IRREGULAR HODGE FILTRATION OF SOME CONFLUENT HYPERGEOMETRIC SYSTEMS

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Abstract. We calculate the irregular Hodge filtration, recently introduced in [Sab15], for the purely irregular hypergeometric $\mathcal{D}$-modules.

1. Introduction

The aim of this paper is to compute some Hodge theoretic invariants of certain classical differential systems in one variable. These are the so-called irregular Hodge numbers, which has been introduced recently by Sabbah [Sab15]. They are called irregular because they are attached to differential systems which may have irregular singular points, a feature that is excluded for classical variation of Hodge structures as well as the more general Hodge modules. The very definition of these numbers rely on the theory of mixed twistor modules of T. Mochizuki ([Moc15a]). Twistor modules generalise Hodge modules, in the sense that the underlying $\mathcal{D}$-module of a twistor module can have irregular singularities. In particular, one can define a version of the Fourier-Laplace transformation functor within the category of mixed twistor modules, which is impossible for mixed Hodge modules. The drawback of this generalization is that one cannot directly assign a filtration to a twistor module, and hence it is not easy to attach numerical invariants, like Hodge numbers to it. In the above mentioned paper [Sab15], Sabbah constructs an intermediate category between mixed Hodge modules and mixed twistor modules which is on the one hand sufficiently large to be stable under all relevant operations that are defined for twistor modules (in particular, the Fourier-Laplace transformation), but which allows one to define a filtration, called irregular Hodge filtration for each of its objects. One of the main results of op. cit. is that rigid irreducible $\mathcal{D}$-modules on the projective line underly objects of this category, and consequently admit a unique irregular Hodge filtration, provided that their formal local monodromies are unitary. Rigid $\mathcal{D}$-modules are particularly interesting since they can be algorithmically constructed from simple objects by an algorithm due to Arinkin and Katz (cf. [Ari10]).

Among the most studied and best understood examples of such rigid $\mathcal{D}$-modules are the classical hypergeometric $\mathcal{D}$-modules. In the regular case, Fedorov has recently given in [Fed15] a closed formula for the Hodge numbers using the work [DS13] of Dettweiler and Sabbah.

The present papers aims at extending Fedorov's work to some special class of irregular hypergeometric modules, where we can explicitly control the (irregular) Hodge filtration and calculate the (irregular) Hodge numbers. Our principal result, Theorem 4.7, gives a very simple formula for these numbers, which is in some sense similar to the shape of Fedorov's formula. The main ingredients are a reduction process to obtain classical hypergeometric $\mathcal{D}$-modules from some higher dimensional ones (present at [BZMW13]), the so-called GKZ-systems, techniques from [Rei14] and [RS15, RS17] (going back to [Sab08]) to understand Hodge module structures on these GKZ-systems as well as a quite explicit solution to the so-called Birkhoff problem that is inspired by calculations in toric mirror symmetry (see again [RS15] as well as [DS04] and also [GMS09]).

While writing this paper, we learned that the same formula has been proved by Sabbah and Yu in the newest version of [Sab15] by different means. Nevertheless, we believe that the geometric approach taken here has some potential to be generalized to arbitrary irregular hypergeometric systems.

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2. Hypergeometric $\mathcal{D}$-modules and dimensional reductions

In this section we will introduce two different kinds of hypergeometric $\mathcal{D}$-modules: classical and GKZ. We will dedicate more time to classical ones, since they form one of the main objects of study of the paper and will end by showing the relation between both types, which will be useful for us in the future.

Definition 2.1. Let $(n, m) \neq (0, 0)$ be a pair of nonnegative integers, and let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$, respectively, n and m elements of $\mathbb{C}$. The (classical) hypergeometric $\mathcal{D}$-module associated with the α_i and the β_j is defined as the quotient of $\mathcal{D}_{\mathbb{G}_m}$ by the ideal generated by the so called hypergeometric operator

$$\prod_{i=1}^{n}(t\partial_t - \alpha_i) - t\prod_{j=1}^{m}(t\partial_t - \beta_j).$$

We will denote it by $\mathcal{H}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m)$, or in an abridged way, $\mathcal{H}(\alpha_i; \beta_j)$.

Remark 2.2. The excluded type $(n, m) = (0, 0)$ corresponds to the punctual delta $\mathcal{D}_{\mathbb{G}_m}$-module on $\mathbb{G}_m$ $\mathcal{H}(\emptyset; \emptyset) = \mathcal{D}_{\mathbb{G}_m}/(1 - x)$.

On the other hand, if $\mathcal{K}_\eta$ denotes the Kummer $\mathcal{D}$-module $\mathcal{D}_{\mathbb{G}_m}/(t\partial_t - \eta)$, then $\mathcal{H}(\alpha_i; \beta_j) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \mathcal{K}_\eta \cong \mathcal{H}(\alpha_i + \eta; \beta_j + \eta)$.

For $n \neq m$, no hypergeometric $\mathcal{D}$-module of type (n, m) has singularities on $\mathbb{G}_m$. If $n > m$ (resp. $m > n$), they have a regular singularity at the origin (resp. infinity) and an irregular singularity at infinity (resp. the origin) of irregularity one and slope $1/|n-m|$ of multiplicity $|n-m|$ (cf. [Kat90, Prop. 2.11.9]). For $n = m$, the hypergeometric modules of type (n, n) are regular, with singularities only at the origin, infinity and 1.

Proposition 2.3 (**Irreducibility**). (cf. [Kat90, Prop. 2.11.9, 3.2]) *Let $\mathcal{H} := \mathcal{H}(\alpha_i; \beta_j)$ be a hypergeometric $\mathcal{D}$-module. It is irreducible if and only if for any pair (i, j) of indexes, $\alpha_i - \beta_j$ is not an integer.*

Proposition 2.4 (**Rigidity**). (cf. [Kat90, Rigidity Thm. 3.5.4, 3.7.3]) *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two irreducible regular holonomic $\mathcal{D}_{\mathbb{G}_m}$-modules of generic rank $n \geq 1$ and Euler-Poincaré characteristic -1 such that the $\mathcal{M}_i \otimes \mathbb{C}((t))$ and the $\mathcal{M}_i \otimes \mathbb{C}((1/t))$ are isomorphic, and either both have a singularity at 1 or none has a singularity there. Then, $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic as $\mathcal{D}_{\mathbb{G}_m}$-modules.*

Proposition 2.5 (**Hodge numbers (regular case)**). (cf. [Fed15, Theorem 1]) *Let $\mathcal{H} = \mathcal{H}(\alpha_i; \beta_j)$ be an irreducible hypergeometric $\mathcal{D}$-module of type (n, n), so that its restriction to $\mathbb{G}_m \backslash \{1\}$ underlies a complex polarizable variation of Hodge structures. Assume that the α_i and the β_j are increasingly ordered real numbers, lying in the interval $[0, 1)$. Set*

$$\rho(k) = |\{j = 1, \ldots, n : \alpha_j < \beta_k\}| - k,$$

for $k = 1, \ldots, n$. Then the Hodge numbers of $\mathcal{H}$ are, up to a shift,

$$h^p = \left|\rho^{-1}(p)\right| = |\{k = 1, \ldots, h : \rho(k) = p\}|.$$

This last result is the analogous one, in the regular case, to theorem 4.7, and served as the main motivation to start this project. Now that we have seen part of the behaviour of classical hypergeometric $\mathcal{D}$-modules, let us continue with the other family of hypergeometric modules mentioned above.

