Toric geometry, hypergeometric D-modules and mirror symmetry

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(joint work with Thomas Reichelt)

In this talk, we describe a version of mirror symmetry for smooth toric varieties with numerically effective anticanonical bundle (e.g. toric Fano manifolds) and also for nef complete intersections in toric varieties. The correspondence is expressed as an equivalence of filtered $\mathcal{D}$-modules. On the A-side of the mirror picture, this is the so-called quantum $\mathcal{D}$-module of the variety X_Σ, that is, a family (parameterized by the space $H^*(X_\Sigma,\mathbb{C})$) of trivial vector bundles on $\mathbb{P}^1$ equipped with an integrable connection with poles along $\{0,\infty\}\times H^*(X_\Sigma,\mathbb{C})$. It is well-known that this object is basically equivalent to the quantum cohomology on $H^*(X_\Sigma,\mathbb{C})$. On the B-side, we consider the Landau-Ginzburg model in the sense of [Giv98] and [HV00], that is, a family of Laurent polynomials parameterized by the Kähler moduli space of X_Σ. The precise definition is as follows.

Definition 1. *Let Σ be a smooth complete n-dimensional fan defining a smooth projective Fano variety X_Σ. Let $A=(\underline{a}_1|\dots|\underline{a}_m)$ be the matrix with columns the primitive integral generators of the rays of Σ. Define*

$$\begin{array}{rcl} \varphi : S\times\Lambda := (\mathbb{C}^*)^n\times\mathbb{C}^m & \longrightarrow & \mathbb{C}_t\times\Lambda \\ (y_1,\dots,y_n),(\lambda_1,\dots,\lambda_m) & \longmapsto & \left(\sum_{i=1}^m \lambda_i \underline{y}^{\underline{a}_i},\lambda_1,\dots,\lambda_m\right) \end{array}$$

where $\underline{y}^{\underline{a}_i} := \prod_{k=1}^n y_k^{a_{ki}}$. This is called the generic family of Laurent polynomials associated to Σ (actually, it depends only on $\Sigma(1)$). On the other hand, there is an (non-canonical) embedding $g:\mathcal{K}_{X_\Sigma}\hookrightarrow\Lambda$, where $\mathcal{K}_{X_\Sigma}$ denotes the ***complexified Kähler moduli space*** *of X_Σ. $\mathcal{K}_{X_\Sigma}$ is an $m-n$-dimensional torus, and a specific choice of a basis of $H^2(X_\Sigma,\mathbb{Z})$ (this choice depends on Σ, not only on $\Sigma(1)$) yields an identification $\mathcal{K}_{X_\Sigma}\cong(\mathbb{C}^*)^{m-n}$. Then we call the family of Laurent polynomials*

$$W := \varphi\circ(\mathrm{id}_S\times g): S\times\mathcal{K}_{X_\Sigma}\to\mathbb{C}_t\times\mathcal{K}_{X_\Sigma}$$

the ***Landau-Ginzburg model*** *of X_Σ.*

Consider the matrix $\widetilde{A}=(\underline{\widetilde{a}}_0,\underline{\widetilde{a}}_1,\dots,\underline{\widetilde{a}}_m)\in\mathrm{Mat}((n+1)\times(m+1),\mathbb{Z})$ where $\underline{\widetilde{a}}_i := (1,\underline{a}_i)\in\mathbb{Z}^{n+1}$ for $i=1,\dots,m$ and $\underline{\widetilde{a}}_0 := (1,\underline{0})$. Then for any $\beta\in\mathbb{Z}^{n+1}$, let $\mathcal{M}^\beta_{\widetilde{A}}$ be the Gelfand-Kapranov-Zelevinsky-hypergeometric $\mathcal{D}_{\mathbb{C}_t\times\Lambda}$-module (see, e.g., [GKZ90]).

