\langle \alpha, \delta_I \rangle = 0$ for all $\alpha \in I$ and hence for all $\alpha \in \mathbb{R}I$. In particular, $\langle \delta_I', \delta_I \rangle = 0$. □

We now begin the proof of Theorem 8.1. Since X is a projective G-homogeneous space, it is isomorphic by Lemma 8.7 to G/P_I for some I. By Theorem 7.4, we know that $\mathscr{L}_{-1}^{\otimes \ell} \cong \omega_X^{\otimes(-k)}$ as G-equivariant line bundles for some integers $k, \ell \neq 0$. Therefore, since $\operatorname{Pic}(X)$ and therefore $\operatorname{Pic}^G(X)$ are torsion-free, and applying Lemma 8.9,

$$\mathscr{L}_{-1} \cong \mathscr{L}_{-\frac{k}{\ell}2\delta_I, P}.$$

Therefore, by Borel–Weil (Theorem 8.8), the representation $V = \Gamma(X, \mathscr{L}_0)^\vee$ has highest weight $\mu = -(-\frac{2k}{\ell}\delta_I) = \frac{2k}{\ell}\delta_I$. Then

$$\frac{2\langle \delta, \mu \rangle}{\langle \mu, \mu \rangle} = \frac{\ell \langle \delta, \delta_I \rangle}{k \langle \delta_I, \delta_I \rangle} = \frac{\ell \langle \delta_I, \delta_I \rangle}{k \langle \delta_I, \delta_I \rangle} = \frac{\ell}{k} = \beta(\mathbf{e}).$$

This completes the proof of Theorem 8.1.

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