Definition 2.6. Let $n \geq m$ two positive integers, and let $d = n - m$. Let $\beta \in \mathbb{C}^d$ be a vector and let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix. Associated with it, we consider the Euler operators $E_i = \sum_j a_{ij}\lambda_j\partial_{\lambda_j}$, for $i = 1, \dots, d$, and the toric ideal

$$\mathcal{I}_A := (\partial_\lambda^u - \partial_\lambda^v : Au = Av)\,.$$

Then, the GKZ-hypergeometric $\mathcal{D}$-module is $\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{A}^n}/(I_A + (E_i - \beta_i : i = 1, \dots, d))$.

We call homogenization of a GKZ-system $\mathcal{M}_A^\beta$ the GKZ-hypergeometric system $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$, where $\widetilde{A}$ is the extended matrix

$$\left(\begin{array}{c|c} 1 & \underline{1}_n \\ \hline \underline{0}_d & A \end{array}\right)$$

and $\widetilde{\beta} = (\beta_0, \beta_1, \dots, \beta_d)^{\mathrm{t}}$, for certain $\beta_0 \in \mathbb{C}$.

GKZ-hypergeometric $\mathcal{D}$-modules have been deeply studied in many aspects, so we can profit from that situation if we can express a classical hypergeometric as an object emerging from a GKZ one. Let us see this procedure in more detail.

Proposition 2.7. *Let $0 \leq r, s \leq m$ be three integers such that $m = r + s \geq 1$, and let $A \in \mathbb{Z}^{(m-1)\times m}$ be an integer matrix of rank $m-1$ and $\kappa \in \mathbb{C}^{m-1}$ such that $\kappa_1 = 0$. Consider the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^m$ given by $t \mapsto (t, 1, \dots, 1)$, and let $B \in \mathbb{Z}^m$ be a Gale dual of A, that is, an integer column matrix which generates $\ker_{\mathbb{Q}} A$. Up to a reorder in the rows of A, assume that B is of the form $(b_1, \dots, b_r, -b_{r+1}, \dots, -b_{r+s})^{\mathrm{t}}$, where the b_i are positive integers and let*

$$\gamma := \prod_{i=1}^{r} b_i^{b_i} \prod_{j=1}^{s} (-b_{r+j})^{-b_{r+j}}.$$

Then, we have $h_\gamma^+\mathcal{H}(\alpha_i;\beta_j) \cong \iota^+\mathcal{M}_A^{A\kappa}$, where h_γ is the homothety of $\mathbb{G}_m$ of ratio γ and

$$(\alpha_i) = \left(-\frac{\kappa_1}{b_1}, -\frac{\kappa_1 - 1}{b_1}, \dots, -\frac{\kappa_1 - b_1 + 1}{b_1}, \dots, -\frac{\kappa_r - b_r + 1}{b_r}\right),$$

$$(\beta_i) = \left(\frac{\kappa_{r+1}}{b_{r+1}}, \frac{\kappa_{r+1} - 1}{b_{r+1}}, \dots, \frac{\kappa_{r+1} - b_{r+1} + 1}{b_{r+1}}, \dots, \frac{\kappa_{r+s} - b_{r+s} + 1}{b_{r+s}}\right).$$

Proof. Let $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the canonical inclusion. Then the statement is just a consequence of applying j^+ to both sides of the isomorphism given in [BZMW13, Thm. 8.1], as long as we rewrite the $\mathcal{D}$-modules in it.

On one hand, $\mathcal{M}_A^{A\kappa}$ is also a lattice basis binomial $\mathcal{D}_{\mathbb{A}^m}$-module (cf. [op. cit., 0.2]). This is because A being of codimension one implies that the toric ideal $\mathcal{I}_A$ is a complete intersection, and then coincides with the lattice basis ideal

$$\mathcal{I}(B) := (\partial^{w_+} - \partial^{w_-} : w = w_+ - w_- \text{ is a column of } B)\,.$$

On the other, the expression we have given for $\mathcal{H}(\alpha_i;\beta_j)$ follows from applying the definition of Horn hypergeometric $\mathcal{D}$-modules given in [op. cit., 0.3] for a column matrix and compare it with Definition 2.1. □

Evidently, the isomorphism above is not the only way to express a hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module *à la* GKZ. Anyway, a systematic approach for any hypergeometric $\mathcal{D}$-module $\mathcal{H}(\alpha_i;\beta_j)$ of type (n, m) with $\alpha_1 = 0$ would be, up to a homothety of ratio -1, if needed, taking

$$A = \left(\begin{array}{c|c|c} \underline{1}_m & \underline{0}_{m\times(n-1)} & \mathrm{Id}_m \\ \hline \underline{1}_{n-1} & -\mathrm{Id}_{n-1} & \underline{0}_{(n-1)\times m} \end{array}\right),$$

$B = (1, \overset{(n)}{\dots}, 1, -1, \overset{(m)}{\dots}, -1)$ and $\kappa = (0, -\alpha_2, \dots, -\alpha_n, \beta_1, \dots, \beta_m)$.

3. Hodge modules and Fourier-Laplace transformation

Let $\alpha_1, \ldots, \alpha_n$ be real numbers in $[0,1)$, and consider the hypergeometric $\mathcal{D}_{\mathbb{G}_m}$-module $\mathcal{H} = \mathcal{H}(\alpha_i; \emptyset)$. The goal of this section is proving Theorem 3.22 below, showing that $\mathcal{H}$ underlies an object of $\mathrm{IrrMHM}(\mathbb{G}_m)$ (the abelian category of exponential Hodge modules as defined in [Sab15]) that can be extended in a unique way to the projective line, allowing us to compute the irregular Hodge filtration of $\mathcal{H}$. To reach the aforementioned theorem we will work with some twistor GKZ-hypergeometric systems, following mainly [Moc15b, RS15, RS17]. Although we will give a direct definition, a comprehensive treatment of that situation can be found in those references.

Consider the special case of definition 2.6 starting with the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & -1 & 0 & \ldots & 0 \\ \vdots & \ldots & & & & \\ 1 & 0 & 0 & 0 & \ldots & -1 \end{pmatrix} \in M\left((n-1) \times n, \mathbb{Z}\right), \tag{1}$$

and where the parameter vector is given by $\alpha = (\alpha_2, \ldots, \alpha_n)^{\mathrm{t}} \in \mathbb{R}^{n-1}$. As before, the extended matrix is

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & -1 & 0 & \ldots & 0 \\ \vdots & \ldots & & & & & \\ 0 & 1 & 0 & 0 & 0 & \ldots & -1 \end{pmatrix} \in M\left(n \times (n+1), \mathbb{Z}\right),$$

and we take a parameter (α_0, α) with $\alpha_0 \in \mathbb{C}$. We could consider the A-hypergeometric systems $\mathcal{M}_A^{\alpha}$ or $\mathcal{M}_{\widetilde{A}}^{\widetilde{\alpha}}$, but we will focus on their twistor versions, as commented at the beginning of the section.

In order to understand better the geometry behind the construction of this section, we will work in a slightly more general setup. We also hope that this approach may be of some use if one wants to consider the irregular Hodge filtration for more general hypergeometric $\mathcal{D}$-modules.