Theorem 2 ([Rei12])**.** *There is an exact sequence in $MHM_{\mathbb{C}_t\times\Lambda}$ (the abelian category of mixed Hodge modules on $\mathbb{C}_t\times\Lambda$)*

$$0\to H^{n-1}(S,\mathbb{C})\otimes\mathcal{O}_{\mathbb{C}_t\times\Lambda}\to\mathcal{H}^0\varphi_+\mathcal{O}_{S\times\Lambda}\to\mathcal{M}^{(0,\underline{0})}_{\widetilde{A}}\to H^n(S,\mathbb{C})\otimes\mathcal{O}_{\mathbb{C}_t\times\Lambda}\to 0$$

For any holonomic $\mathcal{D}_{\mathbb{C}\times\Lambda}$-module $\mathcal{M}$, we denote by $\mathrm{FL}(\mathcal{M})$ the $\mathcal{D}_{\mathbb{C}_z\times\Lambda}$-module obtained by applying a partial Fourier-Laplace transformation (sending t to $z^2\partial_z$ and ∂_t to z^{-1}) to $\mathcal{M}[\partial_t^{-1}] := \mathbb{C}[t,\lambda_1,\dots,\lambda_m]\langle\partial_t,\partial_t^{-1},\partial_{\lambda_0},\dots,\partial_{\lambda_m}\rangle\otimes_{\mathcal{D}_{\mathbb{C}\times\Lambda}}\mathcal{M}$.

Then we have the following corollary of the above result, which can actually be shown independently (and with a considerably simpler proof).

Corollary 3 ([RS10])**.** *There is an isomorphism of holonomic $\mathcal{D}_{\mathbb{C}_z\times\Lambda}$-modules* $\mathrm{FL}(\mathcal{H}^0\varphi_+\mathcal{O}_{S\times\Lambda})\cong\mathrm{FL}(\mathcal{M}_{\widetilde{A}}^{(0,\underline{0})})=:\widehat{\mathcal{M}}_{\widetilde{A}}^{(0,\underline{0})}$.

From these results we can easily deduce a corresponding statement for the Landau-Ginzburg model.

Corollary 4. *There is an isomorphism of holonomic $\mathcal{D}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-modules*

$$\mathrm{FL}(\mathcal{H}^0W_+\mathcal{O}_{S\times\mathcal{K}_{X_\Sigma}})\cong(\mathrm{id}_{\mathbb{C}_z}\times g)^+\widehat{\mathcal{M}}_{\widetilde{A}}^{(0,\underline{0})},$$

and the latter module can be explicit described as a cyclic module (i.e., as quotient of $\mathcal{D}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$).

In order to lift these results into the category of filtered $\mathcal{D}$-modules, we consider the filtration $F_\bullet$ on $\mathcal{M}_{\widetilde{A}}^\beta$ induced by the order filtration on $\mathcal{D}$. This induces a filtration $G_\bullet$ on $\widehat{\mathcal{M}}_{\widetilde{A}}^\beta$, defined as $G_k\widehat{\mathcal{M}}_{\widetilde{A}}^\beta:=\sum_{i\geq 0}\partial_t^{-i}F_{k+i}\mathcal{M}_{\widetilde{A}}^\beta$. In order to simplify the next statements, we restrict from now on to the case where X_Σ is Fano. For nef varieties, the results are basically the same, but slightly more complicated to state.

Theorem 5.
1. *There is an isomorphism of $\mathcal{O}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-modules with connection*

   $$G_0\widehat{\mathcal{M}}_{\widetilde{A}}^{(1,\underline{0})}\cong H^n(\Omega^\bullet[z],zd-dW_1)=:G_0,$$

   where W_1 is the first component of the map W from above. Notice that the right hand side is usually called twisted de Rham cohomology*.*
2. *The module $G_0\widehat{\mathcal{M}}_{\widetilde{A}}^{(1,\underline{0})}$ (and hence also the module $G_0=H^n(\Omega^\bullet[z],zd-d\varphi_1)$) is $\mathcal{O}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-free, and equipped with a connection operator with poles of Poincaré rank 1 along $\{0\}\times\mathcal{K}_{X_\Sigma}$ and no other singularities.*

In order to express the mirror correspondence as an isomorphism of *Frobenius manifolds*, one needs to extend the above objects to a family of trivial vector bundles over $\mathbb{P}^1_z$, such that the connection acquires a logarithmic pole at $z=\infty$. This is known as a *good basis* or a solution to the *Birkhoff problem* (see [Sai89] and also [Sai83]). The result in the present setup is as follows.

