Generalized Fourier transforms for $\mathscr{D}$-modules and applications

Reading group at TU Chemnitz

Spring 2026

1 Introduction

The goal of this reading group is to understand the Fourier transform on connected commutative algebraic group over $\mathbb{C}$, following [Lau85].

2 Outline of the talks

The Fourier transform on vector spaces and on affine tori. Introduce the Fourier transform for $\mathscr{D}$-modules on vector spaces and affine tori, and explain their basic properties following [Lau85, §1 - 2].

Applications for affine tori: Hypergeometric $\mathscr{D}$-modules. Use the Mellin transform from the previous talk to reprove a result by Katz: Every simple holonomic $\mathscr{D}$-module of Euler characteristic one on the 1-dimensional torus $\mathbb{C}^*$ is hypergeometric [LS91, th. 1]. Then present the generalization of this result to higher-dimensional affine tori following [LS91, th. 2 and erratum].

Abelian varieties: Fourier-Mukai for coherent sheaves. Introduce the Fourier-Mukai transform between the derived categories of coherent sheaves on an abelian variety and on the dual abelian variety, following [REF] (optional).

Abelian varieties: Fourier-Mukai for $\mathscr{D}$-modules. Recall the definition of $A^\natural$ and the Fourier-Mukai transformation for $\mathscr{D}$-modules on abelian varieties, following [Lau85, sec. 3] and [Sch15, sec. 8-9]. Show as in [Sch15, th. 9.3] that it induces an equivalence of categories $\mathrm{FM}_A : \mathrm{D}_{\mathrm{coh}}(\mathscr{D}_A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{A^\natural})$, and recall its basic properties [Sch15, th. 9.5].

Applications for abelian varieties I: The structure theorem. The goal of this talk is to prove [Sch15, th. 2.2]. Following [Sch15, sec. 13], define the analog of the Fourier-Mukai transform for constructible sheaves and show that the associated cohomology support loci are algebraic [Sch15, th. 13.6]. Then show the two comparison theorems [Sch15, th. 14.1 and 14.2]. Finally, prove

that the $\mathscr{D}$-module cohomological support loci are algebraic [Sch15, Prop. 15.2] and conclude via Simpson's Theorem 15.1.

Applications for abelian varieties II: Perverse coherent sheaves. The goal of this talk is to prove that the Fourier transform of holonomic $\mathscr{D}$-modules are perverse coherent sheaves. Briefly recall this notion [Sch15, sec. 17], and then prove [Sch15, th. 18.1]. If time permits, we can talk about how the truncation preserves the image of holonomic modules under the Fourier transformation, or discuss conjecture 6.1.

Fourier transform for extensions of an abelian variety by a product of a torus and a vector space. The goal of this talk is to define the Fourier transform $\mathrm{F} : \mathrm{D}^{\mathrm{b}}_{\mathrm{qcoh}}(D_G) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{qcoh}}(D_{G^\natural/(A^\natural,T^\natural)})$ when G is the extension of an abelian variety by the product of a torus and a vector space, and to show that it gives an equivalence of categories [Lau85, th. 6.3]. First treat the case of the extension of an abelian variety by a vector space [Lau85, sec. 4], then by a torus [Lau85, sec. 5], and then do the general case [Lau85, sec. 6].

A unifying framework: 1-motives. The goal of this talk is to define generalized 1-motives and the generalized Fourier transform [Lau85, sec. 7]; this might be split in two talks depending on the amount of details that we want to give. As a motivation, explain how the dualizing functor of [Lau85, Th. 7.3] unifies the definition of $G^\natural$ when G is an abelian variety, a torus, or a vector space (see the examples on page 25).

References

[Lau85] G. Laumon. *Transformation de Fourier géometrique*. Preprint. Unpublished manuscript. IHES, 1985.

[LS91] F. Loeser and C. Sabbah. "Caractérisation des $\mathscr{D}$-modules hypergéométriques irréductibles sur le tore. (Characterization of irreducible $\mathcal{D}$- modules on the torus)". In: *C. R. Acad. Sci., Paris, Sér. I* 312.10 (1991), pp. 735–738.

[Sch15] C. Schnell. "Holonomic D-modules on abelian varieties". In: *Publ. Math., Inst. Hautes Étud. Sci.* 121 (2015), pp. 1–55.