Mirror symmetry, singularity theory and non-commutative Hodge structures

Christian Sevenheck

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Abstract

We review a version of the mirror correspondence for smooth toric varieties with a numerically effective anticanonical bundle. We give a precise description of the so-called B-model, which involves the Gauß-Manin system of a family of Laurent polynomials. We show how to derive from these data a variation of non-commutative Hodge structures and describe general results on period maps and classifying spaces for these generalized Hodge structures. Finally, we explain a version of mirror symmetry as an isomorphism of Frobenius manifolds.

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1 Introduction

The aim of this survey is to describe how classical constructions from singularity theory enter into mirror symmetry. The latter subject evolves from string theory, but has become over the last 20 years one of the main branches of research in pure mathematics, connecting various areas like algebraic, symplectic and differential geometry, integrable systems, linear differential equations, homological algebra and so on. Due to the complexity of the subject, we will limit this survey to a particular aspect of the mirror symmetry picture, in which linear differential equations with complex coefficients (also called $\mathcal{D}$-modules) play a central role.

Let us start this introduction with a well-known motivating example which does not come from physics, but which is a very classical problem in enumerative algebraic geometry. Enumerative geometry is concerned with the question of counting geometric objects of a certain type that satisfy some extra conditions. Here we will be interested in the number of curves in the plane passing through some prescribed set of points. To be more precise, we will look at algebraic curves, that is, vanishing loci

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of a single polynomial in two variables, second, we will work over the complex numbers, that is, we take polynomials with complex coefficients to be sure that our vanishing loci are really 1-dimensional as topological spaces (we do not want things like $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 0\} = \emptyset$), and last we will actually look at projective curves, that is, we consider the zero locus in the projective plane $\mathbb{P}^2$ of a *homogenous* polynomial in three variables. This is the usual approach in algebraic geometry that excludes pathological facts like parallel lines with no intersections. The degree of such a polynomial, henceforth called the degree of the curve, will be a fixed positive integer denoted by d. The problem will consists in determine the number of such curves passing through some fixed points. It is easy to see that we need $3d - 1$ points in general position to have a chance that the number of curves through these points is finite. For small values of d these numbers are known via classical methods of algebraic geometry:

- $d = 1$ and $d = 2$: The number of lines through two points as well as the number of quadrics through 5 general points is known to be one since antiquity.
- The number of cubics (curves of degree three) through 8 general points is 12 (Steiner, 1848).
- For $d = 4$, Zeuthen (1873) computed the number of quartics through 11 points in general position to be 620.

However, for higher d there is no general method to calculate theses number. It came as a surprise when Kontsevich produced a formula that allows one to do this calculation for all d. He used in an essential way ideas from string theory. The precise result is as follows.

Theorem (Kontsevich, 1994)**.** *Denote by N_d the number of rational curves of degree d passing through $3d - 1$ points in $\mathbb{P}^2$ which lie in general position. Then the following recursive formula holds.*

$$N_d = \sum_{d_1+d_2=d;\ d_1,d_2\geq 1} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2} - \sum_{d_1+d_2=d;\ d_1,d_2\geq 1} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}$$

The meaning of this theorem is that we only need to know the first two numbers N_d and then we can calculate all others recursively by the computer. Without giving the details of the proof, let us just outline the strategy: First one re-interprets the numbers N_d as a so-called Gromov-Witten invariant (precise definitions and properties are below in section 2). These give rise to the quantum product on the cohomology space of $\mathbb{P}^2$. One of the main properties of the latter is its associativity, and Kontsevich's proof consists in deriving the above recursive formula from that property.
There is a similar problem in enumerative geometry where classical methods fail to produces results beyond the easiest cases. Namely, it concerns the number of curves of fixed degree on special three-dimensional complex manifold, called Calabi-Yau. Let us give here for future use the precise definition of these and some related varieties.

Definition 1.1. *Let X be a smooth and projective algebraic variety over the complex numbers. Let n be the dimension of X. Denote by K_X its canonical bundle, by definition, $K_X := \bigwedge^n \Omega^1_X$ is the top exterior power of the cotangent bundle, hence, as the latter is of rank n, a line bundle. Then we call X*

1. *Calabi-Yau iff $K_X \cong \mathcal{O}_X$, that is, if it is the trivial line bundle.*
2. *Fano iff $-K_X$ is ample, that is, if there in an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that for some $n \in \mathbb{N}$ we have $K_X^{\otimes n} = i^*\mathcal{O}_{\mathbb{P}^N}(-1)$.*
3. *numerically effective (nef) or sometimes also weak Fano if the intersection of $-K_X$ with any curve is non-negative (recall that to the line bundle $-K_X$ we can associate a divisor which have a well defined intersection product with curves).*

Notice that both Calabi-Yau varieties and Fano varieties are nef.

The most prominent example of a Calabi-Yau threefold is a hypersurface of degree five in $\mathbb{P}^4$ (that this satisfies the Calabi-Yau condition is an immediate consequence of the adjunction formula). For those, [CdlOGP92] gave a formula predicting of the number of curves of fixed degree. This formula is not achieved by direct computation of Gromov-Witten invariants or by using formal properties of the quantum product. Rather, a basic principles of mirror symmetry comes into play: one is given a pair of Calabi-Yau manifolds (or a family thereof), and the quantum cohomology of one of these varieties (called A-model) can be obtained from much more classical invariants (like period integrals) of the other (family of) Calabi-Yau manifold(s), called B-model. To be more precise, the so-called quantum $\mathcal{D}$-module (see section 3 below) of the given Calabi-Yau manifold can be reconstructed as a classical variation of Hodge structures of the mirror family. Let us notice however that in contrast to the case of curves in $\mathbb{P}^2$ discussed above, the enumerative meaning of the Gromov-Witten invariants of the quintic is less clear: The corresponding numbers should be seen as the "virtual number" of curves of fixed degree on X. However, for some degrees they actually coincide with the true numbers, and even this limited result gives interesting enumerative information that was not available by classical algebraic geometry.

It soon turned out that this picture can be extended to Fano and nef varieties. One striking difference to the Calabi-Yau case is that the mirror is no longer a family of compact varieties, but rather an affine morphism, e.g. a family of Laurent polynomials. These are called Landau-Ginzburg models. One aspect that is central to this survey is to explain which kind of Hodge-like structures survive in this more general correspondence. The objects which generalize usual Hodge structures in the correct way that is needed for these more general mirror correspondences are nowadays called non-commutative Hodge structures. This name stems from a (conjectural) relation to the so-called homological mirror symmetry, which very roughly speaking expresses the above mentioned correspondence between A- and B-model through an equivalence of certain categories. However, we are not going to touch upon any aspects of homological mirror symmetry in this survey. Rather, we emphasize the fact that non-commutative Hodge structures, in contrast to classical ones, are certain systems of linear differential equations. For that reason, one may try to express the mirror correspondence by identifying such differential systems on both sides. This is possible at least in the case where the variety on the A-side is toric, then the rather well understood theory of hypergeometric differential equations comes into play.

The structure of this article is as follows: We recall the very basics of quantum cohomology in section 2, where we restrict to the easiest case of genus zero invariants for convex manifolds. This is not quite sufficient for all examples that we are interested in but avoids technical difficulties. Next (section 3) we describe the so-called quantum $\mathcal{D}$-module which defines the differential system on the A-side. We proceed in section 4 by a detailed description of the mirror of a toric Fano resp. numerically effective variety, and show how to define and calculate its associated *Gauß-Manin-system*. The latter will ultimately be the mirror partner for the quantum $\mathcal{D}$-module, this correspondence is worked out in section 7. The section 6 gives definition and some important results on non-commutative Hodge structures and describe how they appear in mirror symmetry.

2 Quantum cohomology of smooth projective varieties

We recall in this section the definition of the quantum cohomology ring of a smooth projective variety. As there are many excellent sources available (e.g. [FP97, Gue08, KV07]), we mainly fix the notation for the later sections. Throughout the first two sections, we will denote by X a smooth projective variety having only even-dimensional cohomology classes.

Before giving precise definitions, let us point out the main idea of the construction of Gromov-Witten invariants and of the quantum product. Suppose that we are interested in an enumerative problem associated to X like those mentioned in the introduction, that is, counting the number of curves on X of a certain degree satisfying some incidence conditions, e.g., passing through some subspaces of X. These subspaces define homology classes on X, and as X is compact, they correspond via Poincaré duality to some cohomology classes. Now the idea is to construct a moduli space of all maps from $\mathbb{P}^1$ to X of a certain degree (this is the degree of the curves to be considered). Our $\mathbb{P}^1$ will also be equipped with some points, called markings, which are allowed to vary in the moduli space. The markings define maps from the moduli space to X (one for each marking) and our enumerative invariants will be obtained by pulling back the aforementioned cohomology classes to the moduli space and then integrating them against the fundamental class of the moduli space. It can be shown that in favorite situations, this construction (with some technical modifications, e.g., in order to obtain compact moduli spaces we need to consider also maps from certain singular curves) can be carried out and the invariants thus defined indeed have the enumerative meaning that we are looking for. Let us notice however that the general construction is very involved, in particular, the ordinary fundamental class of the moduli space may not be the right one

(because the moduli space may have the "wrong" dimension, i.e., its dimension is higher than the so-called "expected one"). In order to circumvent this problem, one needs to construct a *virtual fundamental class*, and this uses rather advanced techniques like stacks, obstruction theories etc. However, as already mentioned above, we will concentrate on the simple case of genus zero Gromov-Witten invariants of *convex* varieties (see below), where these sophisticated techniques are unnecessary.
We now start with a precise description of the moduli spaces involved. First we need to describe what kind of curves on X we are looking at.

Definition 2.1 (stable map)**.** *Let C be a reduced projective curve of genus zero with at most nodal singularities, i.e., singular points that are locally given by an equation $x \cdot y = 0$. Suppose that $x_1, \ldots, x_n \in C$ are distinct smooth points. Let $\beta \in H^2(X, \mathbb{Z})/Tors$. A stable map $f : C \to X$ is a projective morphism such that $f_*([C]) = \beta$ and such that each smooth component of C that is contracted by f to a point in X has at least three marked points.*

Theorem 2.2. *Let X be convex, i.e., for all maps $f : \mathbb{P}^1 \to X$, we have that $H^1(\mathbb{P}^1, f^*(TX)) = 0$, and $\beta \in H^2(X, \mathbb{Z})/Tors$. Then there exists a coarse moduli space $\overline{\mathcal{M}}_{(0,n)}(X, \beta)$ of stable maps which is a projective variety of dimension $n + \dim(X) + \int_\beta c_1(X) - 3$ with at most orbifold singularities. In particular, it carries a well-defined fundamental cycle $[\overline{\mathcal{M}}_{(0,n)}(X, \beta)]$ of degree equal to the dimension of $\overline{\mathcal{M}}_{(0,n)}(X, \beta)$.*

For the definitions and basic results below, we will for simplicity of the exposition always suppose that X is convex. Notice however that not all examples that are going to occur later satisfy this assumption, e.g., the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$ (see end of section 4) are not convex.
We chose once and for all a basis of the cohomology space $H^*(X, \mathbb{C})$ consisting of homogenous classes $T_0, T_1, \ldots, T_r, T_{r+1}, \ldots, T_s$ such that $T_0 = 1 \in H^0(X, \mathbb{C})$ is the Poicaré dual of the fundamental class and such that $T_1, \ldots, T_r$ form a basis of $H^2(X, \mathbb{C})$.

Definition 2.3 (GW-invariant)**.** *Let $\alpha_1, \ldots, \alpha_n$ be cohomology classes in $H^*(X, \mathbb{C})$, then we define the* ***correlator*** *or* ***Gromov-Witten invariant*** *to be*

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,\beta} := \int_{[\overline{\mathcal{M}}_{(0,n)}(X,\beta)]} ev_1^*(\alpha_1) \cup \ldots \cup ev_n^*(\alpha_n).$$

Here for $i \in \{1, \ldots, n\}$ the map $ev_i : \overline{\mathcal{M}}_{(0,n)}(X, \beta) \to X$ sends a class of a stable map $(C, [x_1, \ldots, x_n], f : C \to X) \in \overline{\mathcal{M}}_{(0,n)}(X, \beta)$ to $f(x_i) \in X$.

