Unobstructed lagrangian deformations

Déformations lagrangiennes non-obstruées

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Abstract

We prove that deformations of a lagrangian singularity are unobstructed if the usual (flat) deformations are unobstructed and if a cohomological vanishing condition is satisfied. This gives another application to deformation theory of the lagrangian de Rham complex introduced in [8]. To prove our theorem, we use the T^1-lifting criterion due to Ran, Kawamata and others.

Résumé

On démontre que les déformations d'une singularité lagrangienne ne sont pas obstruées si le foncteur des déformations plates est lisse et si une condition d'annulation cohomologique est satisfaite. Ceci donne une autre application à la théorie de déformations du complexe de de Rham lagrangien introduit dans [8]. Le théorème est prouvé grâce à un critère de relèvement de déformations infinitésimales dû à Ran, Kawamata et d'autres.

Version française abrégée

On considère dans cette note le problème suivant : Soit $(L,0) \subset (\mathbb{C}^{2n},0)$ un germe d'une sous-variété lagrangienne, c'est-à-dire un germe réduit d'un espace complexe L tel que $\omega_{L_{reg}} = 0$ où ω est une forme symplectique holomorphe sur $\mathbb{C}^{2n}$. Une déformation lagrangienne de $(L,0)$ sur $(S,0)$, au sens de [8] est un germe d'une famille plate $(\mathcal{L},0)$ au dessus de $(S,0)$ qui est plongé dans $\mathbb{C}^{2n}\times S$ (défini par un idéal $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^{2n}\times S}$) tel que on a $\{\mathcal{I},\mathcal{I}\} \subset \mathcal{I}$, où $\{\,,\,\}$ est le crochet de Poisson sur $\mathbb{C}^{2n}\times S$ induit par la forme symplectique sur $\mathbb{C}^{2n}$. On peut définir un foncteur de déformation (qui va de la catégorie des $\mathbb{C}$-algèbres artiniennes vers la catégorie d'ensembles) associé à cette situation ; ce foncteur sera noté $LagDef_{L,0}$ par

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la suite. Le problème traité ici consiste à savoir sous quelles conditions ce foncteur est lisse au sens de Grothendieck. Rappelons qu'un foncteur F est lisse (ou non-obstrué) si pour chaque petite extension de $\mathbb{C}$-algèbres artiennes $B \to A$, l'application induite $LagDef_{L,0}(B) \to LagDef_{L,0}(A)$ est surjective.

Dans [8] et [7], pour chaque singularité lagrangienne, ou plus généralement pour chaque famille lagrangienne $(\mathcal{L}, 0)$ (plate sur $(S, 0)$), un complexe de $\mathcal{O}_{\mathcal{L},0}$-modules à différentielle $\mathcal{O}_{S,0}$-linéaire a été introduit. Ce complexe sera appelé *complexe de de Rham lagrangien* et on le note $\mathcal{C}^{\bullet}_{\mathcal{L},0}$. Il peut être vu comme une généralisation du complexe de de Rham d'un $\mathcal{D}$-module cohérent (d'où son nom). Le résultat principal de [8] est l'interprétation du premier groupe de cohomologie de ce complexe (dans le cas d'une seule singularité $(L, 0)$) comme espace tangent du foncteur $LagDef_{L,0}$. La généralisation de ce résultat au cas relatif se fait sans aucune difficulté ([7]). Dans cette note, nous démontrons le résultat suivant.

Théorème 0.1 *Soit* $(L, 0) \subset (\mathbb{C}^{2n}, 0)$ *une singularité lagrangienne telle que le foncteur* $Def_{L,0}$ *de déformations plates de* $(L, 0)$ *soit lisse et que* $H^2(\mathcal{C}^{\bullet}_{L,0}) = 0$. *Alors* $LagDef_{L,0}$ *est aussi lisse.*

Pour prouver ce théorème, on fait appel à une technique introduite par Ran et Kawamata. Il s'agit de démontrer une propriété de relèvement des espaces tangents relatifs associés aux familles sur des bases artiniennes. Plus précisément, si on note $A_n := k[\epsilon]/(\epsilon^{n+1})$ et $B_n := k[x, y]/(x^{n+1}, y^2)$, ce critère affirme qu'un foncteur F de la catégorie des k-algèbres artiniennes vers la catégorie des ensembles satisfaisant les conditions H1 et H2 de Schlessinger ([6]) est lisse si et seulement si le morphisme naturel

$$F(B_{n+1}) \longrightarrow F(B_{n-1}) \times_{F(A_{n-1})} F(A_n)$$

est surjectif. Etant donnée une famille $X_n \in F(A_n)$, ceci est équivalent à la surjectivité du morphisme $T^1_F(X_n/A_n) \to T^1_F(X_{n-1}/A_{n-1})$. Ici on note $T^1_F(X_i/A_i)$ (i arbitraire) l'espace tangent relatif de F de la famille X_i/A_i et $X_{n-1} \in F(A_{n-1})$ la restriction naturelle de X_n induite par la projection $A_n \to A_{n-1}$.