Let $d < n$ be now two positive integers, and take a parameter vector $\beta \in \mathbb{C}^d$. For the Hodge theoretic questions we are interested in, only real parameter vectors are relevant, but we will work with this more general setting until Remark 3.15 below. We will consider an arbitrary matrix $A = (\underline{a}_1, \ldots, \underline{a}_n) \in M(d \times n, \mathbb{Z})$ satisfying the following:

Assumptions 3.1.

1. $\mathbb{Z}A = \mathbb{Z}^d$, here $\mathbb{Z}A := \sum_{i=1}^n \mathbb{Z}\underline{a}_i$,
2. A is saturated, that is, we have $\mathbb{N}A = (\mathbb{R}_{\geq 0}A) \cap \mathbb{Z}^d$,
3. If we denote by $\mathrm{Conv}(\underline{a}_1, \ldots, \underline{a}_n)$ the convex hull in $\mathbb{R}^d$ of the vectors given by the columns of the matrix A, then the origin lies strictly in the interior of $\mathrm{Conv}(\underline{a}_1, \ldots, \underline{a}_n)$.

Notice that these assumptions are fulfilled for the matrix A from equation (1). Let $S_1 = \mathbb{G}_m^d = \mathrm{Spec}(\mathbb{C}[y_1^{\pm}, \ldots, y_d^{\pm}])$ and $S_2 = \mathbb{G}_m^n = \mathrm{Spec}(\mathbb{C}[\lambda_1^{\pm}, \ldots, \lambda_n^{\pm}])$ two algebraic tori, and consider the affine space $V = \mathbb{A}^{n+1}$ with coordinates $\lambda_0, \lambda_1, \ldots, \lambda_n$. We let $V^\vee$ be the dual space with coordinates $w_0, w_1, \ldots, w_n$, and we also set $\tau := -w_0$ and $z := \tau^{-1}$. We decompose $V = \mathbb{A}^1_{\lambda_0} \times W$, and consider $S_2 \subset W$ as an open subset.

For a smooth algebraic variety X, we denote by $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z \times X}$ the sheaf of Rees rings on X (with integrable structure), that is, the subsheaf of non-commutative algebras of $\mathcal{D}_{\mathbb{A}^1_z \times X}$ locally generated by $z^2\partial_z$ and $(z\partial_{x_i})_{i=1,\ldots,n}$, where $(x_1, \ldots, x_n)$ are local coordinates on X. Then we put $\widehat{\mathcal{M}}_A^{(\beta_0,\beta)} = \mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z \times S_2}/\mathcal{I}$, where

$\beta_0 \in \mathbb{C}$ and $\mathcal{I}$ is generated by

$$\begin{array}{l} \prod_{j:l_j>0} (z\partial_{\lambda_j})^{l_j} - \prod_{j:l_j<0} (z\partial_{\lambda_j})^{-l_j},\ \underline{l} \in \mathbb{L} \\ z^2\partial_z + \lambda_1 z\partial_{\lambda_1} + \ldots + \lambda_n z\partial_{\lambda_n} - z\beta_0 \\ \sum_{j=1}^n a_{kj}\lambda_j z\partial_{\lambda_j} - z\beta_k,\ k = 1, \ldots, d \end{array},$$

where $\underline{l} = (l_1, \ldots, l_n)$ and $\mathbb{L} \subset \mathbb{Z}^n$ is the kernel of the linear map $\mathbb{Z}^n \to \mathbb{Z}^d$ given by left multiplication by the matrix A.

In the special case of our original matrix from (1) the generators of $\mathcal{I}$ are

$$\begin{array}{c} (z\partial_{\lambda_1}) \cdot \ldots \cdot (z\partial_{\lambda_n}) - 1 \\ z^2\partial_z + \lambda_1 z\partial_{\lambda_1} + \ldots + \lambda_n z\partial_{\lambda_n} - z\beta_0 \\ \lambda_1 z\partial_{\lambda_1} - \lambda_i z\partial_{\lambda_i} - z\beta_i,\ i = 2, \ldots, n \end{array}.$$

The following is a variant of [RS15, Lem. 2.13].

Lemma 3.2. *$\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$ is locally $\mathcal{O}_{\mathbb{A}^1_z \times S_2}$-free of rank $n!\operatorname{vol}(\Delta(\underline{a}_1, \ldots, \underline{a}_n))$, which equals n in the case of the matrix in equation* (1)*, considering the normalized volume in $\mathbb{R}^n$ such that $[0,1]^n$ has volume one.*

We will also use in this section twistor and ordinary $\mathcal{D}$-modules arising from Kummer modules. Let us present them:

Definition 3.3. Recall the Kummer $\mathcal{D}$-module of parameter β is the quotient $\mathcal{K}_\beta = \mathcal{D}_{\mathbb{G}_{m,t}}/(t\partial_t - \beta)$, for any complex number β. Analogously to Kummer $\mathcal{D}$-modules, we define the Kummer $\mathcal{R}^{\text{int}}_{\mathbb{A}^1_z \times \mathbb{G}_{m,t}}$-module of parameter β as the cyclic $\mathcal{R}^{\text{int}}_{\mathbb{A}^1 \times \mathbb{G}_{m,t}}$-module

$$\widehat{\mathcal{K}}_\beta := \mathcal{R}^{\text{int}}_{\mathbb{A}^1 \times \mathbb{G}_{m,t}}/(z^2\partial_z, zt\partial_t - z\beta).$$

Remark 3.4. Although both Kummer modules can be defined for any complex value of their parameters, in the end we will be interested only in the real case to make use of their Hodge properties. Indeed, since both have no singularities at $\mathbb{G}_{m,t}$, if $\mathcal{K}_\beta$ were a complex Hodge module it would be in fact a complex variation of Hodge structures, and by (the first part of) the proof of [Sch73, Lem. 4.5], β must be real.

On the other hand, $\widehat{\mathcal{K}}_\beta$ is clearly the Rees module of $\mathcal{K}_\beta$ together with the trivial filtration $F_\bullet$ such that $F_k = 0$ for $k < 0$ and $F_k = \mathcal{K}_\beta$ for $k \geq 0$. As described in [Moc15a, Prop. 13.5.4], it gives rise to an integrable pure twistor $\mathcal{D}_{\mathbb{G}_{m,t}}$-module, which obviously belongs as well to $\operatorname{IrrMHM}(\mathbb{G}_m)$. It is also described as a harmonic bundle at [op. cit., §2.1.9].

Definition 3.5. For any $\beta \in \mathbb{C}^d$, we will denote by $\mathcal{O}^\beta_{S_1 \times \star}$, where $\star$ will mean some other variety depending on the context, the $\mathcal{D}_{S_1 \times \star}$-module corresponding to the corresponding structure sheaf twisted by $\underline{y}^{-1-\beta}$, that is,

$$\mathcal{O}^\beta_{S_1 \times \star} := \frac{\mathcal{D}_{S_1}}{(y_k\partial_{y_k} + \beta_k + 1\ :\ k = 1, \ldots, d)} \boxtimes \mathcal{O}_\star =: \mathcal{O}^\beta_{S_1} \boxtimes \mathcal{O}_\star.$$

Note that for any other $\beta' \in \mathbb{C}^d$ such that $\beta - \beta' \in \mathbb{Z}^d$, $\mathcal{O}^\beta_{S_1 \times \star} \cong \mathcal{O}^{\beta'}_{S_1 \times \star}$. For real values of the parameter vector, we will be able to think of them as complex Hodge modules, since they are the corresponding exterior product of the Kummer modules $\mathcal{K}_{\beta_1}, \ldots, \mathcal{K}_{\beta_d}$ and $\mathcal{O}_\star$.