Definition 2.4 (Quantum cohomology)**.** *Denote by $\mathbb{L}_{\mathrm{eff}}$ the set of effective homology classes in $H_2(X, \mathbb{Z})/\mathrm{Tors}$, i.e., classes represented by curves. Denote by $g : H^*(X, \mathbb{C}) \times H^*(X, \mathbb{C}) \to \mathbb{C}$ the Poincaré pairing, which is symmetric (recall that we suppose $H^*(X, \mathbb{C}) = H^{2*}(X, \mathbb{C})$) and non-degenerate. For any triple of cohomology classes $\alpha, \gamma, \eta \in H^*(X, \mathbb{C})$, we define the big quantum product, denoted by $\alpha \circ_\eta \gamma$, by its values under g on any class $\delta \in H^*(X, \mathbb{C})$ using the formula*

$$g(\alpha \circ_\eta \gamma, \delta) := \sum_{n \geq 0; \beta \in \mathbb{L}_{\mathrm{eff}}} \frac{1}{n!} \langle \alpha, \gamma, \underbrace{\eta, \ldots, \eta}_{n-\text{times}}, \delta \rangle_{0,n+3,\beta} \in H^*(X, \mathbb{C}) \qquad (1)$$

Remark: There is a technical obstacle in the definition of the quantum product: Formula (1) does not make sense as such, as we are considering an infinite sum over **both** n and β. Hence it does not even define a formal sum. This problem is usually solved by splitting the contributions of the different homology classes β in $\mathbb{L}_{\mathrm{eff}}$ using the so-called Novikov ring. However, the above definition makes sense once we know that there is a domain in the parameter space on which the quantum product is convergent. Throughout this survey, we will use this assumption without further mentioning. More precise results on the convergence of the quantum product can be found, e.g., in [Iri07].
Let us summarize very briefly some of the most important properties of the quantum product.

Proposition 2.5. *Consider the big quantum product $\circ$ as above.*

1. *$\circ$ is symmetric, associative with unit $1 \in H^0(X, \mathbb{C})$.*

2. *Gromov-Witten invariants have a special behavior with respect to degree 2 classes. More precisely, suppose that $\alpha_1 \in H^2(X,\mathbb{C})$, then we have that*

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{0,n,\beta} = \left(\int_\beta \alpha_1 \right) \cdot \langle \alpha_2, \ldots, \alpha_k \rangle_{0,n-1,\beta}$$

This implies that we can rewrite the definition of the quantum product, that is, formula (1) by decomposing a class $\eta \in H^*(X,\mathbb{C})$ into a sum $\eta = \eta' + \eta''$ with $\eta' \in H^2(X,\mathbb{C})$ and $\eta'' \in H^{\neq 2}(X,\mathbb{C})$. Then we have

$$g(\alpha \circ_\eta \gamma, \delta) := \sum_{n \geq 0; \beta \in \mathbb{L}_{\text{eff}}} \frac{e^{\eta'(\beta)}}{n!} \langle \alpha, \gamma, \underbrace{\eta'', \ldots, \eta''}_{n-\text{times}}, \delta \rangle_{0,n+3,\beta} \in H^*(X,\mathbb{C}) \tag{2}$$

3. *A convenient way to collect all Gromov-Witten invariants is the so-called (genus zero) potential, this is the formal function on $H^*(X,\mathbb{C})$ defined by*

$$\mathcal{F}(\underline{t}) := \sum_{n \geq 3, \beta \in H_2(X,\mathbb{Z})} \frac{1}{n!} \langle \underbrace{\underline{t}, \ldots, \underline{t}}_{n-\text{times}} \rangle_{0,n,\beta}$$

Here $\underline{t} = (t_0, t_1, \ldots, t_s)$ are the coordinates on $H^*(X,\mathbb{C})$ corresponding to the choice of a homogenous basis $T_0, T_1, \ldots, T_s$ from above.

4. *The associativity of the quantum product can be very nicely expressed using the Gromov-Witten potential. It is equivalent to the following system of partial differential equations satisfied by $\mathcal{F}$, which are called WDVV-equations (after Witten, Dijkgraaf, E. Verlinde and H. Verlinde):*

$$\sum_{e,f=0}^{s} (\partial_i \partial_j \partial_e \mathcal{F}) \cdot g^{ef} \cdot (\partial_f \partial_k \partial_l \mathcal{F}) = \sum_{e,f=0}^{s} (\partial_k \partial_j \partial_e \mathcal{F}) \cdot g^{ef} \cdot (\partial_f \partial_i \partial_l \mathcal{F})$$

for any $i,j,k,l \in \{0,\ldots,s\}$, where $(g^{ef})_{e,f\in\{0,\ldots,s\}} := (g(T_e,T_f))^{-1}$.

For many computations, it is sufficient to calculate only a restricted set of Gromov-Witten invariants, which involve the moduli space of stable maps from curves with only three marked points (the so-called three point invariants or correlators). A basic result due to Kontsevich and Manin (see [KM94]) says that often the (big) quantum product can be reconstructed from the small one. The precise definition of the small quantum product is as follows.

Definition 2.6 (The small quantum product)**.** *Let, as before, α, γ and δ be arbitrary classes in $H^*(X,\mathbb{C})$ and take η to be in $H^2(X,\mathbb{C})$. Then we define*

$$g(\alpha \star_\eta \gamma, \delta) := \sum_{\beta \in \mathbb{L}_{\text{eff}}} e^{\eta(\beta)} \langle \alpha, \gamma, \delta \rangle_{0,3,\beta} \tag{3}$$

Remarks: The divisor axiom and the formula (2) that it implies show that we can naturally define the quantum product and the potential on the space $H^0(X,\mathbb{C}) \oplus \left(H^2(X,\mathbb{C})/2\pi i H^2(X,\mathbb{Z})\right) \oplus H^{>2}(X,\mathbb{C})$ (where $2\pi i H^2(X,\mathbb{Z})$ acts on $H^2(X,\mathbb{C})$ by translation), which is a product of an affine space with a torus. Namely, the Gromov-Witten invariants resp. the potential depend on the coordinates $t_1, \ldots, t_r$ on $H^2(X,\mathbb{C})$ only via the exponentials $e^{\eta'(\beta)}$, so that by putting $q_a := e^{t_a}$ $(a = 1, \ldots, r)$, we obtain a function (resp. a tensor) in the variables $t_0, q_1, \ldots, q_r, t_{r+1}, \ldots, t_s$.
At several places below, we will have to use the fact that the quantum product (written in the above q-coordinates) carry an inherent grading. More precisely, consider the first Chern class of X, i.e., the first Chern class of the tangent bundle of X, this is a cohomology class of degree two and hence can be written as $c_1(X) = \sum_{a=1}^{r} d_a T_a$. Write $\deg(T_i) = k$ iff $T_i \in H^{2k}(X,\mathbb{C})$ and put

$$\begin{aligned} \deg(q_a) &= 2 \cdot d_a \\ \deg(t_i) &= 2 - 2 \cdot \deg(T_i) \end{aligned} \tag{4}$$

Then the potential $\mathcal{F}$ has a certain homogeneity property with respect to this grading (to be more precise, the quantum part

$$\mathcal{F}^{quant}(\underline{t}) := \sum_{n\geq 3, \beta\in H_2(X,\mathbb{Z})\setminus\{0\}} \frac{1}{n!}\langle \underbrace{\underline{t},\ldots,\underline{t}}_{n-\text{times}}\rangle_{0,n,\beta}$$

is homogenous of degree $2(3-\dim(X))$).

Example: As a well known and instructive example, we are going to compute here the small quantum product of the projective spaces. The advantage of this case is that the mirror correspondence with the Landau-Ginzburg model can be very explicitly written down (see section 7 below) and this motivates also the general mirror constructions for toric varieties, as explained below.
Let $\mathbb{P}^n$ be the n-dimensional projective space. It is well-known that its ordinary cohomology ring $H^*(\mathbb{P}^n,\mathbb{Z})$ is isomorphic to $\mathbb{Z}[p]/(p^{n+1})$. Here p denotes the degree 2 cohomology class which is Poincaré dual to the class of a hyperplane $H\subset\mathbb{P}^n$. In particular (this is true for any smooth toric variety), the cohomology is generated as a ring by its degree two classes and hence we have $H^*(\mathbb{P}^n,\mathbb{Z}) = H^{2*}(\mathbb{P}^n,\mathbb{Z})$, that is, only even dimensional cohomology classes do appear.
The small quantum cohomology ring is by definition a finitely generated algebra over $\mathbb{C}[q^\pm] := \mathbb{C}[q,q^{-1}]$, where q is the coordinate on $H^2(\mathbb{P}^n,\mathbb{C})/2\pi i H^2(\mathbb{P}^n,\mathbb{Z})$ corresponding to the choice of p as generator of $H^2(\mathbb{P}^n,\mathbb{Z})$. It is graded by $\deg(p)=2$ and $\deg(q)=2c_1(\mathbb{P}^n)=2(n+1)$. As a $\mathbb{C}[q^\pm]$-module, it is isomorphic to $\mathbb{C}[p]/(p^{n+1})\otimes\mathbb{C}[q^\pm]$. The degree preserving property of the quantum product tells us that for any $k\in\{1,\ldots,n\}$, we have

$$\underbrace{p\star\cdots\star p}_{k-\text{times}} \overset{!}{=} \underbrace{p\cup\cdots\cup p}_{k-\text{times}} =: p^k$$

so that it suffices to compute $p^{\star(n+1)}$ which equals $p^{\star n}\star p = p^n\star p$. Notice that here we do not have to put a parameter as index to the small quantum product as we consider it as a family of algebras (see the remark after definition 3.1 below for a more conceptual explanation). We use the definition of the quantum product, i.e., formula (3), saying that we have to compute for any class $\gamma\in H^*(\mathbb{P}^n,\mathbb{C})$ the expression

$$g(p^n\star p,\gamma) = \sum_\beta q^{p(\beta)}\langle p^n,p,\gamma\rangle_{0,3,\beta} = \sum_\beta q^{p(\beta)}\left(\int_{[\mathcal{M}_{0,3}(\mathbb{P}^n,\beta)]} ev_1^*(p^n)\otimes ev_2^*(p)\otimes ev_3^*(\gamma))\right)$$

Notice that β is always a integer multiple of the class of a line $[L]\in H_2(\mathbb{P}^n,\mathbb{Z})$ dual to p, so that we can rewrite the sum as $\sum_{d\geq 0}\langle p^n,p,\gamma\rangle_{0,3,d[L]}\cdot q^d$. The correlators in this sum are nonzero only if the degrees of the classes p^n, p and γ add up to the dimension of $\mathcal{M}_{0,3}(\mathbb{P}^n,d[L])$, that is, to $3+\dim(\mathbb{P}^n)+\int_{d\cdot[L]}(n+1)\cdot\mathrm{PD}([H])-3 = n+d\cdot(n+1)$. Hence only classes γ of degree $\deg(\gamma)=(n+1)d-1$ can give a non-zero correlator. Since $\deg(\gamma)\leq n$, we arrive at the conclusion that d can only take the value 1 and then $\deg(\gamma)=n$. Hence we are left with $\langle p^n,p,p^n\rangle_{0,3,[L]}$, and this is the number of lines in $\mathbb{P}^2$ through two generic points meeting a generic hypersurface. Obviously, there is only one such line, so that we finally arrive at the conclusion that $g(p^{\star(n+1)},\gamma)=q$ if $\gamma=p^n$ and 0 else, meaning that the relation $p^{\star n+1}=q$ holds in the small quantum cohomology ring, i.e., we have the isomorphism

$$SQH(\mathbb{P}^n) = (H^*(X,\mathbb{C}),\star) \cong \mathbb{C}[p,q^\pm]/(p^{n+1}-q) \tag{5}$$

of $\mathbb{C}[q^\pm]$-algebras.
The following rather obvious remark will be of fundamental importance in the sequel. The above description of the small quantum cohomology of $\mathbb{P}^n$ shows that it corresponds to a vector bundle on $\mathbb{C}^*$ (the coordinate on $\mathbb{C}^*$ being q) of rank $n+1$, equipped with a commutative and associative multiplication. There is a canonical extension of that bundle to a bundle over $\mathbb{C}_q$, i.e., over the limit point $q=0$, which is simply given as the $\mathbb{C}[q]$-algebra $\mathbb{C}[p,q]/(p^{n+1}-q)$. Even more, the fibre of this extended bundle on $q=0$ is nothing but the restriction of this algebra to $q=0$, i.e., the zero-dimensional Gorenstein ring $\mathbb{C}[p]/p^{n+1}\cong(H^*(\mathbb{P}^n,\mathbb{C}),\cup)$. This limit behavior is not accidental, the point $q=0$ is called the large radius limit, and it is one of the most prominent features of the quantum product that it degenerates to the ordinary cup product at this limit point. A basic philosophy in the later sections of this paper is to express the mirror correspondence by objects defined on partial compactifications of the parameter spaces which include the large radius limit.

3 Givental's approach: quantum $\mathcal{D}$-module and J-function

One of the central ideas in quantum cohomology that we are going to exploit is that the relations encoded in the WDVV-equation can be rewritten as a system of linear differential equations. This is usually called the quantum $\mathcal{D}$-module. We will encounter some general $\mathcal{D}$-modules below as the so-called Gauß-Manin systems, however, the quantum $\mathcal{D}$-module is merely a vector bundle with a connection. We recall the basic definitions.

Definition 3.1. *Let M be a complex manifold M and $E \to M$ be a holomorphic vector bundle.*

1. *A connection on E is a $\mathbb{C}$-linear map*

$$\nabla : E \longrightarrow E \otimes \Omega^1_M$$

satisfying the Leibniz rule $\nabla(f \cdot s) = f \cdot \nabla(s) + s \otimes df$ for any local sections $f \in \mathcal{O}_M$ and $s \in E$. It is called flat if moreover the $\mathcal{O}_M$-linear map (called curvature) $\nabla^{(2)} \circ \nabla$ vanishes, where $\nabla^{(2)} : E \otimes \Omega^1_M \to E \otimes \Omega^2_M$ is defined by $\nabla^{(2)}(s \otimes \omega) := \nabla(s) \wedge \omega - s \otimes d\omega$.