D'après ce qui a été dit plus haut, pour appliquer ce critère on est donc amené à montrer que si $(\mathcal{L}_n, 0) \subset (\mathbb{C}^{2n} \times Spec(A_n), 0)$ est une déformation lagrangienne sur A_n, le morphisme

$$H^1(\mathcal{C}^{\bullet}_{\mathcal{L}_n,0}) \longrightarrow H^1(\mathcal{C}^{\bullet}_{\mathcal{L}_{n-1},0})$$

est surjectif. Pour cela, nous procédons en deux étapes. On vérifie d'abord que le morphisme correspondant au niveau des complexes $\mathcal{C}^{\bullet}$, c'est-à-dire le morphisme $\mathcal{C}^1_{\mathcal{L}_n,0} \to \mathcal{C}^1_{\mathcal{L}_{n-1},0}$ est surjectif. Ceci est évident sous les hypothèses du théorème, vu que le terme $\mathcal{C}^1$ du complexe lagrangien s'identifie au module normal de L. On en déduit une suite exacte en cohomologie qui permet d'obtenir la propriété de relèvement dès que $H^2(\mathcal{C}^{\bullet}_{L,0})$ s'annule.

1. Introduction

The first step towards a deformation theory of singular lagrangian varieties has been done in [8]. While the tangent spaces of such deformations can be described using the lagrangian de Rham complex introduced in that paper, the problem to know whether an infinitesimal lagrangian deformation extends to higher order (i.e., whether the corresponding local moduli space is smooth at the given point) remains open to a large extent. In fact, this problem is the local analogue of a result in global complex geometry due to Voisin ([10]), namely, that deformations of smooth complex lagrangian submanifolds in holomorphic symplectic compact Kähler manifolds are unobstructed. In this note we give a criterion for unobstructedness of the deformation space which involves the second cohomology group of the lagrangian de Rham complex. In fact, we suppose that usual (flat) deformations are unobstructed. Thus the criterion really measures the obstructions coming from the symplectic nature of the deformation problem. The technical tool used is the so called T^1-lifting principle, which was introduced by Ran and Kawamata to prove smoothness of the moduli space of Calabi-Yau manifolds and further developed by Gross ([5]) and Fantechi and Manetti ([3]).

2. Infinitesimal deformations

The crucial ingredient to apply the T^1-lifting criterion to a deformation problem is a description of the relative tangent space of the deformation functor for a given family $\mathcal{X} \to S$ over *any* base S (in particular if S is artinian). For a lagrangian singularity $(L,0) \subset (\mathbb{C}^{2n},0)$, a cohomological description of the space of infinitesimal lagrangian deformations has been given in [8]. In [7], this description has been extended to the relative case using some generalities from the theory of Lie algebroids. More precisely, the relative tangent space we are interested in occurs as first cohomology group of a complex which can be seen as a symplectic analogue of the de Rham complex of a coherent $\mathcal{D}$-module. We will recall this construction briefly.

Let $f : (\mathcal{L},0) \to (S,0)$ be a germ of a family of lagrangian singularities. This means the following: first there is an embedding $(\mathcal{L},0) \hookrightarrow (\mathbb{C}^{2n} \times S,0)$ such that f is the restriction of the second projection. We denote by $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^{2n},0}\widehat{\otimes}\mathcal{O}_{S,0}$ the defining ideal of $(\mathcal{L},0)$. We see $\mathbb{C}^{2n}$ as holomorphic symplectic manifold with symplectic form ω which induces a Poisson bracket on $\mathcal{O}_{\mathbb{C}^{2n},0}$ and on $\mathcal{O}_{\mathbb{C}^{2n},0}\widehat{\otimes}\mathcal{O}_{S,0}$ making the product $\mathbb{C}^{2n} \times S$ into a Poisson manifold. Then we require the family to be flat of dimension n and and $\mathcal{I}$ to be *involutiv*, i.e. $\{\mathcal{I},\mathcal{I}\} \subset \mathcal{I}$.