We need a slight variant of a comparison result from [RS15, Prop. 2.14] and [RS17, Prop. 3.21]. A more general version, which is not needed here, can be found in [Moc15b, Prop. E.4, E.6]. Consider the following family of Laurent polynomials

$$\begin{array}{rccc} \varphi : & S_1 \times S_2 & \longrightarrow & \mathbb{A}^1_{\lambda_0} \times S_2 \\ & (\underline{y}, \underline{\lambda}) & \longmapsto & \left(-\sum_{i=1}^n \lambda_i \underline{y}^{\underline{a}_i}, \lambda_1, \ldots, \lambda_n \right) \end{array}, \tag{2}$$

which in the case of the matrix from equation (1) becomes

$$\varphi(y_1,\ldots,y_{n-1},\lambda_1,\ldots,\lambda_n)=\left(\frac{\lambda_1}{y_1}+\ldots+\frac{\lambda_{n-1}}{y_{n-1}}+\lambda_n\cdot y_1\cdot\ldots\cdot y_{n-1},\lambda_1,\ldots,\lambda_n\right). \tag{3}$$

Write $f : S_1 \times S_2 \to \mathbb{A}^1_{\lambda_0}$ for the composition of φ with the first projection $\mathbb{A}^1_{\lambda_0} \times S_2 \to \mathbb{A}^1_{\lambda_0}$. With these notations, the following is proved precisely as in [RS17, Prop. 3.21], taking into account the twist of $\mathcal{O}_{S_1\times S_2}$ by $y^{-1-\beta}$.

Proposition 3.6. *There exists an isomorphism of $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times S_2}$-modules*

$$\mathcal{H}^0\left(\pi_{2,*}\Omega^{\bullet+d}_{S_1\times S_2/S_2}[z], z\,(d-\kappa(\beta)\wedge)-df\wedge\right)\longrightarrow \widehat{\mathcal{M}}^{(0,\beta)}_A$$

for any $\beta\in\mathbb{C}^d$, given by sending $\omega := \prod_{j=1}^d y_j/dy_j$ to $[1]\in\widehat{\mathcal{M}}^{(0,\beta)}_A$, where we write $\kappa(\beta)$ for $\sum_{j=1}^d \beta_j dy_j/y_j$ and π_2 for the the second canonical projection $S_1\times S_2\to S_2$.

In order to achieve the comparison results of this section we need to use several variants of the Fourier-Laplace transformations; let us recall here the definitions.

Definition 3.7. Let Y be a smooth algebraic variety, U be a finite-dimensional complex vector space and U' its dual vector space. Denote by $\mathcal{E}$ the trivial vector bundle $\tau : U\times Y\to Y$ and by $\mathcal{E}'$ its dual. Write $\mathrm{can} : U\times U'\to\mathbb{A}^1$ for the canonical morphism defined by $\mathrm{can}(a,\varphi)=\varphi(a)$. This extends to a function $\mathrm{can} : \mathcal{E}\times\mathcal{E}'\to\mathbb{A}^1$. Define $\mathcal{L} := \mathcal{O}_{\mathcal{E}\times_Y\mathcal{E}'}e^{-\mathrm{can}}$, the free rank one module with differential given by the product rule. Consider also the canonical projections $p_1 : \mathcal{E}\times_Y\mathcal{E}'\to\mathcal{E}$, $p_2 : \mathcal{E}\times_Y\mathcal{E}'\to\mathcal{E}'$. The partial Fourier-Laplace transformation is then defined by

$$\mathrm{FL}_Y := p_{2,+}\left(p_1^+\bullet\otimes^{\mathbf{L}}\mathcal{L}\right).$$

If the base Y is a point we recover the usual Fourier-Laplace transformation and we will simply write FL. Notice that although this functor is defined at the level of derived categories, it is exact, i.e., induces a functor $\mathrm{FL}_Y : \mathrm{Mod}_{\mathrm{h}}(\mathcal{D}_{\mathcal{E}})\to\mathrm{Mod}_{\mathrm{h}}(\mathcal{D}_{\mathcal{E}'})$.

Definition 3.8. Keep the notations of the previous definition, and assume moreover that U and U' are one-dimensional, with respective coordinates t and τ. Put $z=\tau^{-1}$, and denote by $j_\tau : \mathbb{G}_{m,\tau}\hookrightarrow\mathbb{A}^1_\tau$ and $j_z : \mathbb{G}_{m,\tau}\hookrightarrow\mathbb{A}^1_z=\mathbb{P}^1_\tau\setminus\{\tau=0\}$ the canonical embeddings. Then the partial localized Fourier-Laplace transformation with respect to τ is defined by

$$\mathrm{FL}^{\mathrm{loc}}_Y := (j_z\times\mathrm{id}_Y)_+(j_\tau\times\mathrm{id}_Y)^+\,\mathrm{FL}_Y\,.$$

In the sequel, we need to consider a partial compactification of the torus S_2 and an extension of φ to a projective morphism. Consider the embedding

$$\begin{array}{rcl} g : S_1 & \hookrightarrow & \mathbb{P}\left(V^\vee\right)=\mathbb{P}^n \\ y & \longmapsto & (w_0:\ldots:w_n)=\left(1:\underline{y}^{\underline{a}_1}:\ldots:\underline{y}^{\underline{a}_n}\right) \end{array} \tag{4}$$

and put $X := \overline{\mathrm{Im}(g)}$. Consider the graph $\Gamma_\varphi\subset S_1\times\mathbb{A}^1\times S_2$, and let ΓX be the closure of Γ_φ in $X\times\mathbb{A}^1\times S_2$. Then we have $\Gamma X\subset Z^*$, where

$$Z^* := \mathcal{V}\left(\sum_{i=0}^n\lambda_i\cdot w_i\right)\subset\mathbb{P}^n\times\mathbb{A}^1\times S_2$$

is the universal hypersurface. We thus have the following commutative diagram

$$\begin{tikzcd} S_1\times S_2 \arrow[r, hook] \arrow[rr, bend left, "k"] \arrow[dr, "\varphi"'] & \Gamma X \arrow[r, hook] \arrow[d] & Z^* \arrow[dl, "p_2"] \\ & \mathbb{A}^1\times S_2 & \end{tikzcd} \tag{5}$$

where the map $S_1 \times S_2 \hookrightarrow \Gamma X$ is the composition of the isomorphism $S_1 \times S_2 \overset{\cong}{\to} \Gamma_\varphi$ with $\Gamma_\varphi \hookrightarrow \Gamma X$ and the map $\Gamma X \hookrightarrow Z^*$ is the closed embedding mentioned above.

We need also the following important geometric property of the families given by φ and ϕ.