2. *Let $D \subset M$ be a simple normal crossing divisor (in most cases that we will meet below, D is simply smooth). Then a meromorphic bundle F is by definition a locally free sheaf of $\mathcal{O}_M(*D)$-modules, and a meromorphic connection on E resp. F is a $\mathbb{C}$-linear operator $\nabla : E \to E \otimes \Omega^1_M(*D)$ resp. $\nabla : F \to F \otimes \Omega^1_M(*D)$ satisfying the Leibniz rule as above. Any meromorphic connection on E defines a meromorphic connection on $E(*D) = E \otimes \mathcal{O}_M(*D)$.*

3. *A meromorphic connection on a holomorpic bundle $E \to M$ is said to have a logarithmic pole along D if it takes values in $E \otimes \Omega^1(\log D)$. Here $\Omega^1(\log D)$ is the sheaf of differential forms with logarithmic poles along D. Locally, by choosing coordinates $z_1, \ldots, z_k, t_{k+1}, \ldots, z_m$ on M such that $D = \{z_1 \cdot \ldots \cdot z_k = 0\}$, the sheaf $\Omega^1_M(\log D)$ is freely generated over $\mathcal{O}_M$ by the forms $dz_1/z_1, \ldots, dz_k/z_k$ and $dz_{k+1}, \ldots dz_m$.*

4. *A connection (E, ∇) which is logarithmic with respect to a smooth divisor $D \subset M$ defines a residue endomorphism Φ^{res} on the restriction $E_{|D}$, locally, if $z_1 = 0$ is the equation of the divisor, it is given by the class of $z_1 \nabla_{\partial_{z_1}} \in \mathcal{E}nd_{\mathcal{O}_D}(E_{|D})$. However, Φ^{res} does not depend on the choice of coordinates. Similarly, but only if we fix a coordinate system $(z_1, \ldots, z_m)$ on M such that $D = \{z_1 = 0\}$, the residue connection ∇^{res} can be defined on $E_{|D}$ in the following way: If $\underline{e} = (e_1, \ldots, e_n)$ is a local basis of E, then ∇ is written as*

$$\nabla(\underline{e}) = \underline{e} \cdot \left(A_1(z_1, \ldots, z_m) \frac{dz_1}{z_1} + \sum_{i \geq 2}^{m} A_i(z_1, \ldots, z_m) \right)$$

then ∇^{res} is defined with respect to $\underline{e}$ by the matrix $A_1(0, z_2, \ldots, z_m)$.

Notice that there is a natural generalization of the notion of the residue endomorphism which applies to meromorphic connections with pole order two (more precisely, with Poicaré rank one), these are the so-called Higgs fields. We refer to [Sab07, section 0.14c] for details.

We can now define the small quantum $\mathcal{D}$-module. It corresponds to the small quantum cohomology ring, and will be sufficient for our purpose. Before giving the formal definition, let us make a remark on the definition of the quantum product (both the big and the small one). A priori, it is defined as a product of two cohomology classes, denoted by α and γ above, depending on a third class, denoted by η (this class has to be of degree two for the small quantum product). An elegant way to eliminate this dependence in the notation is to consider a trivial vector bundle with fibre $H^*(X, \mathbb{C})$ on either $H^*(X, \mathbb{C})$ (big quantum product) or $H^2(X, \mathbb{C})$ (small quantum product). Then any cohomology class gives a (constant) section of this bundle, and we can consider the (small/big) quantum product as a commutative and associative multiplication with unit on this vector bundle. On each fibre of this bundle at a base point η, this multiplication gives back the product $\alpha \circ_\eta \gamma$ resp. $\alpha \star_\eta \gamma$ from above. Below, we will always adopt this convention and write the quantum product as a product of two classes $\alpha \circ \gamma$ resp. $\alpha \star \gamma$.

Definition 3.2. *Let X be smooth projective satisfying $H^*(X,\mathbb{C}) = H^{2*}(X,\mathbb{C})$. Choose a homogenous basis $T_0, T_1, \ldots, T_r, T_{r+1}, \ldots, T_s$ of the cohomology as above. Let $M \subset H^2(X,\mathbb{C})$ be an open subset (with coordinates $t_1, \ldots, t_r$ corresponding to the basis vectors $T_1, \ldots, T_r$) on which the small quantum product is convergent. Consider the projection $\pi : \mathbb{P}^1_z \times M \twoheadrightarrow M$, where we chose z to be a fixed coordinate on the chart of $\mathbb{P}^1$ centered at 0. Let $G := H^*(X,\mathbb{C}) \times M \to M$ be the trivial vector bundle on M with fibre $H^*(X,\mathbb{C})$. It comes equipped with a flat connection ∇^{fl} corresponding to the given trivialization, i.e., such that $\nabla^{fl}(s) = 0$ for any section $s : M \to H^*(X,\mathbb{C}) \times M$ sending any point in M to a constant value $\gamma \in H^*(X,\mathbb{C})$. Put $F := \pi^* G \to \mathbb{P}^1_z \times M$, and define the* Givental connection *∇^{Giv} on F as follows:*

$$\begin{array}{rcl} \nabla^{Giv}_{\partial_{t_k}}(s) & := & \nabla^{fl}_{\partial_{t_k}}(s) - \frac{1}{z} T_k \star s \\ \nabla^{Giv}_{\partial_z}(s) & := & \frac{1}{z}\left(\frac{E \star s}{z} + \mu(s)\right) \end{array} \tag{6}$$

where $s \in F$, $\mu \in Aut_{\mathbb{C}}(H^*(X,\mathbb{C}))$ is the grading operator that takes the value $k \cdot \gamma$ on any class $\gamma \in H^{2k}(X,\mathbb{C})$ and the vector field E is defined as $E := \sum_{i=0}^{s}\left(1 - \frac{\deg(T_i)}{2}\right) t_i \partial_{t_i} + \sum_{a=1}^{r} k_a \partial_{t_a}$, where $\sum_{a=1}^{r} k_a T_a = c_1(X)$ (see also equations (4))
It follows from formula (6) that ∇^{Giv} has poles along $z = \{0, \infty\}$. Its restriction to $\mathbb{C}^*_z \times M$ thus defines a holomorphic connection operator.
By a slight abuse of notation, the object (F, ∇^{Giv}) is called the quantum $\mathcal{D}$-module.

The following is one of the main properties of the Givental connection. It is follows easily from the basic properties of the quantum product. In particular, it implies the WDVV differential equations for the Gromov-Witten potential and hence the associativity of the quantum product.

Proposition 3.3. *The Givental connection is flat, that is, the linear operator $\nabla^2 : F \to \Omega^2_{\mathbb{P}^1_z \times M} \otimes F$ vanishes.*

Remark: By its very definition, the quantum $\mathcal{D}$-module is a vector bundle on $\mathbb{P}^1 \times M$ and the connection has a logarithmic pole along $\{\infty\} \times M$ and a pole of Poincaré rank one along $\{0\} \times M$. Hence we can consider the residue connection ∇^{res} and the residue endomorphism of ∇^{Giv} along $\{\infty\} \times M$ as well as the Higgs field of ∇^{Giv} along $\{0\} \times M$. It also follows from the definition, i.e., from formula (6) that the residue connection is nothing but ∇^{fl}, the residue endomorphism is the grading operator μ and the Higgs field is given by the quantum multiplication. This observation is very useful when studying the object corresponding to the Givental connection via mirror symmetry.
A (rather simple) variant of the Riemann-Hilbert correspondence tells us that the restriction $(F, \nabla^{Giv})_{|\mathbb{C}^*_z \times M}$ defines a local system of flat sections. There is a canonical way to construct such flat sections, starting from the so-called J-function. This is a cohomology valued function, defined in terms of the so-called gravitational descendents.

Definition 3.4 (J-function and fundamental solutions).

1. *Let $\alpha_1, \ldots, \alpha_n \in H^*(X,\mathbb{C})$. Define the genus zero gravitational descendant invariants by*

$$\langle \alpha_1 \psi_1^{k_1}, \ldots, \alpha_n \psi_n^{k_n} \rangle_{0,n,\beta} := \int_{[\overline{\mathcal{M}}_{(0,n)}(X,\beta)]} \psi_1^{k_1} \cup ev_1^*(\alpha_1) \cup \ldots \cup \psi_n^{k_n} \cup ev_n^*(\alpha_n).$$

   where ψ_i are line bundles on $\overline{\mathcal{M}}_{(0,n)}(X,\beta)$ defined in such a way that their fibre at the point $[C,(x_1,\ldots,x_n),f] \in \overline{\mathcal{M}}_{(0,n)}(X,\beta)$ (here C is the projective curve, $x_1,\ldots,x_n$ its marked points and $f : C \to X$ the stable map from C to X) ***is*** *the cotangent line $T^*_{x_i}C$. A more precise definition can be found, e.g., in [RS10, definition 4.1].*

2. *Write $\eta' = \sum_{a=1}^{r} t_a T_a$ and define the $H^*(X_\Sigma,\mathbb{C})$-valued power series J by*

$$J(\eta', z^{-1}) := e^{\frac{\eta'}{z}} \cdot \left[1 + \sum_{\substack{\beta \in \mathrm{Eff}_{X_\Sigma} \setminus \{0\} \\ j=0,\ldots,s}} e^{\eta'(\beta)} \left\langle \frac{T_j}{z - \psi_1}, 1 \right\rangle_{0,2,\beta} T^j \right].$$

here the gravitational descendent GW-invariant $\langle \frac{T_j}{z-\psi_1}, 1\rangle_{0,2,\beta}$ *has to be understood as the formal sum* $\sum_{k\geq 0} z^{-k-1}\langle T_j\psi_1^k, 1\rangle_{0,2,\beta}$ *and* $T^0, T^1, \ldots, T^s$ *is the basis of* $H^*(X_\Sigma, \mathbb{C})$ *which is g-dual to* $T_0, T_1, \ldots, T_s$.

Theorem 3.5. *We have*

$$z\nabla^{Giv}(\partial_{t_k} J) = 0,$$

that is, the partial derivatives of J *form a fundamental system of solutions of the (relative) Givental connection.*

Remark: The derivatives of the J-function are not flat with respect to the "vertical connection" $\nabla^{Giv}_{\partial_z}$. However, one can obtain truly flat sections from the J-function by an easy twist, taking into account the logarithmic pole of ∇^{Giv} on F along $z = \infty$.
Givental has shown how to obtain all interesting Gromov-Witten invariants for nef **toric** varieties. In fact, Giventals result is broader, as it concerns the case of complete intersections in toric varieties. Such a subvariety is not necessarily toric itself. In particular, it includes the case of the quintic Calabi-Yau hypersurface in $\mathbb{P}^4$ mentioned in the introduction. We will restrict to toric varieties in the sequel. A thorough discussion of the B-model of a complete intersection would require the introduction of more notions, from which we refrain here.
The next section contains a detailed reminder on toric geometry, however, let us state Givental's main result here in order to finish the discussion of the quantum $\mathcal{D}$-module. We need one object from toric geometry, which is the so-called I-function.

Definition 3.6. *Let* X_Σ *be a smooth projective toric variety and write* $D_1, \ldots, D_m$ *for its torus invariant divisors. Define* I *to be the* $H^*(X_\Sigma, \mathbb{C})$*-valued formal power series*

$$I = e^{\eta'/z} \cdot \sum_{\underline{l}\in\mathbb{L}_{\text{eff}}} q^{\underline{l}} \cdot \prod_{i=1}^{m} \frac{\prod_{\nu=-\infty}^{0}([D_i] + \nu z)}{\prod_{\nu=-\infty}^{l_i}([D_i] + \nu z)} \in H^*(X_\Sigma, \mathbb{C})[z][[q_1, \ldots, q_r]][[z^{-1}]].$$

Now Givental's theorem takes the following form.

Theorem 3.7. *[Giv98, theorem 0.1] Let* X_Σ *be a smooth projective toric variety with a numerical effective anticanonical bundle* $-K_{X_\Sigma}$. *There is a formal coordinate change* $\kappa \in (\mathbb{C}[[q_1, \ldots, q_r]])^r)$ *called the mirror map, such that*

$$I = (\mathrm{id}_{\mathbb{C}_z} \times \kappa)^* J$$

If X_Σ *is Fano, that is, if* $-K_{X_\Sigma}$ *is ample, then* $\kappa = \mathrm{id}$.