Proposition 6. *Let X_Σ smooth toric and Fano. Consider the Landau-Ginzburg model $W:S\times\mathcal{K}_{X_\Sigma}\to\mathbb{C}_t\times\mathcal{K}_{X_\Sigma}$ and the $\mathcal{O}_{\mathbb{C}_z\times\mathcal{K}_{X_\Sigma}}$-locally free module G_0 from above. Let $\overline{\mathcal{KM}}_{X_\Sigma}=\mathbb{C}^{m-n}$ be the natural partial compactification of $\mathcal{K}_{X_\Sigma}$ induced by the choice of coordinates (i.e., by the identification $\mathcal{K}_{X_\Sigma}\cong(\mathbb{C}^*)^{m-n}$ defined by the choice of a basis of $H^2(X_\Sigma,\mathbb{Z})$). There is an extension $\overline{G}_0\to\mathbb{P}^1_z\times U$ of $(G_0)_{|\mathbb{C}_z\times U}$, where $U\subset\overline{\mathcal{KM}}_{X_\Sigma}$ is Zariski open and contains the origin. $\overline{G}_0$ has the following properties:*

1. *It is fibrewise trivial, i.e. $p^*p_*\overline{G}_0\cong\overline{G}_0$ if $p:\mathbb{P}^1_z\times U\to U$ is the projection.*
2. *The connection extends with a logarithmic pole along the normal crossing divisor $(\{\infty\}\times U)\cup\big(\mathbb{P}^1\times(U\backslash\mathcal{K}_{X_\Sigma})\big)$.*

From this, we deduce the following construction theorem of Frobenius manifolds.

Theorem 7. *Put $\mu := \dim_{\mathbb{C}} H^*(X_\Sigma, \mathbb{C})$. There is a germ of a canonical Frobenius structure on $\mathbb{C}^{\mu-(m-n)} \times U$ associated to W, which has logarithmic poles (in the sense of* [Rei09]*) along $\mathbb{C}^{\mu-(m-n)} \times (\overline{\mathcal{KM}}_{X_\Sigma} \backslash U)$. It is isomorphic to the big quantum cohomology of X_Σ.*

In the case of a nef complete intersection $Y \subset X_\Sigma$ (i.e., X_Σ is toric smooth projective as before and Y is the zero locus of a generic section of a split vector bundle $\mathcal{E} = \oplus_{j=1}^c \mathcal{L}_j \to X_\Sigma$ where $\mathcal{L}_j \in \mathrm{Pic}(X_\Sigma)$ are ample and such that $-K_{X_\Sigma} - \sum_{j=1}^c c_1(\mathcal{L}_j)$ is nef), we can construct a **non-affine** Landau-Ginzburg model, which is a projective morphism $\Pi : Z \to \mathbb{C}_z \times \mathcal{K}_{X_\Sigma}$ from a quasi-projective variety Z (which is not smooth in general). Then the result is as follows.

Theorem 8 ([RS12])**.**

Let $(X_\Sigma, \mathcal{L}_1, \ldots, \mathcal{L}_c)$ define a nef complete intersection Y in X_Σ. Consider the ambient (or reduced) quantum $\mathcal{D}$-module $\mathrm{QDM}(X_\Sigma, \mathcal{E} := \oplus_{j=1}^c \mathcal{L}_j)$ of Y, as defined in [MM11]. Then we have

$$\left(\mathrm{FL}\left(\mathcal{H}^0 DR^{-1}(R\Pi_* IC_Z)\right)\right)_{|\mathbb{C}_z \times B_\varepsilon} \cong (id_{\mathbb{C}_z} \times \mathrm{Mir})^* \mathrm{QDM}(X_\Sigma, \mathcal{E})(*\{0\} \times \mathcal{K}_{X_\Sigma})_{|\mathbb{C}_z \times B_\varepsilon},$$

where IC_Z is the intersection complex of Z, DR^{-1} denotes a complex of $\mathcal{D}$-modules corresponding to a given constructible complex via the Riemann-Hilbert correspondence, B_ε is a small ball in $\mathcal{K}_{X_\Sigma}$ around the origin in $\overline{\mathcal{KM}}_{X_\Sigma}$ and Mir is Givental's **mirror map**.

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