Lemma 3.9. *The morphism ϕ is stratified smooth on its boundary $\Gamma X \setminus (S_1 \times S_2)$. As a consequence, for any parameter value $\underline{\lambda} \in S_2$, the restricted morphism $f_{\underline{\lambda}} : S_1 \to \mathbb{A}^1$ is cohomologically tame in the sense of [Sab06, §8].*

Proof. Recall that the projective toric variety X is stratified by $X = \bigcup_{\Gamma \subset \Delta} X_\Gamma$, where Δ is the convex hull in $\mathbb{R}^d$ of the columns of the matrix A, Γ is a face of Δ, and X_Γ is a torus orbit associated with the face Γ (cf., for instance, [GKZ94, Prop. 5.1.9]). That stratification induces another one on ΓX. By definition, a morphism is stratified smooth if its restriction to each strata is a non-singular map. Recall that $f_{\underline{\lambda}}$ is non-degenerate for any parameter value $\underline{\lambda} \in S_2$ by [RS15, Lem. 2.8]. Hence so is $F_{\underline{\lambda}}$ and $\phi_{|\Gamma X \setminus \operatorname{im} j}$ is stratified smooth with respect to the toric stratification.

Fix then a $\underline{\lambda} \in S_2$. The function $f_{\underline{\lambda}}$ is cohomologically tame if the sheaf $\mathbf{R}g_* \mathbb{Q}_{S_1}$ has no vanishing cycles outside of S_1 with respect to any $F_{\underline{\lambda}} - c$ with $c \in \mathbb{C}$, seeing g as an embedding $g : S_1 \hookrightarrow X$ and extending $F_{\underline{\lambda}}$ to X. Now, since $F_{\underline{\lambda}} - c$ is stratified smooth on $X \setminus S_1$ for any $c \in \mathbb{C}$, the vanishing cycles with respect to it of any stratified local system, as $\mathbf{R}g_* \mathbb{Q}_{S_1}$, have no support outside of S_1 by [Dim04, Prop. 4.2.8]. In conclusion, $f_{\underline{\lambda}}$ is cohomologically tame. ☐

As an intermediate goal we want to compare the left-hand side of the isomorphism of Proposition 3.6 with an object derived from a certain intersection cohomology $\mathcal{D}$-module on Z^*.

Recall that for a smooth algebraic variety Y, and an open subvariety $j : U \hookrightarrow Y$, we call intersection cohomology module with coefficients in some $\mathcal{D}_U$-module $\mathcal{N}$ is the intermediate extension $j_{\dagger+}\mathcal{N} := \operatorname{im}(j_\dagger \mathcal{N} \to j_+ \mathcal{N})$. Its name comes from the fact that, if $\mathcal{N}$ is smooth and corresponds to a local system $\mathcal{L}$ on U under the Rieman-Hilbert correspondence, $j_{\dagger+}\mathcal{N}$ corresponds to the intersection cohomology complex on Y with coefficients in $\mathcal{L}$, $IC(Y, \mathcal{L})$.

We will also make use of different Radon transformations. Let us define them and state some properties, following [Rei14]:

Definition 3.10. Denote by $Z \subset \mathbb{P}(V^\vee) \times V$ the universal hyperplane given with equation $\sum_{i=0}^n w_i \lambda_i = 0$ and by $U := (\mathbb{P}(V^\vee) \times V) \setminus Z$ its complement. Consider the following diagram

$$\begin{tikzcd}
 & U \arrow[dl, "\pi_1^U"'] \arrow[d, hook, "j_U"] \arrow[dr, "\pi_2^U"] & \\
\mathbb{P}(V^\vee) & \mathbb{P}(V^\vee) \times V \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & V, \\
 & Z \arrow[ul, "\pi_1^Z"] \arrow[u, hook', "i_Z"'] \arrow[ur, "\pi_2^Z"'] &
\end{tikzcd}$$

The Radon transformations are functors from $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}(V^\vee)})$ to $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_V)$ given by

$$\begin{aligned}
\mathcal{R} &:= \pi^Z_{2,+} \pi_1^{Z,+} \cong \pi_{2,+} i_{Z,+} i_Z^+ \pi_1^+, \\
\mathcal{R}^\circ &:= \pi^U_{2,+} \pi_1^{U,+} \cong \pi_{2,+} j_{U,+} j_U^+ \pi_1^+, \\
\mathcal{R}^\circ_c &:= \pi^U_{2,\dagger} \pi_1^{U,+} \cong \pi_{2,+} j_{U,\dagger} j_U^+ \pi_1^+, \\
\mathcal{R}_{cst} &:= \pi_{2,+} \pi_1^+.
\end{aligned}$$

Proposition 3.11. *Let g be as in 4. Then, for every $\beta \in \mathbb{C}^d$, we have the following two exact sequences of regular holonomic $\mathcal{D}_V$-modules, which are dual to each other:*

$$0 \longrightarrow \mathcal{H}^{-1}\mathcal{R}_{cst} g_+ \mathcal{O}^\beta_{S_1} \longrightarrow \mathcal{H}^0 \mathcal{R} g_+ \mathcal{O}^\beta_{S_1} \longrightarrow \mathcal{H}^0 \mathcal{R}^\circ_c g_+ \mathcal{O}^\beta_{S_1} \longrightarrow \mathcal{H}^0 \mathcal{R}_{cst} g_+ \mathcal{O}^\beta_{S_1} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{H}^0\mathcal{R}_{cst} g_\dagger \mathcal{O}^{-\beta}_{S_1} \longrightarrow \mathcal{H}^0 \mathcal{R}^\circ g_\dagger \mathcal{O}^{-\beta}_{S_1} \longrightarrow \mathcal{H}^0 \mathcal{R} g_\dagger \mathcal{O}^{-\beta}_{S_1} \longrightarrow \mathcal{H}^1 \mathcal{R}_{cst} g_\dagger \mathcal{O}^{-\beta}_{S_1} \longrightarrow 0.$$

Moreover, the $\mathcal{D}_V$-modules $\mathcal{H}^i\mathcal{R}_{cst}g_\star\mathcal{O}^\beta_{S_1}$, for $i \in \{-1,0,1\}$ and $\star \in \{+,\dagger\}$ that appear in the above sequences are $\mathcal{O}_V$-free. Consequently, for any $\beta \in \mathbb{C}^d$, calling j_{V^*} the canonical inclusion $V^* := \mathbb{A}^1_{\lambda_0} \times S_2 \hookrightarrow V$, we have isomorphisms of $\mathcal{D}_{\mathbb{A}^1_z \times S_2}$-modules

$$\begin{array}{rcl} \mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}^\circ_c g_+\mathcal{O}^\beta_{S_1} & \cong & \mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}g_+\mathcal{O}^\beta_{S_1} \\ \mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}^\circ g_\dagger\mathcal{O}^\beta_{S_1} & \cong & \mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}g_\dagger\mathcal{O}^\beta_{S_1} \end{array}.$$

Proof. Following the notation of the previous definition, since Z is smooth, the excision triangle ([HTT08, Prop. 1.7.1]) corresponding to the diagram $U \hookrightarrow \mathbb{P}(V^\vee) \times V \leftarrow Z$ gives rise to the following triangles of Radon transformations for any $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_{\mathbb{P}(V^\vee)})$:

$$\mathcal{R}_{cst}\mathcal{M} \longrightarrow \mathcal{R}^\circ\mathcal{M} \longrightarrow \mathcal{R}\mathcal{M} \longrightarrow, \tag{6}$$

$$\mathcal{R}\mathcal{M} \longrightarrow \mathcal{R}^\circ_c\mathcal{M} \longrightarrow \mathcal{R}_{cst}\mathcal{M} \longrightarrow, \tag{7}$$

where the second triangle is dual to the first.