4 Landau-Ginzburg models of toric Fano varieties

In the sequel of this survey, we will concentrate on the case of a toric variety with a nef anticanonical divisor. We will describe how to associate a certain family of Laurent polynomials to such a variety. These are called Landau-Ginzburg models. Our presentation below follows mainly [Iri09].
Whereas these Landau-Ginzburg models can be obtained very explicitly in a rather elementary way in the toric case, it is a largely unsolved problem how to construct them for more general varieties. Most of the known construction somehow come back to the toric case, like the technique of toric degenerations ([Bat04]). In order to orient the reader, let us give some reminders on basics of toric geometry that are relevant for the present paper. As a basic reference for the facts discussed below, the reader may consult [Ful93] or the more recent [CLS11].
Let $N = \oplus_{k=1}^n \mathbb{Z}n_k$ be a free abelian group of rank n and $\Sigma \subset N \otimes \mathbb{R}$ be a fan. This means that Σ is a collection of cones $\{\sigma \in \Sigma\}$, where any σ is a strongly convex (i.e., $\sigma \cap (-\sigma) = \{0\}$) polyhedral cone (i.e., $\sigma = \sum \mathbb{R}_{\geq 0} b_i$ for some b_i's in N). Being a fan means that for any $\sigma \in \Sigma$, any face of σ is again a cone in Σ, and that for any two cones $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of both τ and σ. The fan Σ defines a toric variety X_Σ. Recall that X_Σ is covered by affine charts $X_\sigma := \text{Spec } \mathbb{C}[M \cap \sigma^\vee]$, here $M := Hom_\mathbb{Z}(N, \mathbb{Z})$ and $\sigma^\vee := \{m \in M \otimes \mathbb{R} \,|\, m(n) \geq 0 \quad \forall n \in N \otimes \mathbb{R}\}$, and that X_Σ is obtained from these affine pieces by gluing X_σ and X_τ along $X_{\sigma\cap\tau}$.
We will suppose for simplicity that the fan Σ is *smooth* and *complete*, which means by definition that any cone $\sigma \in \Sigma$ can be generated by elements b_i which can be completed to a $\mathbb{Z}$-basis of N and that

the *support* $\mathrm{Supp}(\Sigma) = \bigcup_{\sigma\in\Sigma} \sigma$ is all of $N \otimes \mathbb{R}$. It is well-known that this translates into X_Σ being smooth and complete. The smoothness condition can by weakened by requiring Σ to be only *simplicial*, which means that the generators of each cone are linearly independent over $\mathbb{Q}$. In this case X_Σ can have quotient singularities, i.e., it is the underlying topological space of a an orbifold. The question how to extend the results presented here to the orbifold case is in the focus of current reasearch on quantum cohomology and mirror symmetry, however, we will restrict to the smooth case in this survey for simplicity.
We have an exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow N \longrightarrow 0 \tag{7}$$

where $\Sigma(1)$ are the one-dimensional cones of Σ, called rays, the last map sends a generator e_i of $\mathbb{Z}^{\Sigma(1)}$ to a primitive integral generator $b_i \in N$ of a ray, and where the lattice $\mathbb{L}$ is the free submodule of $\mathbb{Z}^{\Sigma(1)}$ of relations between the elements $b_i \in N$. Dualizing yields the sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathbb{L}^\vee \longrightarrow 0.$$

It is well known (see, e.g., [Ful93, p. 106]) that for a smooth toric manifold X_Σ, we have $H^2(X_\Sigma, \mathbb{Z}) \cong \mathbb{L}^\vee$. Inside $\mathbb{L}^\vee \otimes \mathbb{R}$ we have the cone $K(X_\Sigma)$ of *Kähler classes*, which can be defined by saying that $a \in K(X_\Sigma)$ iff $a(\beta) \geq 0$ for all effective 1-cycles in $H_2(X_\Sigma, \mathbb{R})$ (The latter set of cycles also forms a cone, called the Mori cone). We write $K^0(X_\Sigma)$ for the interior of $K(X)$, i.e., for all elements $a \in \mathbb{L}^\vee$ with $a(\beta) > 0$. Write $D_i \in \mathbb{L}^\vee$ for the components of the map $\mathbb{L} \hookrightarrow \mathbb{Z}^{\Sigma(1)}$, then the anticanonical divisor $-K_{X_\Sigma}$ is $\sum_{i=1}^m D_i \in \mathbb{L}^\vee$. As we already mentioned in definition 1.1, X_Σ is called a Fano variety iff $-K_{X_\Sigma}$ is ample, i.e., if it lies in $K^0(X_\Sigma)$. If $-K_{X_\Sigma} \in K(X_\Sigma)$, then X_Σ is nef . Notice that a Calabi-Yau manifold (i.e., $K_{X_\Sigma} = 0$) is obviously nef, however, it is easy to see that in this case the defining fan can never be complete.
The projection $\mathbb{Z}^{\Sigma(1)} \twoheadrightarrow N$ is given by a matrix $(a_{ki})_{k=1,\ldots,n;i=1,\ldots,m}$ with respect to the basis (n_k) of N. Moreover, we will chose once and for all a basis $(p_a)_{a=1,\ldots,r}$ of $\mathbb{L}^\vee$ (with $r = m - n$ and $m = |\Sigma(1)|$) which consists of nef classes (i.e., classes lying inside of $K(X)$) and such that the anti-canonical class $-K_{X_\Sigma}$ lies in the cone $\sum_{a=1}^r \mathbb{R}_{\geq 0} p_a$. Then the map $\mathbb{L} \hookrightarrow \mathbb{Z}^{\Sigma(1)}$ is given by a matrix $(m_{ia})_{i=1,\ldots,m;a=1,\ldots,r}$ with respect to the dual basis $(p_a^\vee)$.
Applying the functor $Hom_\mathbb{Z}(-, \mathbb{C}^*)$ (where $\mathbb{Z}$ acts on $\mathbb{C}^*$ by exponentiating) to the exact sequence (7) yields

$$1 \longrightarrow Hom_\mathbb{Z}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n \xrightarrow{\alpha} (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{\beta} Hom_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \longrightarrow 1 \tag{8}$$

where $\alpha(y_1, \ldots, y_k) = (w_i := \prod_{k=1}^n y_k^{a_{ki}})_{i=1,\ldots,m}$ and $\beta(w_1, \ldots, w_m) = (q_a := \prod_{i=1}^m w_i^{m_{ia}})_{a=1,\ldots,r}$, here $(q_a)_{a=1,\ldots,r}$ are the coordinates on $Hom_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*)$ corresponding to the basis (p_a) of $\mathbb{L}^\vee$, $(w_i)_{i=1,\ldots,m}$ are the standard coordinates on $(\mathbb{C}^*)^{\Sigma(1)}$ and $(y_k)_{k=1,\ldots,m}$ are the coordinates on $Hom_\mathbb{Z}(N, \mathbb{C}^*)$ corresponding to the basis $(n_k^\vee)$ of M.

Definition 4.1. *Let $W = \sum_{i=1}^m w_i$. The* Landau-Ginzburg *model of X_Σ is defined to be the restriction of W to the fibres of the map $\beta : (\mathbb{C}^*)^{\Sigma(1)} \to (\mathbb{C}^*)^r$.*

Notice that the choice of the basis (p_a) of $\mathbb{L}^\vee$ and hence the isomorphism $Hom_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r$ is part of the data of the Landau-Ginzburg model, as the map $(\mathbb{C}^*)^{\Sigma(1)} \to Hom_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*)$ depend only on $\Sigma(1)$, not on Σ itself.
The following construction allows us to rewrite the restriction of W to the fibres of β as a family of Laurent polynomials. Chose a section $l : \mathbb{L}^\vee \to \mathbb{Z}^{\Sigma(1)}$ of the projection $l : \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \mathbb{L}^\vee$, given, with respect to the above bases, by a matrix (l_{ia}). This yields a section (denoted abusively by the same letter)

$$l : (\mathbb{C}^*)^r \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \tag{9}$$

which sends $(q_1, \ldots, q_r)$ to $(w_i := \prod_{a=1}^r q_a^{l_{ia}})$. Then putting $\Psi : Hom_\mathbb{Z}(N, \mathbb{C}^*) \times (\mathbb{C}^*)^r \to (\mathbb{C}^*)^{\Sigma(1)}$ where $\Psi(\underline{y}, \underline{q}) := \left(w_i := \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}} \right)_{i=1,\ldots,r}$ yields a coordinate change on $(\mathbb{C}^*)^m$ such that β becomes the projection $p_2 : Hom_\mathbb{Z}(N, \mathbb{C}^*) \times (\mathbb{C}^*)^r \twoheadrightarrow (\mathbb{C}^*)^r$. Then we put

$$\begin{aligned} \widetilde{W} := W \circ \Psi : Hom_\mathbb{Z}(N, \mathbb{C}^*) \times (\mathbb{C}^*)^r \quad &\longrightarrow \quad \mathbb{C} \\ (y_1, \ldots, y_k, q_1, \ldots, q_a) \quad &\longmapsto \quad \sum_{i=1}^m \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}} \end{aligned}$$

which is a family of Laurent polynomials on $Hom_{\mathbb{Z}}(N, \mathbb{C}^*)$ parameterized by $(\mathbb{C}^*)^r$.
Recall ([Kou76]) that a single Laurent polynomial $\widetilde{W}_{\underline{q}} := \widetilde{W}(-, \underline{q}) \in \mathcal{O}_{Hom(N,\mathbb{C}^*)}$ is called convenient iff 0 lies in the interior of its Newton polyhedron, and non-degenerate iff for any proper face τ of its Newton polyhedron, the Laurent polynomial $(\widetilde{W}_{\underline{q}})_\tau = \sum_{b_i \in \tau} \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}}$ does not have any critical point on $Hom_{\mathbb{Z}}(N, \mathbb{C}^*)$. If we consider the whole family $\widetilde{W}$, the following holds

Proposition 4.2. *1. $\widetilde{W}_{\underline{q}}$ is convenient for any $\underline{q} \in Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$*

2. *There is an algebraic subvariety $Z \subset (\mathbb{C}^*)^r$ such that $\widetilde{W}_{\underline{q}}$ is non-degenerate for all $\underline{q} \notin Z$. Write $\mathcal{M}^0 := (\mathbb{C}^*)^r \backslash Z$.*
3. *If X_Σ is Fano, then $Z = \emptyset$.*
4. *If X_Σ is weak Fano, then there exists an $\epsilon > 0$, such that for all $\underline{q} \in Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \subset \mathbb{C}^r$ with $|q| < \epsilon$, we have $\underline{q} \notin Z$. Here the inclusion $Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \subset \mathbb{C}^r$ and the metric $|\cdot|$ refer to the chosen coordinates (q_a) on $Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$.*

Examples: In order to make the above construction more transparent, let us consider some simple but important examples.

1. The mirror of projective spaces:

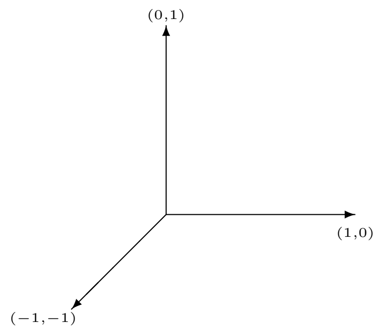

Figure 1: The fan of $\mathbb{P}^2$

The fan of $\mathbb{P}^n$ consists of $n+1$ rays (see figure 1 for the case n=2), namely, the standard vectors e_i for $i = 1, \dots, n$ in $\mathbb{Z}^n$ and the additional vector $\sum_{i=1}^n -e_i$. Hence the exact sequence (7) reads

$$0 \longrightarrow \mathbb{L} \cong \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}} \mathbb{Z}^{n+1} \xrightarrow{\begin{pmatrix} 1 & 0 & \dots & \dots & -1 \\ 0 & 1 & \dots & \dots & -1 \\ \vdots & \vdots & \ddots & \dots & -1 \\ 0 & 0 \dots & 0 & 1 & -1 \end{pmatrix}} \mathbb{Z}^n \longrightarrow 0$$

where we have chosen a basis of $\mathbb{L}^\vee$ corresponding to the Poincaré dual of a hyperplane. Hence by dualizing and tensoring with $\mathbb{C}^*$ we obtain the Landau-Ginzburg model given by the restriction of the linear function $W = w_0 + \ldots + w_n$ to the fibres of the fibration $\beta : (\mathbb{C}^*)^{n+1} \to \mathbb{C}^*$, which sends $(w_0, \dots, w_n)$ to $w_0 \cdot \ldots \cdot w_n$. Choosing the section $l : \mathbb{Z} \cong \mathbb{L}^\vee \to \mathbb{Z}^{n+1}$, $l(m) = (m, 0, \dots) \in \mathbb{Z}^n$ we obtain that $\widetilde{W}_{\mathbb{P}^n}(y_1, \dots, y_n, q) = y_1 + \ldots + y_n + \frac{q}{y_1 \cdot \ldots \cdot y_n}$.

2. The mirror of the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$:

   Recall that for any $k \in \mathbb{N}$, the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1})$ is called the k-th Hirzebruch surface and denoted by $\mathbb{F}_k$. However, these surfaces are Fano only for $k = 0$ (this is the rather trivial case $\mathbb{P}^1 \times \mathbb{P}^1$) and $k = 1$. For $k = 2$, $\mathbb{F}_2$ has a nef anticanonical divisor, but this is no longer true for the higher $\mathbb{F}_k$'s. Hence we can construct Landau-Ginzburg models for $\mathbb{F}_0$, $\mathbb{F}_1$ and $\mathbb{F}_2$. Let us concentrate on the last two cases. These are toric varieties defined by the fans shown in figure 2.

