Übungsaufgaben zur Katastrophentheorie

1. *(4 Punkte)* Sei $k \in \mathbb{N}$, $k > 3$. Sei $f \in \mathcal{E}_2$. Zeigen Sie
   (a) Falls $f = x^3 + g$, $g \in \mathbf{m}_{\mathcal{E}_2}^k$ gilt, dann gibt es $l \in \mathbb{N}$ mit $l \geq k$ so dass f rechtsäquivalent ist zu $\varepsilon^l(x^3 + y^l)$ oder zu $x^3 + \varepsilon^l xy^{l-1} + g'$ mit $g' \in \mathbf{m}_{\mathcal{E}_2}^{l+1}$ und $\varepsilon \in \{-1, 1\}$.
   (b) Falls $f = x^2y + g$, $g \in \mathbf{m}_{\mathcal{E}_2}^k$ gilt, dann gibt es $l \in \mathbb{N}$ mit $l \geq k$ so dass f rechtsäquivalent ist zu $x^2y + \varepsilon y^l$.

2. *(2 Punkte)* Sei $f \in \mathbf{m}_{\mathcal{E}_n}^2$ endlich bestimmt mit Bestimmtheit k. Zeigen Sie, dass

$$\mu(f) \leq \binom{n+k}{n}$$

   gilt.

3. *(4 points)* Extend the classification of germs of smooth functions to the case $\mu = 6$ in the following way:
   (a) Put $k := \text{corank}(f) = 1$, then show that $k \in \{1, 2\}$.
   (b) If $k = 1$, conclude that f is stably right equivalent to x^7.
   (c) If $k = 2$, show that f is stably right equivalent to $g \in \mathbf{m}_{\mathcal{E}_2}^3$ which is 5-determined.
   (d) Write g as $g = p + h$ with $p \in \mathbb{R}[x, y]_3$ and $h \in \mathbf{m}_{\mathcal{E}_2}^4$, Assume the following estimate for the Milnor number: for any $f \in \mathbf{m}_{\mathcal{E}_n}^k$, and any $l \in \mathbb{N}$ we have

$$\mu(f) \geq \binom{n+k+l-1}{k+l-1} - n\binom{n+l}{l}$$

   Using this, show that up to a linear transformation $\phi_A \in \mathcal{G}_2$ for some $A \in \text{Gl}(2, \mathbb{R})$, we have $p \in \{x^3, x^2y\}$.
   (e) If $p = x^3$, then use exercise 1(a) to show that f can only be equivalent to $\pm(x^3 + y^4)$, $x^3 + y^5$,$x^3 \pm xy^4$ and $x^3 + xy^3 + j$ with $j \in \mathbf{m}_{\mathcal{E}_2}^5$.
   (f) If $p = x^2y$, then use exercise 1(b) to show that f can only be equivalent to $x^2y \pm y^4$ or $x^2y \pm y^5$.
   (g) Finally, deduce from the previous results that any germ $f \in \mathbf{m}_{\mathcal{E}_2}^3$ with $\mu(f) = 7$ is equivalent to either $\pm(x^3 + y^4)$, $x^2y + y^5$ or $x^2y + y^5$.