Note that $\mathbb{D}\mathcal{O}^\beta_{S_1} \cong \mathcal{O}^{-1-\beta}_{S_1} \cong \mathcal{O}^{-\beta}_{S_1}$. It suffices then to show the existence of the first exact sequence of the statement of the proposition and the fact that the $\mathcal{H}^i\mathcal{R}_{cst}g_\star\mathcal{O}^\beta_{S_1}$ are constant. The idea of the proof is similar to that of proposition 3.6; the existence of the exact sequences here follows by just a variation of [Rei14, Prop. 2.8], but considering the twist by $\underline{y}^{-1-\beta}$ of $\mathcal{O}_{S_1}$. That needs in turn [op. cit., Prop. 2.5, 2.7, Lem. 2.6], but almost every argument is functorial and anyway they can be easily adapted to our context. The constancy of $\mathcal{H}^i\mathcal{R}_{cst}g_\star\mathcal{O}^\beta_{S_1}$ can be proved as the second point of [op. cit., Lem. 2.7]. $\square$

Now is when we can properly state and prove the next comparison result of this section:

Proposition 3.12. *For any $\beta \in \mathbb{C}^d$, we have the following isomorphism of $\mathcal{D}_{\mathbb{A}^1_z \times S_2}$-modules:*

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0 p_{2,+}k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \xrightarrow{\sim} \mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0\varphi_+\mathcal{O}^\beta_{S_1\times S_2}.$$

Moreover, this isomorphism is induced from the canonical morphism $k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \hookrightarrow k_+\mathcal{O}^\beta_{S_1\times S_2}$ by applying the functor $\mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0 p_{2,+}$.

Proof. Call p_1 the restriction to Z^* of the projection $\mathbb{P}^n \times \mathbb{A}^1_{\lambda_0} \times S_2 \to \mathbb{P}^n$, abusing a bit of the notation. Consider the cartesian diagram

$$\begin{tikzcd} S_1 \times S_2 \arrow[r, "\pi_1"] \arrow[d, "k"'] \arrow[dr, phantom, "\square"] & S_1 \arrow[d, "g"] \\ Z^* \arrow[r, "p_1"'] & \mathbb{P}^n \end{tikzcd},$$

By base change, we have that $k_+\mathcal{O}^\beta_{S_1\times S_2} \cong p_1^+g_+\mathcal{O}^\beta_{S_1}$ for any $\beta \in \mathbb{C}^d$. Now, since p_1 is smooth, we have the analogous isomorphism $k_\dagger\mathcal{O}^\beta_{S_1\times S_2} \cong p_1^+g_\dagger\mathcal{O}^\beta_{S_1}$ for every β as well. Now note that because of p_1 being non-characteristic, it is easy to show (cf. the proof of the third point of [RS17, Prop. 2.22]) that for every β

$$k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \cong p_1^+g_{\dagger+}\mathcal{O}^\beta_{S_1}.$$

Consider now the commutative diagram

$$\begin{tikzcd} & Z^* \arrow[r, "p_2"] \arrow[dl, "p_1"'] \arrow[d, "j_{Z^*}"] & V^* \arrow[d, "j_{V^*}"] \\ \mathbb{P}^n & Z \arrow[l, "\pi_1^Z"] \arrow[r, "\pi_2^Z"'] & V \end{tikzcd},$$

where the square is also cartesian. By applying base change again, we obtain the natural transformations of functors from $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_{\mathbb{P}^n})$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_{V^*})$

$$p_{2,+}p_1^+ \cong p_{2,+}j^+_{Z^*}\pi_1^{Z,+} \cong j^+_{V^*}\pi^Z_{2,+}\pi_1^{Z,+} = j_{V^*}\mathcal{R}. \tag{8}$$

Therefore, it follows that

$$p_{2,+}k_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S_2} \cong j^+_{V^*}\mathcal{R}g_{\dagger+}\mathcal{O}^{\beta}_{S_1}, \tag{9}$$

for any $\beta \in \mathbb{C}^d$.

We need now to relate the various Radon transformations with the Fourier-Laplace transformation $FL : \mathrm{Mod_h}(\mathcal{D}_{V^\vee}) \to \mathrm{Mod_h}(\mathcal{D}_V)$. This is possible due to the fundamental result of d'Agnolo and Eastwood [DE03, Prop. 1]. We quote the formulation from the proof of [RS17, Lem. 2.12]. Let $Bl_0(V^\vee) \subset \mathbb{P}^n \times V^\vee$ be the blow-up of $V^\vee$ at the origin and consider the following diagram

$$\begin{tikzcd}
V^* \arrow[r, "h"] \arrow[dr, "\tilde{h}"] \arrow[dd, "\pi_{V^*}"'] & V^\vee & Bl_0(V^\vee) \arrow[l, "p"'] \arrow[ddl, "q"] \\
 & V^\vee \setminus \{0\} \arrow[u, "j_0"'] \arrow[d, "\pi"'] & \\
S_1 \arrow[r, "g"] & \mathbb{P}^n &
\end{tikzcd},$$

where π is the canonical morphism $V^\vee \setminus \{0\} \to \mathbb{P}(V^\vee)$, π_{V^*} is the second projection, and h and $\tilde{h}$ are given by $(y_0, \underline{y}) \mapsto (y_0, y_0\underline{y}^{\underline{a}_1}, \ldots, y_0\underline{y}^{\underline{a}_d})$. Then we have the natural transformations

$$\mathcal{R}^\circ_c g_+ \cong \mathrm{FL}\, h_+\pi^+_{V^*} \ \text{ and } \ \mathcal{R}^\circ g_\dagger \cong \mathrm{FL}\, h_\dagger\pi^+_{V^*}, \tag{10}$$

of functors from $\mathrm{Mod_h}(\mathcal{D}_{S_1})$ to $\mathrm{Mod_h}(\mathcal{D}_V)$. Indeed, although in general $\mathcal{R}^\circ_c$ and $\mathcal{R}^\circ$ are not exact, the compositions $\mathcal{R}^\circ_c g_+$ and $\mathcal{R}^\circ g_\dagger$ are so precisely due to the above isomorphisms.

Applying $\mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}$ to the natural transformations in 10 we obtain the following ones of functors from $\mathrm{Mod_h}(\mathcal{D}_{S_1})$ to $\mathrm{Mod_h}(\mathcal{D}_{\mathbb{A}^1_z\times S_2})$:

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\mathcal{R}^\circ_c g_+ \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\,\mathrm{FL}\, h_+\pi^+ \ \text{ and } \ \mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\mathcal{R}^\circ g_\dagger \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\,\mathrm{FL}\, h_\dagger\pi^+ \tag{11}$$

We are now closer to the isomorphism of the statement; let us rewrite in a different way the functors above to obtain it. Namely,

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\,\mathrm{FL} \cong j^+_{V^*}\,\mathrm{FL}^{\mathrm{loc}}_W\,\mathrm{FL} \cong j^+_{V^*}(j_z\times\mathrm{id}_W)_+(j_\tau\times\mathrm{id}_W)^+\,\mathrm{FL}_W\,\mathrm{FL} \cong j^+_{V^*}(j_z\times\mathrm{id}_W)_+(j_\tau\times\mathrm{id}_W)^+\,\mathrm{FL}_{\mathbb{A}^1_{w_0}}.$$