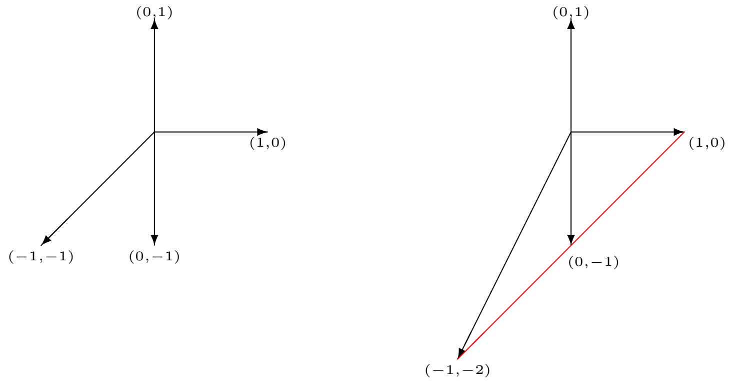

Figure 2: The fans of $\mathbb{F}_1$ and $\mathbb{F}_2$

The exact sequence (7) takes the following form for $\mathbb{F}_1$

$$0 \longrightarrow \mathbb{L} \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

so that the Landau-Ginzburg model is the following two parameter family of Laurent polynomials:

$$\widetilde{W}_{\mathbb{F}_1} = x + y + \frac{q_1 \cdot q_2}{xy} + \frac{q_2}{y}$$

where we have chosen the section of the map $\mathbb{Z}^4 \to \mathbb{L}^\vee$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $\mathbb{F}_2$, we have the following exact sequence

$$0 \longrightarrow \mathbb{L} \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

and we obtain:

$$\widetilde{W}_{\mathbb{F}_2} = x + y + \frac{q_1 \cdot q_2^2}{xy^2} + \frac{q_2}{y}$$

where the section is given by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Notice that for $q_1 = \frac{1}{4}$, $W_{\mathbb{F}_2}(x, y, \frac{1}{4}, q_2)$ is degenerate: the Laurent polynomial $x + \frac{q_2^2}{4xy^2} + \frac{q_2}{y}$ has critical points on the torus $(\mathbb{C}^*)^2$. This reflects the fact that $\mathbb{F}_2$ is nef but not Fano, and can be seen on its fan from the fact that there is a lattice point on the boundary of the convex hull defined by the rays of the fan of $\mathbb{F}_2$ (the point $(1, -1)$ on the red line).

5 Gauß-Manin systems and hypergeometric differential equations

In this section we describe how to associate a system of differential equations to the Landau-Ginzburg models defined above. These systems will ultimately be equal to the quantum $\mathcal{D}$-module, and this is precisely the kind of mirror correspondences we are interested in. However, the differential equations we are going to consider are interesting in their own right, and have been studied since a long time. They are related to the classical *Gauß-Manin connection* but they are more general in two respects: First, one has to take into account singularities which occur at the critical points of the Laurent polynomials. The corresponding object is called Gauß-Manin system, and is constructed in a functorial way using the general notion of direct image in the category of $\mathcal{D}$-modules. Equivalently, and this is the point of view that we are going to adapt below, it is obtain as a twisted de Rham cohomology group. The second difference to the classical setup is that in order to mach with the quantum $\mathcal{D}$-module, we have to consider a variant of the Gauß-Manin system, which is obtained by a partial Fourier transformation. The solutions of the transformed systems can be obtained as oscillating integrals, whereas the original Gauß-Manin system consists of differential equations satisfied by period integrals over vanishing cycles. We will not explain in detail this more analytic point of view, one can find in [Her03, chapter 8] some explanations for the related case of germs of functions with isolated singularities. Notice also that the construction described below is carried out in the analytic category in [Iri09].
In order to establish the mirror correspondence via differential equations satisfied by oscillating integrals, one needs to have a concrete description of these $\mathcal{D}$-modules. Luckily, such a description is available in the toric case, and the systems obtained are said to have a hypergeometric structure. Hypergeometric functions and hypergeometric differential equations have a long history, starting at least with Gauß. We will not review here these developpements (one may consult, e.g., [Sti07] for some classical aspects of the theory). Instead, we start with the following definition of the so-called GKZ-systems (after Gelfand, Kapranov and Zelevinski) taken from [GGZ87] and [GZK89] (see also the more recent reference [Ado94]). Any system of hypergeometric equation can be rewritten as a GKZ-system (or a reduction of it). We also discuss the main properties of these $\mathcal{D}$-modules, and for that purpose we recall some of the most important notions related to algebraic $\mathcal{D}$-modules.

Definition 5.1 (GKZ- or A-hypergeometric system). *Consider a lattice $\mathbb{Z}^n$ and vectors $\underline{a}_1,\ldots,\underline{a}_m \in \mathbb{Z}^n$ which we also write as a matrix $A=(\underline{a}_1,\ldots,\underline{a}_n)$. Moreover, let $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{C}^n$. Write $\mathbb{L}$ for the module of relations of A and $\mathcal{D}_{\mathbb{C}^m}$ for the sheaf of rings of algebraic differential operators on $\mathbb{C}^m$ (where we choose $x_1,\ldots,x_m$ as coordinates). Define*

$$\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{C}^s} / \left((\Box_{\underline{l}})_{\underline{l}\in\mathbb{L}} + (Z_k)_{k=1,\ldots t} \right), \tag{10}$$

where

$$\begin{aligned} \Box_{\underline{l}} &:= \textstyle\prod_{i:l_i<0} \partial_{x_i}^{-l_i} - \prod_{i:l_i>0} \partial_{x_i}^{l_i} \\ Z_k &:= \textstyle\sum_{i=1}^s b_{ki} x_i \partial_{x_i} + \beta_k \end{aligned}$$

$\mathcal{M}_A^\beta$ is called hypergeometric system.

Notice that although the definition of $\mathcal{M}_A^\beta$ involves infinitely many operators (one for each $\underline{l}\in\mathbb{L}$ plus the finite number of operators Z_k), the denominator of in formula (10) is of course generated by a finite number of elements of $\mathcal{D}_{\mathbb{C}^s}$. However, and this is one important feature of the theory of GKZ-systems, in order to generate the ideal $(\Box_{\underline{l}})_{\underline{l}\in\mathbb{L}}$, it is in general not not sufficient to take operators $\Box_{\underline{l}}$ where $\underline{l}$ runs through a basis of $\mathbb{L}$.
Next we are going to describe some general properties of GKZ-systems. In order to do this, we first recall some basic notions from the general theory of algebraic $\mathcal{D}$-modules. As an example for a reference, the interested reader may consult [HTT08] for the proofs and many more results.

Definition-Lemma 5.2. *Let X be a smooth algebraic variety over $\mathbb{C}$, $\mathcal{D}_X$ the sheaf of algebraic differential operators on X and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Consider the usual filtration $F_\bullet$ on $\mathcal{D}$ by orders of operators. The associated graded (sheaf of) rings $gr_\bullet(\mathcal{D})$ equals the structure sheaf on the cotangent bundle T^*X.*

1. *A filtration $F_\bullet\mathcal{M}$ is good if it is compatible with $F_\bullet\mathcal{D}$, i.e., if $F_k\mathcal{D}\cdot F_l\mathcal{M}\subset F_{k+l}\mathcal{M}$ holds for all k,l and if this is an equality for l sufficiently large and if moreover we have that $gr_\bullet(\mathcal{M})$ is $gr_\bullet(\mathcal{D})=\mathcal{O}_{T^*X}$-coherent.*

2. *The characteristic variety* $\mathrm{char}(\mathcal{M})$ *of a $\mathcal{M}$ is the reduced support of $gr_\bullet(\mathcal{M})$ in T^*X. This subvariety does not depend on the choice of the good filtration $F_\bullet\mathcal{M}$.*

3. *$\mathcal{M}$ is holonomic iff* $\mathrm{char}(M)$ *is a Lagrangian subvariety of $T^*\mathbb{C}^m$ for its natural symplectic structure, that is, iff the restriction of the symplectic form to all tangent spaces of smooth points of* $\mathrm{char}(M)$ *vanishes. Equivalently, $\mathcal{M}$ is holonomic iff $\mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for all $p\neq n$.*

4. *Let $\pi:T^*X\to X$ be the canonical projection. Let* $\mathrm{char}(\mathcal{M})=\bigcup C_i$ *be the decomposition of* $\mathrm{char}(\mathcal{M})$ *into irreducible components. Suppose that the zero section T^*_XX of T^*X is a component of* $\mathrm{char}(\mathcal{M})$ *and that it is equal to C_1. Define the singular support* $\mathrm{Sing}(\mathcal{M})$ *to be* $\pi(\mathrm{char}(M)\backslash C_1)$*, if $T^*_XX\not\subset$* $\mathrm{char}(\mathcal{M})$*, then* $\mathrm{Sing}(\mathcal{M})=\mathrm{Supp}(\mathcal{M})$*. The restriction $\mathcal{M}_{|X\backslash C_1}$ is $\mathcal{O}_X$-locally free of rank k ($k=0$ if* $\mathrm{Supp}(\mathcal{M})\subsetneq X$*), and k is called the holonomic rank of $\mathcal{M}$. It is equal to the dimension of the space of (say, holomorphic) local solutions of $\mathcal{M}$ near a point in* $X\backslash\mathrm{Sing}(\mathcal{M})$*.*

5. *$\mathcal{M}$ is regular if its restriction to any curve $C\subset X$ is so, and this last condition can be reduced to the usual condition of regularity for linear systems of differential equations in one variable. A precise definition of regularity can be found, e.g., in [HTT08, chapter 6].*

With all these notions in mind, we can describe the main properties of the GKZ-systems.

Proposition 5.3. *Let A, β and $\mathcal{M}_A^\beta$ be as above.*

1. *$\mathcal{M}_A^\beta$ is holonomic for any A and any β. For generic β, the holonomic rank of $\mathcal{M}_A^\beta$ is $n!\cdot$* $\mathrm{vol}(\Delta(\underline{a}_1,\dots,\underline{a}_n))$*, where $\Delta(\underline{a}_1,\dots,\underline{a}_n)$ denotes the convex hull of $\underline{a}_1,\dots,\underline{a}_n$ in $\mathbb{R}^n$ and* $\mathrm{vol}(-)$ *is the normalized volume, which takes the value 1 on the hypercube $[0,1]^n\subset\mathbb{R}^n$. In particular, if β is generic, then $n!\cdot\mathrm{vol}(\Delta(\underline{a}_1,\dots,\underline{a}_n))$ is the dimension of the solution space of the differential system defined by $\mathcal{M}_A^\beta$ at a generic point of $\mathbb{C}^m$.*

2. *$\mathcal{M}_A^\beta$ is regular if and only if $\underline{a}_1,\dots,\underline{a}_n$ is contained in an affine hyperplane of $\mathbb{Z}^n$.*

3. *The singular locus equals the degeneracy locus of the Laurent polynomial $\sum_{i=1}^m x_i\cdot\underline{y}^{\underline{a}_i}$, where $\underline{y}^{\underline{a}_i}:=\prod_{k=1}^n y_k^{a_{ki}}$, i.e., is equal to*

$$\left\{(\lambda_1,\dots,\lambda_m)\in\mathbb{C}^m\,|\,\forall\tau\in\partial\Delta(\underline{a}_1,\dots,\underline{a}_n),\sum_{j:\underline{a}_j\in\tau}\lambda_j\underline{y}^{\underline{a}_j}\text{ has a critical point in }(\mathbb{C}^*)^n\right\}$$

The differential systems that will appear in the mirror correspondence that we are going to explain are variants of special GKZ-systems. First, one starts with a regular GKZ-system, this is achieved by forcing the columns of the defining matrix to be contained in an affine hyperplane (see point (2) in the above proposition). Next there are two modifications to be carried out: A restriction of the parameter space, this corresponds to the chosen embedding l from equation (9), and finally a Fourier-Laplace transformation which introduces irregular singularities. Let us explain these steps in some more detail. For the restriction just mentioned, we have to use the inverse image functor of $\mathcal{D}$-modules, which we do not explain here (see again [HTT08] for details).

