Variation of twistor structures and hypersurface singularities

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We are interested in applications of the theory of (mixed) twistor structures to hypersurface singularities. The relation between these subjects stems from a construction which has its origin in [**?**] and which was formalized in [**?**] under the name TERP-structure. Roughly speaking, given a hypersurface singularity or a tame function on an affine manifold, a Fourier-Laplace transformation of the Brieskorn lattice produces a holomorphic bundle H on $\mathbb{C}$ equipped with a flat connection ∇ having a pole of order at most 2 at zero, and a real subbundle $H'_{\mathbb{R}}$ of maximal rank on $\mathbb{C}^*$. This allows to construct a $\mathbb{P}^1$-bundle: for a section s of H near zero, consider the flat shift of $\overline{s}$ to infinity along the circular involution $z \mapsto \overline{z}^{-1}$. These sections define an extension to infinity which produces a twistor (i.e., a holomorphic $\mathbb{P}^1$-bundle). The notion and a good part of the theory of twistor structures goes back to the work of Simpson ([**?**], [**?**], [**?**], [**?**]).

The same construction makes sense in the relative case (e.g., given an unfolding of a singularity), then a variation of twistor structures is obtained. The striking feature of these variations is that they are holomorphic in the $\mathbb{P}^1$-direction, but depend only smoothly (or real analytically) on the parameters.

There is another ingredient in a TERP-structure, which plays the role of a polarization. It can be shown that the Fourier-Laplace transformation of the Brieskorn lattice comes equipped with a pairing between opposite fibres, i.e., a form $P : H_z \times H_{-z} \to \mathbb{C}$ for all $z \in \mathbb{C}^*$ with meromorphic behavior at zero. In the case of local singularities, it is essentially the form one gets from K. Saito's higher residue pairings. The twistor constructed above is naturally equipped with a hermitian form h derived from P.

By the classical construction of Steenbrink and Scherk ([**?**]), the Brieskorn lattice allows to define a Hodge filtration on the cohomology of the Milnor fibre of the singularity making up a polarized mixed Hodge structure. Up to a twist, this also works if one starts with the Fourier-Laplace-transformed object, at least in the local case. In the case of tame functions, it was shown in [**?**] that a modified construction also gives rise to a mixed Hodge structure. In this way TERP/twistor structures are considered as generalization of Hodge structures: Any TERP-structure gives rise to a filtration on the space of nearby cycles, and we can identify a particular class (those generated by "elementary sections") which are in fact equivalent to (i.e, can be reconstructed from) their induced filtration. The nice feature of this geometric point of view is that if we look at the twistor corresponding to a such a filtration, then it is very easy to tell whether this filtration gives rise to a Hodge structure: This is precisely the case if the twistor is a trivial (or more generally semi-stable) bundle on $\mathbb{P}^1$, these twistors are called pure, and pure polarized if the form h is positive definite. For the semi-universal unfolding of a hypersurface singularity, the corresponding variation of twistor structures is pure outside a real analytic hypersurface and this complement has connected components on which the signature of h is constant.

As in classical Hodge theory, we are interested in studying degenerations of twistor structures, which leads to the notion of a (polarized) mixed twistor: a bundle on $\mathbb{P}^1$ equipped with an increasing filtration of subbundles such that the quotients are semi-stable of appropriate weight.

Pursuing this analogy further, one might ask for a generalization of nilpotent orbits of Hodge structures which are considered in the work of Schmid ([**?**]). It turns out that this leads to a particularly important class of variations of twistor/TERP structures, which are obtained by a rather simple procedure: Start with a single TERP-structure (without parameters) and rescale the coordinate on $\mathbb{C}$. This yields a variation on $\mathbb{C}^*$ (the space of the rescaling parameter) and similarly one gets a variation of twistor structures. If these twistors are pure polarized for sufficiently small parameters, then this variation is called nilpotent orbit of TERP/resp. twistor structures.

There is a classical correspondence due to Cattani, Kaplan and Schmid [**?**] between nilpotent orbits of Hodge structures and polarized (limit) mixed Hodge structures. The following two results, taken from [**?**] are the appropriate generalizations to TERP/twistor structures.

The first one concerns the case of TERP-structures where (H, ∇) has a regular singularity at zero. We denote by $\pi_r : \mathbb{C} \to \mathbb{C}; z \mapsto r \cdot z$ for $r \in \mathbb{C}^*$ the rescaling map.

Theorem 1. *A regular singular TERP structure $(H, \nabla, H'_{\mathbb{R}}, P)$ induces a nilpotent orbit (i.e., $\pi_r^*(H, \nabla, H'_{\mathbb{R}}, P)$ is pure polarized for all $|r| \ll 1$) if and only if the filtration $F^\bullet$ induced by (H, ∇) on $H^\infty := \psi_z(H, \nabla)$ gives rise to a polarized mixed Hodge structure.*

One direction of this theorem is proved in [**?**]. The other direction, which can be found in [**?**], uses a fundamental result of Mochizuki ([**?**]) which states that for a variation of polarized twistor structures on a complement of a normal crossing divisor with tame behavior, there exists a limit object which is a polarized mixed twistor.

In the general (i.e., irregular case), we can prove one part of this correspondence. The main ingredients are the decomposition theorem for irregular connections and a precise discussion of the associated Stokes structure. It turns out that under a compatibility condition between the Stokes and the real structure, each regular singular piece that appears in the decomposition induces a filtration as in the regular singular case. The result is then as follows.

Theorem 2. *Consider any (possibly irregular) TERP-structure $(H, \nabla, H'_{\mathbb{R}}, P)$. Suppose that the Stokes structure is compatible with the real structure defined by $H'_{\mathbb{R}}$. Suppose further that the filtrations defined by the regular singular pieces give rise to a polarized mixed Hodge structures. Then $(H, \nabla, H'_{\mathbb{R}}, P)$ induces a nilpotent orbit.*

The missing converse direction in this irregular case is conjectured to be true.

Let us point out two rather direct applications of these results. Both are concerned with the example alluded to above, that is, functions with isolated singularities.

Corollary 3. *Consider the semi-universal unfolding M of a holomorphic function with isolated critical points. Then for any $t \in M$, the restriction of the variation of TERP structures to the orbit of the Euler field passing through t is a nilpotent orbit of TERP-structures, in particular, if one follows this orbit sufficiently far enough, the variation of twistors is pure polarized.*

The second corollary deals with the global case. As already mentioned, a modified procedure of the Scherk-Steenbrink construction due to Sabbah produces a filtration $F^\bullet_{Sab}$ which gives rise to a mixed Hodge structure, but which is not polarized. Using the above correspondence and a recent, fundamental result of Sabbah ([**?**]), we obtain:

Corollary 4. *Let $f : Y \to \mathbb{A}^1$ be a tame function on an affine manifold. Consider the Hodge filtration $F^\bullet_{Sab}$ on H^∞ as defined in [**?**]. There exists an automorphism $G \in Aut(H^\infty)$ which induces the identity on the quotients of weight filtration such that $G^{-1}F^\bullet_{Sab}$ gives rise to a* ***polarized*** *mixed Hodge structure.*

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