This condition ensures that we have brackets

$$\{\ ,\ \} : \mathcal{I}/\mathcal{I}^2 \times \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2$$
$$\{\ ,\ \} : \mathcal{I}/\mathcal{I}^2 \times \mathcal{O}_{\mathcal{L},0} \longrightarrow \mathcal{O}_{\mathcal{L},0}$$

where $\mathcal{I}/\mathcal{I}^2$ is the conormal module of $\mathcal{L}$. Moreover, $\mathcal{I}/\mathcal{I}^2$ acts on $\mathcal{O}_{\mathcal{L}}$ by derivations, i.e., we have $\{i, h\cdot g\} = \{i,h\}\cdot g + h\cdot\{i,g\}$ for any $i \in \mathcal{I}/\mathcal{I}^2$, $h,g \in \mathcal{O}_{\mathcal{L}}$. We can use this structure to define the main object we are working with in this note, the lagrangian de Rham complex.

Definition 2.1 *Set*

$$\mathcal{C}^p_{\mathcal{L},0} := \mathcal{H}om_{\mathcal{O}_{\mathcal{L},0}}\left(\bigwedge^p \mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{L},0}\right)$$

and define a ($\mathcal{O}_{S,0}$-linear) differential $\delta : \mathcal{C}^p_{\mathcal{L},0} \to \mathcal{C}^{p+1}_{\mathcal{L},0}$ *by*

$$\begin{aligned}(\delta(\phi))(h_1\wedge\ldots\wedge h_{p+1}) \;&:=\; \sum_{i=1}^{p+1}(-1)^i\left\{h_i, \phi\left(h_1\wedge\ldots\wedge\widehat{h}_i\wedge\ldots h_{p+1}\right)\right\}\\ +\sum_{1\le i<j\le p+1}&(-1)^{i+j-1}\,\phi\left(\{h_i,h_j\}\wedge h_1\wedge\ldots\wedge\widehat{h}_i\wedge\ldots\wedge\widehat{h}_j\wedge\ldots\wedge h_{p+1}\right)\end{aligned}$$

Let us point out that the structure on the conormal module described above is known under the name *Lie algebroid.* There is an extension of $\mathcal{D}$-module theory to Lie algebroids, which can be used in our case to give a more intrinsic definition of the complex $\mathcal{C}^\bullet_{\mathcal{L}}$. We refer to [7] for more details.

The following theorem describes one of the main results of [8] and [7], namely, the relationship of the deformation theory of the family $\mathcal{L} \to S$ with the cohomology of $\mathcal{C}^\bullet_{\mathcal{L},0}$.

Theorem 2.2 – *The space of infinitesimal deformations of the family $\mathcal{L} \to S$ denoted by $T^1_{LagDef}(\mathcal{L}/S)$ is given by $H^1(\mathcal{C}^\bullet_{\mathcal{L},0})$. Note that this is only an $\mathcal{O}_{S,0}$-module.*

– *Suppose that the following condition holds for each fibre $L_s := f^{-1}(s)$ (here we choose a representative for the germ of f): Let $\{0\} = L_s^{(0)} \subset L_s^{(1)} \subset \ldots \subset L_s^{(n)}$ be the canonical stratification by embedding dimension, i.e., $L_s^{(i)} = \{x \in L_s \,|\, \mathrm{embdim}(L_s,x) = 2n-i\}$, then $\dim(L_s^{(i)}) \le i$. Under this condition (which was called “Condition P” in [8]) $H^1(\mathcal{C}^\bullet_{\mathcal{L},0})$ is finitely generated over $\mathcal{O}_{S,0}$.*

Note that the finiteness theorem was stated and proved in [8] only in the absolute case. As explained in [7], the relative case can be treated by the same methods giving the above result. This has also be done in the preprint [4], where another application of the finiteness of $H^1(\mathcal{C}^\bullet_{\mathcal{L},0})$ is proved.

For a single lagrangian singularity $(L,0) \subset (\mathbb{C}^{2n},0)$ this implies using Schlessinger's theorem ([6]) that a formal semi-universal deformation exists if the condition on the dimension of the strata is satisfied.