Definition-Lemma 5.4. 1. *For a given matrix $A\in M(n\times m,\mathbb{Z})$ with columns $\underline{a}_1,\dots,\underline{a}_m$, let $\widetilde{\underline{a}}_i:=(1,\underline{a}_i)\in\mathbb{Z}^{n+1}$ for $i=1,\dots,m$ and $\widetilde{\underline{a}}_0=(1,\underline{0})$. Write $\widetilde{A}$ for the matrix with columns $\widetilde{\underline{a}}_0,\widetilde{\underline{a}}_1,\dots,\widetilde{\underline{a}}_m$ and consider the hypergeometric systems $\mathcal{M}_{\widetilde{A}}^\beta$ for $\beta\in\mathbb{C}^{n+1}$.*

2. *For any $\mathbb{C}[\lambda_0,\dots,\lambda_m]\langle\partial_0,\dots,\partial_m\rangle$-module $\mathcal{M}$, define $\mathrm{FL}_{\lambda_0}^{z^{-1}}(\mathcal{M})[z]$ to be the operation of replacing ∂_0 by z^{-1}, λ_0 by $z^2\partial_z$ and by inverting z^{-1}, i.e., by tensoring with $\mathbb{C}[z^\pm,\lambda_1,\dots,\lambda_m]$ over $\mathbb{C}[z^{-1},\lambda_1,\dots,\lambda_m]$. Then $\mathrm{FL}_{\lambda_0}^{z^{-1}}(\mathcal{M})[z]$ is a $\mathbb{C}[z^\pm,\lambda_1,\dots,\lambda_m]\langle\partial_z,\partial_1,\dots,\partial_m\rangle$-module.*

3. *Consider the chosen section* $l : Hom_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \to (\mathbb{C}^*)^{\Sigma(1)} \cong (\mathbb{C}^*)^m$ *from equation* (9). *Define*
$$\mathcal{QM}_{\widetilde{A}} := \Big((\mathrm{id}_z, l)^+ \, \mathrm{FL}_{\lambda_0}^{z^{-1}} (\mathcal{M}_{\widetilde{A}}^{(1,\underline{0})})[z] \Big)$$
then $\mathcal{QM}_{\widetilde{A}}$ *is given as the quotient of* $\mathbb{C}[z^{\pm}, q_1^{\pm}, \ldots, q_r^{\pm}]\langle \partial_z, \partial_{q_1}, \ldots \partial_{q_r} \rangle$ *by the left ideal generated by*
$$\widetilde{\Box}_{\underline{l}} := \prod_{a: p_a(\underline{l})>0} q_a^{p_a(\underline{l})} \prod_{i: l_i<0} \prod_{\nu=0}^{-l_i-1} (\sum_{a=1}^r m_{ia} z q_a \partial_{q_a} - \nu z) - \prod_{a: p_a(\underline{l})<0} q_a^{-p_a(\underline{l})} \prod_{i: l_i>0} \prod_{\nu=0}^{l_i-1} (\sum_{a=1}^r m_{ia} z q_a \partial_{q_a} - \nu z)$$
for any $\underline{l} \in \mathbb{L}$ *and by the single operator*
$$z^2 \partial_z - \sum_{a=1}^r K_{X_\Sigma}(p_a^\vee) q_a z \partial_{q_a}.$$
4. *Denote by* ${}_0\mathcal{QM}_{\widetilde{A}} \subset \mathcal{QM}_{\widetilde{A}}$ *the* $\mathbb{C}[z, q_1^{\pm}, \ldots, q_r^{\pm}]$*-subalgebra generated by* $z^2\partial_z$ *and* $z q_a \partial_{q_a}$ *where* $a = 1, \ldots, r$. *Then* ${}_0\mathcal{QM}_{\widetilde{A}}$ *is* $\mathbb{C}[z, q_1^{\pm}, \ldots, q_r^{\pm}]$*-free of rank* $n! \cdot \mathrm{vol}(\Delta(\underline{a}_1, \ldots, \underline{a}_m))$ *and it comes equipped with a connection operator with a pole of Poincaré rank one along* $z = 0$.

There are many results in the literature concerning solutions of hypergeometric differential equations. In our setup, the I-function introduced above (definition 3.6) will yield (cohomology valued) solutions of the GKZ-systems defined by a toric variety.

Proposition 5.5. *Put* $\widetilde{I} := z^{K_{X_\Sigma}} \cdot z^\mu \cdot I$ *(μ is the grading operator on cohomology classes) and write* $\widetilde{I} = \sum_{t=0}^s \widetilde{I}_t \cdot T_t$*, where* $T_0, T_1, \ldots, T_s$ *of* $H^*(X_\Sigma, \mathbb{C})$ *is a homogeneous basis of* $H^*(X_\Sigma, \mathbb{C})$ *as above. Then the components* $\widetilde{I}_t$ *yield solutions of the differential system* $\mathcal{QM}_{\widetilde{A}}$ *over a subset of* $\mathbb{C}_z^* \times (\mathbb{C}^*)^r$ *on which the* I*-function is convergent. For a precise statement, see [RS10, proposition 3.12 and corollary 3.13].*

The next step is to explain how we can associate a (version of a) GKZ-system to the Landau-Ginzburg models defined in section 4. As mentioned in the beginning of this section, this is done using the so-called twisted de Rham cohomology, which is a version of the more general Gauß-Manin system. Here is the corresponding definition, which is simplified to fit to our purpose.

Definition 5.6. *Let* U, K *be a smooth affine algebraic varieties with* $\dim_{\mathbb{C}}(U) = n$ *and* $\varphi = (F, pr) : U \times K \to \mathbb{C} \times K$ *be an affine morphism, where* $F \in \mathcal{O}_{U \times K}$ *and* $pr : U \times K \to K$ *is the projection. Let* z *be a new variable and consider the following complex of* $\mathcal{O}_{U\times K}$*-modules with a* $\mathcal{O}_{\mathbb{C}\times K}$*-linear differential*
$$\big(\Omega^\bullet_{pr}[z], zd - d\varphi\wedge \big)$$
where $\Omega^\bullet_{pr} := \Omega^\bullet_U \otimes_{\mathcal{O}_U} \mathcal{O}_{K\times U}$ *are the differential forms relative to the projection map* pr. *Call* $H^n(\varphi) := H^n(\Omega^\bullet_{pr}[z], zd - d\varphi\wedge)$ *the twisted de Rham cohomology of* φ *(more precisely, it is the de Rham complex of* U *with the differential twisted by* φ*). Notice that in the examples we are interested in, all other cohomology groups* $H^i(\varphi)$ *for* $i \neq 0$ *will vanish. Moreover, define a connection operator* $\nabla : H^n(\varphi) \to H^n(\varphi) \otimes \Omega^1_{\mathbb{C}\times U}(*\{0\} \times U)$ *by*
$$\begin{aligned} \nabla_{\partial_z}(\omega) &:= -z^{-2} \cdot F \cdot \omega \\ \nabla_X(\omega) &:= \mathrm{Lie}_X(\omega) + z^{-1} \cdot X(F)\omega \end{aligned}$$
where $\omega \in \Omega^k_{pr}$ *and* $X \in \mathcal{T}_K$ *and where the above formulas are extended to the whole module* $H^n(\varphi)$ *by the Leibniz rule.*

An appropriate version of the GKZ-system can be used to compute the twisted de Rham complex of the Landau-Ginzburg models we are interested in. Hence, let $(\widetilde{W}, pr) : (\mathbb{C}^*)^n \times (\mathbb{C}^*)^r \to \mathbb{C} \times (\mathbb{C}^*)^r$ be a morphism as in section 4. Then we have the following.

Theorem 5.7 ([RS10, corollary 3.3], [Iri09])**.** *There is an isomorphism of* $\mathcal{O}_{\mathbb{C}_z\times(\mathbb{C}^*)^r}$*-modules with connections on* $\mathbb{C}_z \times (\mathbb{C}^*)^r$
$$H^n(\widetilde{W}, pr) \cong {}_0\mathcal{QM}_{\widetilde{A}}. \tag{11}$$
${}_0\mathcal{QM}_{\widetilde{A}}$ *(and hence also* $H^n(\widetilde{W}, pr)$*) is locally free of rank* $n! \cdot \mathrm{vol}(\Delta(\underline{a}_1, \ldots, \underline{a}_m))$ *when restricted to the complement of the degeneracy locus* Z *defined in proposition 4.2. In particular, both objects are locally free over the whole of* $\mathbb{C}_z \times (\mathbb{C}^*)^r$ *if* X_Σ *is Fano.*

Examples: Using the last theorem, we can give an explicit expression for the twisted de Rham cohomology for the examples considered in section 4.

1. $X_\Sigma = \mathbb{P}^n$: Recall that $h^2(\mathbb{P}^n) = 1$ and hence $(\widetilde{W}_{\mathbb{P}^n}, pr) : (\mathbb{C}^*)^n \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}^*; (x_1, \ldots, x_n, q) \mapsto x_1 + \ldots + x_n + q/(x_1 \cdot \ldots \cdot x_n)$. Then we have

$$H^n(\widetilde{W}, pr) \cong \frac{\mathbb{C}[z, q^{\pm}]\langle z^2\partial_z, qz\partial_q\rangle}{((zq\partial_q)^{n+1} - q, z^2\partial_z + (n+1)q\partial_q)} \tag{12}$$

2. $X_\Sigma = \mathbb{F}_1$: We have $h^2(\mathbb{F}_1) = 2$, $(\widetilde{W}_{\mathbb{F}_1}, pr) : (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \to \mathbb{C} \times (\mathbb{C}^*)^2; (x, y, q_1, q_2) \mapsto x + y + \frac{q_1 \cdot q_2}{xy} + \frac{q_2}{y}$ and

$$H^n(\widetilde{W}, pr) \cong \frac{\mathbb{C}[z, , q_1^{\pm}, q_2^{\pm}]\langle z^2\partial_z, z\partial_{q_1}, q_2 z\partial_{q_2}\rangle}{((q_1 z\partial_{q_1})^2 - q_1(q_2 z\partial_{q_2} - q_1 z\partial_{q_1}), q_2 z\partial_{q_2} \cdot q_1 z\partial_{q_1} - q_2, z^2\partial_z + q_1 z\partial_{q_1} + 2q_2 z\partial_{q_2}))}$$

3. $X_\Sigma = \mathbb{F}_2$: We have $h^2(\mathbb{F}_1) = 2$, $(\widetilde{W}_{\mathbb{F}_1}, pr) : (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \to \mathbb{C} \times (\mathbb{C}^*)^2; (x, y, q_1, q_2) \mapsto x + y + \frac{q_1 \cdot q_2}{xy^2} + \frac{q_2}{y}$ and

$$H^n(\widetilde{W}, pr) \cong \mathbb{C}[z, q_1^{\pm}, q_1^{\pm}]\langle z^2\partial_z, z\partial_{q_1}, q_2 z\partial_{q_2}\rangle / I$$

where I is the left ideal generated by

$$(q_1 z\partial_{q_1})^2 - q_1(q_2 z\partial_{q_2} - 2q_1 z\partial_{q_1})(q_2 z\partial_{q_2} - 2q_1 z\partial_{q_1} - 1)$$
$$(zq_2\partial_{q_2})(zq_2\partial_{q_2} - 2zq_1\partial_{q_1}) - q_2$$
$$z^2\partial_z + 2q_2 z\partial_{q_2}$$

6 Non-commutative Hodge structures

In this section, which can be read almost independently of the other parts of the text, we will discuss some results on abstract non-commutative Hodge structures (called ncHodge structures for short in the sequel). The ultimate aim is to use ncHodge structures very much like ordinary ones, in particular, one would like to study period maps, Torelli problems etc. However, for the moment these kind of techniques are available only for a restricted class of ncHodge-structures (namely, the so-called *regular* ones). However, these are in a certain sense the building blocks for more general (irregular) ncHodge structures, which, as we will see, occur in mirror symmetry. In that sense any result for the regular case will certainly also be of importance for ncHodge structures defined by Landau-Ginzburg models.
We start with the very definition of a non-commutative Hodge structure. For simplicity, we suppress any notion of weights, that is, we consider only ncHodge structures of weight 0. Such structures almost never exists in (commutative or non-commutative) geometry, however, they are technically slightly simpler to treat and the adaption to the general case is not very difficult. We also omit the grading present in the definition in [KKP08], as we are not going to discuss the (conjectural) construction of an ncHodge structures from a category, as described in loc.cit.

Definition 6.1 (**ncHodge structure**, [HS07b], [KKP08], [Sab11])**.** *A real resp. rational non-commutative Hodge structure (of weight 0) consists of the following data:*

1. *An algebraic vector bundle $\mathcal{H}$ on $\mathbb{C}_z$ (z being a fixed coordinate on $\mathbb{C}$) of rank μ.*

2. *A $\mathbb{K}$-local system $\mathcal{L}$ on $\mathbb{C}^*$ (with $\mathbb{K}$ being either $\mathbb{R}$ or $\mathbb{Q}$), together with an isomorphism*

$$\text{iso} : \mathcal{L} \otimes_{\mathbb{K}} \mathcal{O}_{\mathbb{C}^*} \longrightarrow \mathcal{H}_{|\mathbb{C}^*}$$

such that the connection ∇ induced by iso *has a pole of order at most 2 at $z = 0$ and a regular singularity at $z = \infty$.*

3. *A polarizing symmetric form $P : \mathcal{L} \otimes j^*\mathcal{L} \to \underline{\mathbb{K}}_{\mathbb{C}^*}$ (where $j(z) = -z$), which induces a non-degenerate pairing*

$$P : \mathcal{H} \otimes_{\mathcal{O}_\mathbb{C}} j^*\mathcal{H} \to \mathcal{O}_\mathbb{C}$$

In particular, we have an induced non-degenerate pairing $[P] : \mathcal{H}/z\mathcal{H} \times \mathcal{H}/z\mathcal{H} \to \mathbb{C}$.