From the third of the assumptions 3.1 we know that we we can decompose the morphism h as

$$h : V^* \xhookrightarrow{h_1} \mathbb{G}_{m,w_0} \times W^\vee \xhookrightarrow{j_\tau\times\mathrm{id}_{W^\vee}} \mathbb{A}^1_{w_0} \times W^\vee = V^\vee \quad ,$$

where h_1 is a closed embedding. It follows that

$$(j_\tau \times \mathrm{id}_W)^+\,\mathrm{FL}_{\mathbb{A}^1_{w_0}}\, h_+ \cong \mathrm{FL}_{\mathbb{A}^1_{w_0}}(j_\tau \times \mathrm{id}_{W^\vee})^+h_+ \cong \mathrm{FL}_{\mathbb{A}^1_{w_0}}(j_\tau \times \mathrm{id}_{W^\vee})^+(j_\tau \times \mathrm{id}_{W^\vee})_+h_{1,+} \cong \mathrm{FL}_{\mathbb{A}^1_{w_0}}\, h_{1,+}.$$

Summarizing, we obtain that

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\,\mathrm{FL}\, h_\star \cong j^+_{V^*}(j_z \times \mathrm{id}_W)\,\mathrm{FL}_{\mathbb{A}^1_{w_0}}\, h_{1,+},$$

where $\star \in \{+, \dagger\}$, because $h_{1,+} = h_{1,\dagger}$ for h_1 is proper. In particular, we have $\mathrm{FL}^{\mathrm{loc}}_{S_2} \circ j^+_{V^*} \circ \mathrm{FL} \circ h_+ \cong \mathrm{FL}^{\mathrm{loc}}_{S_2} \circ j^+_{V^*} \circ \mathrm{FL} \circ h_\dagger$. As a consequence, we can claim using the isomorphisms in 11 that

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\mathcal{R}^\circ_c g_+ \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\mathcal{R}^\circ g_\dagger,$$

and so, from applying this last natural transformation to $\mathcal{O}^{\beta}_{S_1}$,

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\mathcal{R}g_{\dagger+}\mathcal{O}^{\beta}_{S_1} \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\, j^+_{V^*}\mathcal{R}g_+\mathcal{O}^{\beta}_{S_1}.$$

The final step towards the statement uses the first point of [Rei14, Prop. 2.7]. Although its statement has to do only with the structure sheaf $\mathcal{O}_{S_1}$, as we said in the proof of Proposition 3.11

almost every argument follows by functoriality and it is straightforward to generalise them to work with $\mathcal{O}^\beta_{S_1}$ to get that

$$j^+_{V^*}\mathcal{R}g_+\mathcal{O}^\beta_{S_1} \cong \varphi_+\mathcal{O}^\beta_{S_1\times S_2}. \tag{12}$$

In conclusion, we finally obtain, by using the isomorphism from 9, that

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0p_{2,+}k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \cong \mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}g_{\dagger+}\mathcal{O}^\beta_{S_1} \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0\varphi_+\mathcal{O}^\beta_{S_1\times S_2}.$$

The last statement is an easy consequence of isomorphisms 8, 9 and 12, noting that

$$\mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0p_{2,+}k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0 j^+_{V^*}\mathcal{R}g_{\dagger+}\mathcal{O}^\beta_{S_1} \cong \mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}g_+\mathcal{O}^\beta_{S_1} \cong \mathrm{FL}^{\mathrm{loc}}_{S_2}\mathcal{H}^0\varphi_+\mathcal{O}^\beta_{S_1\times S_2}$$

is induced from the canonical morphism $g_{\dagger+}\mathcal{O}^\beta_{S_1} \hookrightarrow g_+\mathcal{O}^\beta_{S_1}$ via $\mathrm{FL}^{\mathrm{loc}}_{S_2} j^+_{V^*}\mathcal{R}$. □

Consider the space $P := \mathbb{P}^n \times (\mathbb{A}^1_{\lambda_0} \times S_2)$, together with the canonical closed embedding $Z^* \hookrightarrow P$. We have the diagram

$$\begin{tikzcd} Z^* \arrow[r, hook] \arrow[d, "p_2"] & P \arrow[dl, "p_2"] \\ \mathbb{A}^1_{\lambda_0} \times S_2 & \end{tikzcd},$$

where we denote the restriction of the second projection $p_2 : P \to \mathbb{A}^1_{\lambda_0} \times S_2$ to Z^* by the same letter (as was done before). Let $l : S_1 \times S_2 \to P$ be the composition of the map $k : S_1 \times S_2 \to Z^*$ with the closed embedding $Z^* \hookrightarrow P$. Since the latter is proper, we have

$$\mathcal{H}^0p_{2,+}k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \cong \mathcal{H}^0p_{2,+}l_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \text{ and } \mathcal{H}^0\varphi_+\mathcal{O}^\beta_{S_1\times S_2} \cong \mathcal{H}^0p_{2,+}k_+\mathcal{O}^\beta_{S_1\times S_2} \cong \mathcal{H}^0p_{2,+}l_+\mathcal{O}^\beta_{S_1\times S_2}.$$

so that we can also write the morphism $\mathcal{H}^0p_{2,+}k_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \xrightarrow{\sim} \mathcal{H}^0\varphi_+\mathcal{O}^\beta_{S_1\times S_2}$ from Proposition 3.12 as $\mathcal{H}^0p_{2,+}l_{\dagger+}\mathcal{O}^\beta_{S_1\times S_2} \longrightarrow \mathcal{H}^0p_{2,+}l_+\mathcal{O}^\beta_{S_1\times S_2}$. We will also need to consider the affine chart

$$P^* := \mathbb{A}^n \times (\mathbb{A}^1_{\lambda_0} \times S_2) \subset P$$

where $\mathbb{A}^n \subset \mathbb{P}^n$ is given by $w_0 \neq 0$.

In order to proceed, we will have to take into account certain group actions on the spaces $S_1 \times S_2$, $\mathbb{P}^n \times (\mathbb{A}^1_{\lambda_0} \times S_2)$ and $\mathbb{A}^1_{\lambda_0} \times S_2$ as well as some equivariance properties of the various sheaves of modules on these spaces. For the readers convenience, we recall some facts from [RS17, §2.4].

We consider the action

$$\begin{array}{ccc} S_1 \times (S_1 \times S_2) & \longrightarrow & S_1 \times S_2 \\ \underline{t}, (\underline{t}, \lambda_1, \dots, \lambda_n) & \longmapsto & (t_1y_1, \dots, t_dy_d, \underline{t}^{-\underline{a}_1}\lambda_1, \dots, \underline{t}^{-\underline{a}_n}\lambda_n), \end{array}$$

the action

$$\begin{array}{ccc} S_1 \times P & \longrightarrow & P \\ \underline{t}, (w_0 : \dots : w_n, \lambda_0, \lambda_1, \dots, \lambda_n) & \longmapsto & (w_0 : \underline{t}^{\underline{a}_1}w_1 : \dots : \underline{t}^{\underline{a}_n}w_n, \lambda_0, \underline{t}^{-\underline{a}_1}\lambda_1, \dots, \underline{t}^{-\underline{a}_n}\lambda_n), \end{array}$$

as well as the induced action $S_1 \times P^* \to P^*$ and the action

$$\begin{array}{ccc} S_1 \times \left(\mathbb{A}^1_{\lambda_0} \times S_2\right) & \longrightarrow & \mathbb{A}^1_{\lambda_0} \times S_2 \\ \underline{t}, (\lambda_0, \lambda_1, \dots, \lambda_n) & \longmapsto & (\lambda_0 : \underline{t}^{-\underline{a}_1}\lambda_1, \dots, \underline{t}^{-\underline{a}_n}\lambda_n). \end{array}$$

It can be shown as in loc. cit. that all these four actions are free (basically because the action $S_1 \times S_2 \to S_2$ which sends $(\underline{t}, \underline{\lambda})$ to $\underline{t}^{-\underline{a}_1}\lambda_1, \dots, \underline{t}^{-\underline{a}_n}\lambda_n$ is free) and hence have smooth geometric quotients. These are described by the following result, which we cite from [RS17, §2.4]. For a free action of an algebraic group G on a smooth variety X, we write X/G for the geometric quotient, which is again a smooth algebraic variety.