3. The T^1-lifting principle

Before coming to our theorem, we will first recall a version of the T^1-lifting criterion as stated in [3]. The criterion concerns functors from the catogery of Artin rings (which we suppose to be $\mathbb{C}$-algebras for our purpose) to sets satisfying Schlessingers conditions (H1) and (H2). We use the following notations, let $A_n := k[\epsilon]/(\epsilon^{n+1})$, and $B_n := k[x,y]/(x^{n+1},y^2)$. Then the T^1-lifting criterion takes the form:

Lemma 3.1 *Let F be a functor satisfying H1 and H2. Suppose that for given $n>0$, the natural map*

$$F(B_{n+1}) \longrightarrow F(B_{n-1}) \times_{F(A_{n-1})} F(A_n)$$

is surjective. Then also $F(A_{n+1}) \to F(A_n)$ is surjective.

The lemma shows that a functor F satisfying the above condition for any n does not have so-called curvilinear obstructions. This implies its unobstructedness, if F has a hull, this is obvious, otherwise, it follows from the factorization theorem of Fantechi and Manetti (see [2]).

In order to use this result, we need the following reinterpretation. Let F be a functor as above, $X_n \in F(A_n)$ and let $T^1_F(X_n/A_n) := \{Y_n \in F(B_n) \,|\, F(\pi_n)(Y_n) = X_n\}$ where $\pi_n : B_n \to A_n$ is the canonical projection. Of course, this is nothing else than the relative tangent space of the family $X_n \to Spec(A_n)$.

Corollary 3.2 *Consider the projection $\alpha_{n-1} : A_n \to A_{n-1}$ and let $X_{n-1} := F(\alpha_n)(X_n)$. Then the T^1-lifting property from above holds if and only if the map $T^1_F(X_n/A_n) \longrightarrow T^1_F(X_{n-1}/A_{n-1})$ is surjective.*

4. Lifting of lagrangian deformations

We are going to apply the T^1-lifting principle to deformations of lagrangian singularities using the description of the relative tangent spaces from the section 2. The result is the following.

Theorem 4.1 *Let $(L,0) \subset (\mathbb{C}^{2n},0)$ be lagrangian and denote by $Def_{L,0}$ the functor of flat deformations of the germ $(L,0)$. Suppose that $Def_{L,0}$ is smooth and that $H^2(\mathcal{C}^\bullet_{L,0})$ is zero. Then the functor $LagDef_{L,0}$ is also smooth.*

Proof: Consider the functor $EmbDef_{L,0}$ of *embedded* deformations of the analytic algebra $\mathcal{O}_{L,0}$. It is classically known (see, e.g., [1]), that the natural transformation $EmbDef_{L,0} \to Def_{L,0}$ is smooth. Therefore the functor $EmbDef_{L,0}$ is smooth under the hypothesis of the theorem. Now let $(\mathcal{L}_n,0) \in LagDef_{L,0}(A_n)$ be a lagrangian family. Denote the defining ideal of $\mathcal{L}_n$ by $\mathcal{I}_n \subset \mathcal{O}_{\mathbb{C}^{2n},0}\widehat{\otimes}A_n$. Reduction modulo ϵ^n yields a family $\mathcal{O}_{\mathcal{L}_{n-1},0}$ over A_{n-1} (which lies of course in $LagDef_{L,0}(A_{n-1})$). We are going to compare the relative lagrangian de Rham complexes $\mathcal{C}^\bullet_{\mathcal{L}_n,0}$ and $\mathcal{C}^\bullet_{\mathcal{L}_{n-1},0}$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow 0$$

Tensoring with the flat A_n-module $\mathcal{O}_{\mathcal{L}_n,0}$ yields

$$0 \longrightarrow \mathcal{O}_{L,0} \longrightarrow \mathcal{O}_{\mathcal{L}_n,0} \longrightarrow \mathcal{O}_{\mathcal{L}_{n-1},0} \longrightarrow 0$$

Now we apply the functor $Hom_{\mathcal{O}_{\mathcal{L}_n,0}}(\bigwedge^\bullet \mathcal{I}_n/\mathcal{I}_n^2, -)$ (which is not right exact in general) to this sequence to obtain

$$0 \longrightarrow \mathcal{C}^\bullet_{L,0} \longrightarrow \mathcal{C}^\bullet_{\mathcal{L}_n,0} \longrightarrow \mathcal{C}^\bullet_{\mathcal{L}_{n-1},0}$$