4. *There is an isomorphism*

$$\mathcal{H} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}_{\mathbb{C}}[*\{0\}] \cong \oplus_{i=1}^{k} (\mathcal{R}_i, \nabla_i) \otimes e^{u_i/z}$$

   *where $u_1, \ldots, u_k \in \mathbb{C}$, where $\widehat{\mathcal{O}}_{\mathbb{C}}$ denotes the completion of $\mathcal{O}_{\mathbb{C}}$ at $z = 0$ and where $(\mathcal{R}_i, \nabla_i)$ are formal meromorphic bundles (i.e., locally free $\widehat{\mathcal{O}}_{\mathbb{C}}(*\{0\})$-modules) equipped with a connection with **regular** singularity at $z = 0$.*

   If all u_i in this decomposition are equal to zero, then $\mathcal{H} \otimes \mathcal{O}_{\mathbb{C}}(*0)$ is a regular $\mathcal{D}_{\mathbb{C}_z}$-module, and we say that the $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ is regular in this case.

5. *Consider the morphism $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\gamma(z) = 1/\overline{z}$, and glue the bundle $\mathcal{H}$ on $\mathbb{C}$ and the bundle $\overline{\gamma^*\mathcal{H}}$ on $\mathbb{P}^1\backslash\{0\}$ via an identification of the local systems $\mathcal{L} \otimes_{\underline{\mathbb{K}}} \underline{\mathbb{C}}$ and $\overline{\gamma^*\mathcal{L} \otimes_{\underline{\mathbb{K}}} \underline{\mathbb{C}}}$ on $\mathbb{C}^*$. Using the flat structure, it suffices to define this identification on S^1 only and here it is given by complex conjugation called τ, that is, by conjugation with respect to $\mathbb{K}$-structure $\mathcal{L}$ in $\mathcal{L}_{\mathbb{C}}$. Call the resulting holomorphic bundle $\widehat{\mathcal{H}} \to \mathbb{P}^1$. Then we call $\mathcal{H}$ pure iff $\widehat{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^1}^{\mu}$ and pure polarized if it is pure, and if the hermitian form*

$$h := P(-, \tau-) : H^0(\mathbb{P}^1, \widehat{\mathcal{H}}) \times H^0(\mathbb{P}^1, \widehat{\mathcal{H}}) \longrightarrow \mathbb{C}$$

   is positive definite. Notice that τ induces an anti-linear involution on the space of global sections of the trivial bundle $\widehat{\mathcal{H}}$.

The following result gives a partial explanation of the term "ncHodge": namely, it shows how ordinary Hodge structures can be seen as ncHodge-structures.

Proposition 6.2 ([KKP08, lemma 2.9],[HS07b, section 5])**.** *The functor sending a real resp. rational Hodge structure $(V_{\mathbb{K}}, F^\bullet V_{\mathbb{C}}, w)$ to the ncHodge structure given by $\mathcal{H} := \oplus_{k\in\mathbb{Z}} z^{-k} F^k V_{\mathbb{C}} \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ (where $\mathcal{L} \cong V_{\mathbb{K}} \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$) is fully faithful. Its image are the ncHodge structures where ∇ has a logarithmic pole (i.e., a pole of order 1) on $\mathcal{H}$ at zero and such that the monodromy of ∇ is trivial.*

Remark: We will not give any more details on the origins of the name non-commutative Hodge structures. The basic idea (which is described in [KKP08] but which is for the moment merely a collection of conjectures) is that one can find such structures starting from a certain triangulated category. The object which is supposed to underly a non-commutative Hodge structure is the so-called negative cyclic homology of this category. In some special cases, one expects to get back ncHodge structures constructed directly from geometric input data via this general categorical setup. See, e.g. [Shk11] for the case of isolated hypersurface singularities (the example from theorem 6.9 below).
A very important feature of the theory is the study of families of ncHodge structures, very much similar to the case of ordinary Hodge structures. The classical notion of *Griffiths transversality* is expressed as a certain pole order property of a family of ncHodge structures.

Definition 6.3. *A variation of (pure resp. pure polarized) ncHodge structures on a complex manifold M is a vector $\mathcal{H}$ bundle on $\mathbb{C}_z \times M$, equipped with a connection operator with poles along $\{0\} \times M$ such that*

1. *For any vector field X on M, $\mathcal{H}$ is invariant under the operator $z\nabla_X$.*

2. *For any point $c \in M$, the restriction of $\mathcal{H}$ to $\mathbb{C}_z \times \{c\}$ is a (pure resp. pure polarized) ncHodge structure in the sense of definition 6.1*

One checks that for a variation of ncHodge structures coming from an ordinary one, that is, such that the restriction to each point in the parameter space lies in the essential image of the functor considered in proposition 6.2, the pole order property (1.) in the above definition is equivalent to the Griffiths transversality property of the variation of Hodge structures we started with.
A regular ncHodge structure has a set of discrete invariants, the so called spectral numbers. They will be used below for the construction of classifying spaces. We give the definition together with some of the main properties.

Definition-Lemma 6.4. *Let $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ be a* ***regular*** *ncHodge structure and suppose that the monodromy of the local system $\mathcal{L}$ is quasi-unipotent, i.e, that the monodromy operator T on the space of (flat multivalued) sections of $\mathcal{L}$ satisfies $(T^n - Id)^m = 0$ for some non-negative integers m, n. This condition is equivalent to the fact that the eigenvalues of T are roots of unity, and it is satisfied in virtually all examples coming from geometry.*

1. *Define the spectrum (which is actually an invariant of $\mathcal{H}$ and ∇ only) by*

$$\mathrm{Sp}(\mathcal{H}, \nabla) = \sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{C}} \left(\frac{Gr_V^{\alpha} \mathcal{H}}{Gr_V^{\alpha} z\mathcal{H}} \right) \cdot \alpha \in \mathbb{Z}[\mathbb{Q}],$$

   where $V^\bullet \mathcal{H}(*D)$ is the canonical V-filtration, also called Kashiwara-Malgrange filtration, on the $\mathcal{D}_{\mathbb{C}_z}$-module $\mathcal{H}(*0)$ (see, e.g. [Her02, section 7.2]). It induces a filtration $V^\bullet\mathcal{H}$ on the lattice $\mathcal{H} \subset \mathcal{H}(*0)$, which is used in the above definition through its graded parts $Gr_V^{\alpha}\mathcal{H}$. Notice that the V-filtration is indexed by $\mathbb{Q}$, which corresponds to the quasi-unipotency of $\mathcal{L}$.

   We also write $\mathrm{Sp}(H, \nabla)$ as a tuple $\alpha_1, \ldots, \alpha_\mu$ of μ numbers (with $\mu = \mathrm{rank}(H)$), ordered by $\alpha_1 \leq \ldots \leq \alpha_\mu$.

2. *α is a spectral number, that is, we have $\dim_{\mathbb{C}} (Gr_V^{\alpha}(\mathcal{H}/z\mathcal{H})) > 0$, only if $e^{-2\pi i \alpha}$ is an eigenvalue of the monodromy operator T.*

3. *The spectrum satisfies $\alpha_i = -\alpha_{\mu+1-i}$ (more generally, if we allow weights, then we have $\alpha_i + \alpha_{\mu+1-i} = w$ for an ncHodge structure of weight w).*

A fundamental tool in the study of Hodge structures is the theory of classifying spaces and period maps associated to a variation of Hodge structures. A similar result exists in the non-commutative case, and can be expressed as follows.

Theorem 6.5. 1. *[HS10, theorem 7.3] Fix a quasi-unipotent $\mathbb{K}$-local system $\mathcal{L}$ on $\mathbb{C}^*$ and the polarizing form $P : \mathcal{L} \otimes j^*\mathcal{L} \to \underline{\mathbb{K}}_{\mathbb{C}^*}$. Fix also a rational number α_1 such that $e^{-2\pi i \alpha_1}$ is an eigenvalue of the monodromy of $\mathcal{L}$. Moreover, suppose that $\alpha_1 \leq 0$ (or, more generally, that $\alpha_1 \leq \frac{w}{2}$, where w is a fixed integer that will be the weight of the ncHodge structures to consider). Put*

$$\begin{aligned} \mathcal{M} \;\; := \;\; & \big\{ (\mathcal{H}, \nabla_z) \,|\, \mathcal{H} \to \mathbb{C}_z \textit{ vector bundle, } \nabla_z \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_{|\mathbb{C}_z^*}) \text{ connection }, (z^2\nabla_z)(\mathcal{H}) \subset \mathcal{H}, \\ & \mathrm{Sp}(\mathcal{H}, \nabla) \subset [\alpha_1, -\alpha_1] \cap \mathbb{Q}, \exists \text{ iso} : \mathcal{L} \otimes_{\underline{\mathbb{K}}_{\mathbb{C}^*}} \mathcal{O}_{\mathbb{C}^*} \xrightarrow{\cong} \mathcal{H}_{|\mathbb{C}^*}, P(\mathcal{H}, \mathcal{H}) \subset z^w \mathcal{O}_{\mathbb{C}_z} \textit{ non-degenerate} \big\} \end{aligned}$$

   Then $\mathcal{M}$ is a projective variety which is stratified by locally closed smooth subvarieties parameterizing bundles with connection with fixed spectral numbers.

   $\mathcal{M}$ comes equipped with a universal bundle $\mathcal{H}^{\mathcal{M}} \to \mathbb{C}_z \times \mathcal{M}$ with a relative connection ∇_z, and a polarizing form defined by P.

2. *[HS10, section 8] Define*

$$\mathcal{M}^{pp} := \big\{ x \in \mathcal{M} \,|\, (\mathcal{H}^{\mathcal{M}}, \mathcal{L}, \mathrm{iso}, P)_{|\mathbb{C} \times \{x\}} \text{ is a pure polarized ncHodge structure} \big\}$$

   (this is an open subvariety of $\mathcal{M}$), then the tangent sheaf $\Theta_{\mathcal{M}^{pp}}$ of $\mathcal{M}^{pp}$ can be endowed with a positive definite hermitian metric h, which defines a distance function d_h on $\mathcal{M}^{pp}$.

3. *[HS10, theorem 8.6] The metric space $(\mathcal{M}^{pp}, d_h)$ is complete.*

The following result describes the period maps which are analogues of the classical period maps for variations of ordinary Hodge structures.

Proposition 6.6. *Let $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ be a variation of regular pure polarized ncHodge structures on a simply connected manifold M, and let α_1 be the smallest spectral number of the restriction of $(\mathcal{H}, \nabla)$ to a generic point of M. Then there is a period map $\phi_{ncHodge} : M \to \mathcal{M}^{pp}$ satisfying $\phi^*_{ncHodge}(\mathcal{H}^{\mathcal{M}}) \cong \mathcal{H}$. If the spectrum of $(\mathcal{H}, \nabla)$ is constant on M, then the holomorphic sectional curvature κ of the metric h on $\Theta_{\mathcal{M}^{pp}}$ will be negative and bounded from above by a negative number on the image $\mathrm{Im}(d\phi_{ncHodge})$ of the derivative $d\phi_{ncHodge} : \mathcal{T}_M \to \phi^*_{ncHodge}\Theta_{\mathcal{M}^{pp}}$ of the period map.*

Using standard tools from complex hyperbolic analysis, we obtain the following two consequences.

Corollary 6.7. *1. [HS07a, corollary 4.5] Let $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ be a variation of pure polarized regular ncHodge-structures on $\mathbb{C}^n$ with constant spectrum. Then the associated period map $\phi_{ncHodge}$ is constant, in other words $\mathcal{H}$ is stable under ∇. One says that $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ is a trivial variation of ncHodge structures in this case.*

2. [HS10, theorem 9.5] Let X be a complex manifold, $Z \subset X$ a complex space of codimension at least two. Suppose that the complement $Y := X \backslash Z$ is simply connected. Let $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ be a variation of pure polarized regular ncHodge-structures on the complement Y which has constant spectral numbers. Then this variation extends to the whole of X, with possibly jumping spectral numbers over Z.

Remarks:

1. The first statement from the above corollary even extends to the irregular case. However, if we do not suppose that the connection ∇ is regular along $z = 0$, then we need some kind of regularity along the boundary of the parameter space, e.g. along $\mathbb{P}^n \backslash \mathbb{C}^n$. Such a property exists, and is called *tameness* of the associated *harmonic bundle*. See [HS10, corollary 6.3] for a precise statement.

2. The second statement treats extensions of regular ncHodge structures over codimension two subvarieties. The question how to extend a variation over a *divisor* is perhaps even more important. In that case, we need the full power of the limit statements for harmonic bundle, due to Mochizuki (see [Moc07]). We also need to take care of the possible monodromy along the boundary divisor, this can be done by adding the structure of a lattice (i.e., a $\mathbb{Z}$-local subsystem of $\mathcal{L}$). A precise formulation of the result for extensions over divisors can be found in [HS10, theorem 9.7].

The following fundamental theorem shows how ncHodge structures occur in geometry. It concerns a certain type of regular functions on smooth affine varieties, called cohomologically tame. Without giving the precise definition of this notion (see [Sab06]) let us just mention that such functions have isolated critical points and satisfy moreover an assumption concerning their behavior at infinity (in the fibres of an appropriate compactification). In particular, convenient and non-degenerate Laurent polynomials, like the Landau-Ginzburg model of a toric Fano manifold are cohomologically tame.

