Seminar on: "Hodge theory, local cohomology and the du Bois complex"

SS 2025

The aim of this seminar is to study some recent developments in Hodge theory for singular subvarieties in smooth algebraic varieties. These are mainly due to Popa and Mustaţă (see, e.g., [MP22] and some subsequent papers, partly with other co-authors). Among the topics treated in these papers are: (new) characterization of local cohomological dimension (resp. defect) of singular subvarieties, local vanishing theorems for log resolutions, characterization of (higher) du Bois singularities in terms of comparison of Hodge and order filtration on the local cohomology sheaves (at least in the l.c.i. case) etc. These aspects, together with some older material on the du Bois complex and on the (more classical) notion of du Bois singularities will be discussed in the seminar.
Below is an outline of a program, as usual, this will need to be adapted as we go on. Some of the topics might need more time than one talk. Especially the last talks depend on each other, and for certain proofs some reorganization may be necessary.

1 Introduction (Christian Sevenheck), 02.05.2025

In this talk a few aspects of the seminar should be highlighted, without going into details. This includes: Basics from Hodge theory (compact Kähler/smooth projective versus non-compact resp. singular varieties), degenerations, Hodge modules, Hodge ideals, birational aspects. Some of the material in [Pop24, Section 1] can be used, e.g. the discussion of rationality (in particular, [Pop24, Example 1.2.10] treating cones over smooth hypersurfaces), and the provisional definition of du Bois singularities ([Pop24, Definition 1.3.1]). One could also mention (without any details) the construction of MHM on the cohomology of singular varieties by Deligne (see [PS08, Theorem 5.33]) and/or the definition of the du Bois complex ([Pop24, Definition 2.2.1] or [PS08, Section 7.3.1]), by admitting the existence of hyperresolutions or cubical resolutions, which is treated in the following talk. If time permits, local cohomology can be mentioned, within the framework of Hodge modules, as well as the connection to the graded pieces of the du Bois complex ([Pop24, Proposition 4.4.2]).

2 Hyperresolutions, cubical resolutions and definition of the du Bois complex (Andreas Hohl), 16.05.2025

Go through [Pop24, Sections 2.1 and 2.2] and present essentially all statements, in particular the definition of hyper- resp. cubical resolutions. [PS08, Sections 5.1, 5.2] can be used as an additional reference giving much more details, in particular, the definition of geometric realization of simplicial resp. cubical spaces, the theory of sheaves on them, and ([PS08, Definition 5.6]) the definition of being of cohomological descent. Give some hints on the proof of Theorem [Pop24, 2.1.14] as presented in [PS08, Theorem 5.26], and discuss the example [Pop24, 2.1.15]. Then define the du Bois complex $\underline{\Omega}^\bullet_X$ (called "filtered de Rham complex" in [Pop24]) and its graded pieces $\underline{\Omega}^p_X \in D^b_{coh}(X)$ (called "p-th du Bois-complex"). Discuss in particular (some of) the examples [Pop24, 2.2.6 to 2.2.11] (one may also consult [PS08, Section 7.3] for more material/examples).

3 Properties of the du Bois complex and more examples (Paul Görlach), 23.05.2025

Discuss first the "elementary" properties of $\underline{\Omega}^\bullet_X$ in [Pop24, Section 2.3]. Most of these are relatively formal consequences of the definition. As to the more elaborate properties in [Pop24, Section 2.4], state the vanishing assertions [Pop24, 2.4.1, 2.4.2] without proof. Mention the case of rational singularities (where the celebrated paper [KS21] relates $\underline{\Omega}^p_X$ with the extendable forms $f_*\Omega^p_{\widetilde{X}}$ for a resolution f) and discuss in detail the case of SNC divisors ([Pop24, 2.4.(11)]). Say something about the Mayer-Vietoris property in [Pop24, Proposition 2.4.8] and deduce Steenbrink's birational description of the p-th du Bois complex in [Pop24, Proposition 2.4.10] (see also [Mus24, Proposition 6.1]). Finally, discuss (with as many details as time permits) the two examples of quotient singularities ([Pop24, Theorem 2.5.1]) and cones over smooth projective varieties ([Pop24, Proposition 2.5.3]).