The relative tangent space of *EmbDef* of a family $\mathcal{L}_k$ (k arbitrary) is given by the normal module $Hom_{\mathcal{O}_{\mathcal{L}_k}}(\mathcal{I}_k/\mathcal{I}_k^2, \mathcal{O}_{\mathcal{L}_k}) = \mathcal{C}^1_{\mathcal{L}_k,0}$. As a smooth functor obviously satisfies the T^1-lifting property we get that the map $\mathcal{C}^1_{\mathcal{L}_n,0} \to \mathcal{C}^1_{\mathcal{L}_{n-1},0}$ occurring in the above sequence of complexes is surjective. This means that we obtain a connecting homomorphism $H^1(\mathcal{C}^\bullet_{\mathcal{L}_{n-1},0}) \xrightarrow{\delta} H^2(\mathcal{C}^\bullet_{L,0})$ and a sequence

$$H^1(\mathcal{C}^\bullet_{\mathcal{L}_n,0}) \longrightarrow H^1(\mathcal{C}^\bullet_{\mathcal{L}_{n-1},0}) \xrightarrow{\delta} H^2(\mathcal{C}^\bullet_{L,0})$$

In particular, if $H^2(\mathcal{C}^\bullet_{L,0}) = 0$ we get the required surjectivity which allows to apply the T^1-lifting method. ⊡

We note two simple cases where smoothness of $Def_{L,0}$ is known.

Corollary 4.2 *Let $(L, 0) \subset (\mathbb{C}^{2n}, 0)$ be lagrangian and either a complete intersection or a Cohen-Macaulay surface singularity. Then $LagDef_{L,0}$ is smooth if $H^2(\mathcal{C}^\bullet_{L,0}) = 0$.*

To finish, we would like to point out possible improvements of this result. Let us first notice that it would be desirable to weaken the assumption that $Def_{L,0}$ is smooth, as we do not care about non-lagrangian deformations. It is a priori not necessary to suppose that they are unobstructed to deduce smoothness of $LagDef_{L,0}$. Moreover, the above sequence used to prove the lifting property of the relative tangent spaces can be continued and it is sufficient to ask that $H^2(\mathcal{C}^\bullet_{L,0}) \to H^2(\mathcal{C}^\bullet_{\mathcal{L}_n,0})$ is injective which is considerably weaker. This would follow from the more general result that for any family $\mathcal{L} \to S$, the modules $H^i(\mathcal{C}^\bullet_{\mathcal{L},0})$ are *free* $\mathcal{O}_{S,0}$-modules. Partial results in this direction are known, but do not cover the case in question here.

On the other hand, the methods from [9], which consist essentially in studying sections of the cohomology of $\mathcal{C}^\bullet_L$ with support in a subspace, might be used in order to deduce smoothness of the deformation space of a lagrangian singularity if we know something about the smoothness of deformations of a transversal section. As an example, consider a general point of the singular locus, then in a neighborhood of this point the singularity decomposes into a product of a smooth germ with a lagrangian curve the deformations of which are obviously unobstructed. We will treat this question in a subsequent paper.

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References

[1] Artin M., *Lectures on deformations of singularities. Notes by C. S. Seshadri, Allen Tannenbaum*, Lectures on Mathematics and Physics, vol. 54, Tata Institute of Fundamental Research, Bombay, 1976.

[2] Fantechi B., Manetti M., *Obstruction calculus for functors on Artin Rings, I*, Journal of Algebra **202** (1998), 541–576.

[3] Fantechi B., Manetti M., *On the T^1-lifting theorem*, Journal of Algebraic Geometry **8** (1999), 31–39.

[4] Garay M., *A rigidity theorem for lagrangian deformations*,preprint alg-geom/0305394, 2003.

[5] Gross M., *Deforming Calabi-Yau threefolds.*, Math. Ann. **308** (1997), no. 2, 187–220.

[6] Schlessinger M., *Functors of Artin rings*, Trans. Am. Math. Soc. **130** (1968), 208–222.

[7] Sevenheck Ch., *Lagrangian singularities*, Ph.D. thesis, Johannes-Gutenberg-Universität Mainz and Ecole Polytechnique, Palaiseau, 2003, available at http://www.dma.ens.fr/~sevenhec, p. 177.

[8] Sevenheck Ch., van Straten D., *Deformation of singular lagrangian subvarieties*, Mathematische Annalen **327** (2003), no. 1, 79–102.

[9] Sevenheck Ch., van Straten D., *Rigid and complete intersection lagrangian singularities*, preprint alg-geom/0311503, 2003.

[10] Voisin C., *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, Complex projective geometry, Lect. Note, vol. 179, Lond Math. Soc., 1992, pp. 294–303.