Theorem 6.8 ([Sab08])**.** *Let U be a smooth affine manifold and $f : U \to \mathbb{C}$ a cohomologically tame function. Then the twisted de Rham cohomology $H^n(f)$ underlies a pure polarized non-commutative Hodge structure.*

There is another important class of examples where the twisted de Rham cohomology can be equipped with an ncHodge structure. These are germs of holomorphic functions with isolated critical points also called isolated hypersurface singularities. However, and this is one of the main issues of study in this so-called local case, the corresponding structure does not necessarily satisfy the condition (5.) in definition 6.1. Nevertheless, we have the following result.

Theorem 6.9 ([HS07b, corollary 11.4])**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated hypersurface singularity. Then for a sufficiently large real number r, the twisted de Rham cohomology of an appropriate representative of the germ $r \cdot f$ underlies a pure polarized ncHodge structure.*

7 Mirror symmetry statements

Using all the objects introduced above we can now express mirror correspondences as isomorphisms of systems of differential equations. This identification relies on Givental's theorem (theorem 3.7 above), and yields an isomorphims of vector bundle with connections on $\mathbb{C}_z \times (\mathbb{C}^*)^r$. This is the first result of this section. However, from a physical point of view, we would like to express the mirror correspondence as an isomorphism of so-called **Frobenius manifolds**, which appear as moduli spaces of two dimensional topological field theory in physics. The cohomology of any, say smooth projective, variety carries a Frobenius structure which is defined precisely using the quantum multiplication. We introduce this notion here and show briefly (refering to [RS10] for more details) how to construct such a manifold

from the Landau-Ginzburg model. The final result then says that there is an isomorphism of Frobenius manifolds between the A-model and the B-model, and this can be considered to be the culminating point of this version of mirror symmetry for smooth toric varieties with ample or nef anticanonical bundle.
Let us start with the very definition of a Frobenius manifold. For our purpose, we also need an extended version called logarithmic Frobenius manifold, which takes into account the degeneration behavior of the quantum multiplication at the large radius volume limit.

Definition 7.1. *Let M be a complex manifold.*

1. *A Frobenius structure on M is given by two tensors $\circ \in (\Omega_M)^{\otimes 2} \otimes_{\mathcal{O}_M} \mathcal{T}_M$, $g \in (\Omega_M)^{\otimes 2}$ and two vector fields $E, e \in \mathcal{T}_M$ subject to the following relations.*
   - *(a) $\circ$ defines a commutative and associative multiplication on $\mathcal{T}_M$ with unit e.*
   - *(b) g is bilinear, symmetric and non-degenerate.*
   - *(c) For any $X, Y, Z \in \mathcal{T}_M$, $g(X \circ Y, Z) = g(X, Y \circ Z)$.*
   - *(d) g is flat, i.e., locally there are coordinates $t_1, \ldots, t_\mu$ on M such that the matrix of g in the basis $(\partial_{t_1}, \ldots, \partial_{t_\mu})$ is constant.*
   - *(e) Write ∇ for the Levi-Civita connection of g, then the tensor $\nabla\circ$ is totally symmetric.*
   - *(f) $\nabla(e) = 0$.*
   - *(g) $\mathrm{Lie}_E(\circ) = \circ$, $\mathrm{Lie}_E(g) = D \cdot g$ for some $D \in \mathbb{C}$*
2. *Now suppose that $\dim_{\mathbb{C}}(M) > 0$ and let $D \subset M$ be a simple normal crossing divisor. Suppose that $(M\backslash D, \circ, g, e, E)$ is a Frobenius manifold. Then we say that it has a logarithmic pole along D (or that $(M, D, \circ, g, e, E)$ is a logarithmic Frobenius manifold for short) if $\circ \in \Omega^1_M(\log D)^{\otimes 2} \otimes \mathcal{T}_M(\log D)$, $g \in \Omega^1_M(\log D)^{\otimes 2}$, $E, e \in \mathcal{T}(\log D)$ and if g is non-degenerate on $\mathcal{T}_M(\log D)$. Here $\Omega^1(\log D)$ resp. $\mathcal{T}(\log D)$ are the sheaves of logarithmic differential forms resp. logarithmic vector fields along D.*

The following is the basic result which explains why Frobenius structures enter into the mirror symmetry picture.

Theorem 7.2 (see, e.g., [Man99])**.** *Let X be smooth projective and convex (this last assumption is not essential). Define a multiplication $\circ$ on the tangent bundle of the cohomology space $H^*(X, \mathbb{C})$ as was done above before definition 3.2. Define a constant (hence flat) pairing $g(-,-)$ on $TH^*(X, \mathbb{C})$ by the non-degenerate Poincaré metric on $H^*(X, \mathbb{C})$. Put $e = 1 \in H^0(X, \mathbb{C})$ and recall from the definition of the quantum $\mathcal{D}$-module (equations (6)) that $E = \sum_{i=0}^{s} \left(1 - \frac{\deg(T_i)}{2}\right) t_i \partial_{t_i} + \sum_{a=1}^{r} k_a \partial_{T_a}$, where $\sum_{a=1}^{r} k_a T_a = c_1(X)$. Then the tuple $(H^*(X, \mathbb{C}), \circ, g, e, E)$ defines a formal germ of a Frobenius manifold at $t_0 = t_1 = \ldots = t_s = 0$.*

The fact that we do not know in general whether the quantum product is convergent forces us to restrict to formal germs in the last theorem. However, as explained above, we do not worry about convergence questions in this paper, so that we simply assume that there is a certain subspace of $H^*(X, \mathbb{C})$ on which we have a holomorphic Frobenius structure, and we also want this subspace to contain a neighborhood of the large radius limit. Then the divisor axiom for Gromov-Witten invariants yields

Lemma 7.3 ([Rei09, section 2.1.2])**.** *Let $U \subset H^0(X, \mathbb{C}) \oplus \left(H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})\right) \oplus H^{>2}(X, \mathbb{C})$ be a domain of convergence of the quantum product, and assume that a point $(t_0, \underline{q}, \underline{t})$ is contained in U if it is small enough in the standard hermitian metric of $\mathbb{C} \times (\mathbb{C}^*)^r \times \mathbb{C}^{s-r-1}$. Let $\overline{U} \subset \mathbb{C}^s$ be the closure (i.e., including points where $q_a = 0$ for some $a = 1, \ldots, r$). Then the Frobenius structure on U extends to a logarithmic Frobenius structure on $(\overline{U}, D)$, where $D = \bigcup_{a=1}^{r} D_a$, with $D_a = \{q_a = 0\}$.*

Our next task is to explain how one can construct a Frobenius structure (which will also acquire logarithmic poles along a normal crossing divisor) starting from a Landau-Ginzburg model, that is, from a family of Laurent polynomials $\widetilde{W} : (\mathbb{C}^*)^n \times (\mathbb{C}^*)^r \to \mathbb{C}$. We use the twisted de Rham cohomology constructed in section 5, and the isomorphism (11) expressing it by hypergeometric differential equations. The main step towards the construction of Frobenius manifolds is contained in the following proposition. In order to keep notations simple, we restrict to the Fano case.

Proposition 7.4 ([RS10, proposition 3.10]). *Suppose that X_Σ is smooth toric and Fano. There is a Zariski open subset $\overline{U} \subset \mathbb{C}^r$ including the limit point $\{\underline{q} = \underline{0}\} \in \mathbb{C}^r$ and an extension $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}} \to \mathbb{P}^1_z \times \overline{U}$ of ${}_0\mathcal{QM}_{\widetilde{A}}$ which is a family of trivial $\mathbb{P}^1$-bundles and such that the connection extends with a logarithmic pole along $\{z = \infty\} \cup \bigcup_{a=1}^r \{q_a = 0\}$. Moreover, the restriction $\left(\widehat{{}_0\mathcal{QM}_{\widetilde{A}}}\right)_{|\{z=0,q_a=0\}}$ is canonically equipped with a multiplication and is isomorphic as an algebra to the classical cohomology ring of X_Σ.*

Example: We give here the simplest example for which the mirror correspondence can be established directly, namely, that of the projective spaces. For the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$, a similar computation can be carried out. In fact, the representation of the twisted de Rham cohomology of the Landau-Ginzburg model of $\mathbb{P}^n$, i.e., formula (12) already gives the desired extension to $z = \infty$. More precisely, put $\omega_i := (zq\partial_q)^i$ for $i = 0, \dots, n$, then there is an isomorphism $H^n(\widetilde{W}_{\mathbb{P}^n}, pr) \cong \oplus_{i=0}^n \mathcal{O}_{\mathbb{C}_z \times (\mathbb{C}^*)^r} \omega_i$, and we have a connection

$$\nabla(\underline{\omega}) = \underline{\omega} \cdot \left[(A_0 \frac{1}{z} + A_\infty) \frac{dz}{z} - A_0 \cdot \frac{dq}{n \cdot z \cdot q} \right] \tag{13}$$

where $\underline{\omega} = (\omega_0, \dots, \omega_n)$, where

$$A_0 := \begin{pmatrix} 0 & 0 & \dots & 0 & c \cdot q \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

with $c \in \mathbb{C}^*$ and where $A_\infty = \mathrm{diag}(0, 1, \dots, n)$.
Then we can simply define $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}} := \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1_z \times (\mathbb{C}^*)^r} \omega_i$, so that $U = (\mathbb{C}^*)^r$ in this case and it is easily seen from equation 13 that the connection ∇ on $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}}$ has poles with the desired properties.

We now deduce from Givental's theorem two types of mirror statements. The first one concerns the small quantum $\mathcal{D}$-module G from definition 3.2.

Theorem 7.5. *[Iri09, proposition 4.8], [RS10, proposition 4.10] Let X_Σ be Fano. There is an isomorphism of bundles with connection on $\mathbb{P}^1 \times \overline{U}$*

$$\widehat{{}_0\mathcal{QM}_{\widetilde{A}}} \cong F.$$

The second mirror correspondence will be an isomorphism of Frobenius manifolds. There is a general strategy to construct Frobenius manifolds starting from families of trivial vector bundles on $\mathbb{P}^1$. Results of this kind are due to Malgrange, Dubrovin and Hertling-Manin (see [HM04] and the references therein). The version that we need here (taking into account logarithmic poles) can be found in [Rei09], and this gives the following result.

Theorem 7.6 (Mirror symmetry for smooth toric Fano varieties). *Let X_Σ be smooth projective and Fano. Let $\widetilde{W}_{X_\Sigma}$ be its Landau-Ginzburg model. There is a germ $((M,0),(\widetilde{D},0))$ of a canonical logarithmic Frobenius structure associated to $\widetilde{W}_{X_\Sigma}$. Here $M = \overline{U} \times \mathbb{C}^k$ and $\widetilde{D} = D \times \mathbb{C}^k$, where $k = n!\,\mathrm{vol}(\Delta(\underline{a}_1, \dots, \underline{a}_m)) - r$. The Frobenius structure is defined by a family of $\mathbb{P}^1$-bundles on M which restricts to $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}}$ on $\overline{U} \times \{0\}$. This Frobenius structure is isomorphic to the one from theorem 7.2 (i.e. to the quantum cohomology of X_Σ) near the limit point $\underline{q} = 0$.*

The very last statement can be considered as the final version of the mirror symmetry for smooth toric Fano varieties. The nef case can also be treated by these methods, and the result is basically the same, with some small technical modifications.
Finally, let us once again come back to the example of the projective spaces. Consider the twisted de Rham cohomology (i.e., either formula (12) or formula (13)), then we easily see (and this is of course a

general fact) that the restriction $H^n(\widetilde{W}_{\mathbb{P}^n}, pr)_{|z=0}$ is a family of $\mathbb{C}[q^{\pm}]$-algebras. More precisely, write p for the class of $zq\partial_q$, then formula (12) gives that

$$H^n(\widetilde{W}_{\mathbb{P}^n}, pr)_{|z=0} \cong \frac{\mathbb{C}[p, q^{\pm}]}{(p^{n+1} - q)}$$

and we see from the isomorphism (5) that this is precisely the small quantum cohomology of $\mathbb{P}^n$. In the same way, it is easy to see that the quantum-$\mathcal{D}$-module of $\mathbb{P}^n$ is exactly given by the connection operator in formula (13). Hence, in this simple case there is an explicit identification of the differential systems on the two sides of the mirror correspondence.
We finish this survey by mentioning the following corollary, which follows directly from theorem 7.5 and theorem 6.8 above.

Corollary 7.7. *Let X_Σ be Fano. Then the (restriction to $\mathbb{C}_z \times (\mathbb{C}^*)^r$ of the) quantum $\mathcal{D}$-module F of X_Σ underlies a variation of a pure polarized non-commutative Hodge structures.*

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Lehrstuhl für Mathematik VI
Institut für Mathematik
Universität Mannheim, A 5, 6
68131 Mannheim
Germany

Christian.Sevenheck@math.uni-mannheim.de