4 Definition of du Bois singularities, weak normality, relation to other singularity types (Yichen Qin), 06.06.2025

Present everything from [Pop24, Section 3.1] (in particular, the definition 3.1.1. of du Bois singularities, and the examples following it). Then discuss the notion of weak/semi-normality in [Pop24, Section 3.2] (one might also use [PS08, Example 7.23, 2], see also [Sta25, Tag 0EUK]). Do a reasonable subset of the material from [Pop24, Sections 3.3]: In particular, present statement and proof of [Pop24, Proposition 3.3.2]. The final aim of this talk is then to show that rationality implies du Bois ([Pop24, Theorem 3.3.3]). Its proof relies on the technical result [Pop24, Theorem 3.3.4]. The full proof of the latter is certainly our of reach for time limitations, but parts of it should be explained.

5 Reminders on (mixed) Hodge modules and on Hodge ideals (Carolina Tamborini), 13.06.2025

Discuss a few basics on mixed Hodge modules from the standard sources [Sai88], [Sai90], [Sch19]. Then turn to Hodge ideals and give an overview of the main results of [MP19] and the subsequent papers [MP20a, MP20b, MOP20] (maybe the overview [Pop18] is helpful). Concentrate on relation of Hodge filtration and pole order filtration, generating level, birational description and (local) vanishing theorems as well as definition and properties of the minimal exponent. A good summary to consult might be [Mus24, Sections 5.1, 5.3, 5.4].

6 Hodge theory and local cohomology, Hodge module interpretation of the du Bois complex (Henry Dakin), 20.06.2025

Discuss local cohomology sheaves of a subvariety $Z \subset X$, both in general and from the Hodge theoretic point of view. For generalities, one may consult [ILL$^+$07] (or also [MP22, Section B.2] for a short review), for the Hodge theoretic interpretation, follow [Mus24, section 5.2] (see also [MP22, Section B.3]). Mention the relation of the Hodge filtration $F_\bullet \mathcal{H}^q_Z(X)$ to both the order and the Ext-filtration on $\mathcal{H}^q_Z(X)$ ([Mus24, section 5.2.2], for more thorough information on this point, one might need to consult the original paper [MP22]). Then go to the characterization of the p-th du Bois complexes via local cohomology in [Mus24, section 6.1.2]. In particular, present [Mus24, Corollary 6.4] and its proof (which needs Steenbrink's description of the p-th du Bois complex, see talk 3). Compare also [Pop24, Section 4.4].

7 Higher du Bois singularities (Christian Sevenheck), 27.06.2025

Present parts of [Pop24, Section 4.1], in particular the definition of (pre-)m-du Bois singularities ([Pop24, Definitions 4.1.1 and 4.1.2]). Discuss some examples ([Pop24, 4.1.11-4.1.14]). Then mention [Pop24, Theorem 4.1.15] (which is also [Mus24, Theorem 6.7]) for the hypersurface case. Its proof is probably out of reach, but some aspects can be discussed. Mention as much as time permits the generalization [MP22, Theorem F] (referring to the results of the previous talks).
A potential application (that certainly would need an extra talk) is to mention [FL24, Theorem 1.2] about the local freeness of higher direct images under flat proper maps with higher du Bois singularities in the fibres.

8 Local cohomological dimension resp. defect (Christian Lehn), 11.07.2025

Recall previous results on the local cohomological dimension (see, e.g. [Ogu73] or [RSW23]). Discuss the relation of generating level on the top non-zero $\mathcal{H}^q_Z(X)$ to local vanishing of log forms ([Mus24, Theorem 5.30 and Corollary 5.33]) together with its proof. As application, discuss parts of [MP22, Section 11], in particular, the case of quotient singularities [MP22, Corollary 11.22] and the case semi-group rings [MP22, Corollary 11.26]. Notice that the proof of the former uses [Pop24, Corollary 12.6], but this follows from the criterion [Mus24, Corollary 5.33] resp. [MP22, Theorem E] using the aforementioned description by Steenbrink of the p-th du Bois complex (see again talk 3 above).

9 The Hodge-du Bois diamond, Hodge theory of singular varieties, to be scheduled

This talk is an option to dive deeper into the applications of the material we have learned so far. One should essentially follow [Pop24, Section 4.3]. Details of this talk can be worked out if we make it until here.

References

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