Proposition 3.13. *In the above situation, put $S^{\mathrm{red}} := (\mathbb{G}_{m,t})^{n-d}$. Then the geometric quotients $(S_1 \times S_2)/S_1$, P/S_1 and $(\mathbb{A}^1_{\lambda_0} \times S_2)/S_1$ are given, respectively, by the spaces*

$$S_1 \times S^{\mathrm{red}},\ \mathbb{P}^n \times \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}},\ \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}}$$

There is a canonical embedding $S^{\mathrm{red}} \hookrightarrow S_2$ inducing embeddings (all denoted by ι)

$$\begin{array}{rcl} S_1 \times S^{\mathrm{red}} & \hookrightarrow & S_1 \times S_2 \\ \mathbb{P}^n \times \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}} & \hookrightarrow & P \\ S_1 \times \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}} & \hookrightarrow & S_1 \times \mathbb{A}^1_{\lambda_0} \times S_2. \end{array}$$

In the sequel, we will always consider S^{red} as a subspace of S_2 (as well as $\mathbb{P}^n \times \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}}$ as a subspace of P etc.). We define reduced versions of the maps φ and l by the cartesian diagrams

$$\begin{tikzcd} S_1 \times S_2 \arrow[r, "\varphi"] & \mathbb{A}^1_{\lambda_0} \times S_2 \\ S_1 \times S^{\mathrm{red}} \arrow[u, hook, "\iota"] \arrow[r, "\varphi^{\mathrm{red}}"] & \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}} \arrow[u, hook, "\iota"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S_1 \times S_2 \arrow[r, "l"] & P \\ S_1 \times S^{\mathrm{red}} \arrow[u, hook, "\iota"] \arrow[r, "l^{\mathrm{red}}"] & \mathbb{P}^n \times \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}} \arrow[u, hook, "\iota"'] \end{tikzcd}$$

In order to illustrate these statements, consider the main case of interest, where we have

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \dots & & & & \\ 1 & 0 & 0 & 0 & \dots & -1 \end{pmatrix} \in M\left((n-1) \times n, \mathbb{Z}\right),$$

Then $d = n-1$, $S^{\mathrm{red}} = \mathbb{G}_{m,t}$ and we have

$$\begin{array}{rcl} \varphi^{\mathrm{red}} : S_1 \times \mathbb{G}_{m,t} & \longrightarrow & \mathbb{A}^1_{\lambda_0} \times \mathbb{G}_{m,t} \\ (y_1, \dots, y_{n-1}, t) & \longmapsto & \left(\frac{1}{y_1} + \dots + \frac{1}{y_{n-1}} + t \cdot y_1 \cdot \dots \cdot y_{n-1}, t\right) \end{array} \tag{13}$$

We also need the following result from loc. cit. showing that ι behaves well with respect to all modules in question. To simplify our notation we will write

$$M_{\dagger+,S_2} := \mathcal{H}^0 p_{2,+} l_{\dagger+} \mathcal{O}^\beta_{S_1 \times S_2} \qquad \text{and} \qquad M_{S_2} := \mathcal{H}^0 \varphi_+ \mathcal{O}^\beta_{S_1 \times S_2} = \mathcal{H}^0 p_{2,+} l_+ \mathcal{O}^\beta_{S_1 \times S_2},$$

and analogously,

$$M_{\dagger+} := \mathcal{H}^0 p_{2,+} l^{\mathrm{red}}_{\dagger+} \mathcal{O}^\beta_{S_1 \times S^{\mathrm{red}}} \qquad \text{and} \qquad M := \mathcal{H}^0 \varphi^{\mathrm{red}}_+ \mathcal{O}^\beta_{S_1 \times S^{\mathrm{red}}} = \mathcal{H}^0 p_{2,+} l^{\mathrm{red}}_+ \mathcal{O}^\beta_{S_1 \times S^{\mathrm{red}}}.$$

Proposition 3.14 ([RS17, Prop. 2.22., Lem. 6.4.])**.** *The morphism $\iota : \mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}} \hookrightarrow \mathbb{A}^1_{\lambda_0} \times S_2$ is non-characteristic for the modules $M_{\dagger+,S_2}$ and M_{S_2}. Consequently, we have the morphism of $\mathcal{D}_{\mathbb{A}^1_{\lambda_0} \times S^{\mathrm{red}}}$-modules $\phi^{\mathrm{red}} : M_{\dagger+} \to M$ and an isomorphism of $\mathcal{D}_{\mathbb{A}^1_z \times S^{\mathrm{red}}}$-modules $\mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}} M_{\dagger+} \cong \mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}} M$.*

Remark 3.15. From now on we will assume that the β_i are real numbers, since we will use some Hodge theoretic constructions, which are not valid for arbitrary complex β_i, as commented in Remark 3.4. Next we will introduce certain filtrations on the $\mathcal{D}$-modules considered above.

Recall that an easy calculation decomposing φ^{red} as a graph embedding followed by a projection shows that the direct image complex $\varphi^{\mathrm{red}}_+ \mathcal{O}^\beta_{S_1 \times S^{\mathrm{red}}}$ can be represented by

$$\left(\varphi^{\mathrm{red}}_* \Omega^{\bullet+d}_{S_1 \times S^{\mathrm{red}}/S^{\mathrm{red}}}[\partial_{\lambda_0}], d - \kappa(\beta) \wedge -(df \wedge) \otimes \partial_{\lambda_0}\right),$$

where f is still the first component of φ^{red}.

We consider the filtration on each $\varphi_*^{\mathrm{red}}\Omega^{l+d}_{S_1\times S^{\mathrm{red}}/S^{\mathrm{red}}}[\partial_{\lambda_0}]$ given by

$$F_{k+l}\varphi_*^{\mathrm{red}}\Omega^{l+d}_{S_1\times S^{\mathrm{red}}/S^{\mathrm{red}}}[\partial_{\lambda_0}] = \sum_{i=0}^{k+l}\varphi_*^{\mathrm{red}}\Omega^{l+d}_{S_1\times S^{\mathrm{red}}/S^{\mathrm{red}}}\partial^i_{\lambda_0}, \tag{14}$$

and we put in the induced complex

$$F_k\Big(\varphi_*^{\mathrm{red}}\Omega^{\bullet+d}_{S_1\times S^{\mathrm{red}}/S^{\mathrm{red}}}[\partial_{\lambda_0}], d-(\kappa(\beta)\wedge)-(df\wedge)\otimes\partial_{\lambda_0}\Big) := \Big(F_{k+\bullet}\varphi_*^{\mathrm{red}}\Omega^{\bullet+d}_{S_1\times S^{\mathrm{red}}/S^{\mathrm{red}}}[\partial_{\lambda_0}], d-(\kappa(\beta)\wedge)-(df\wedge)\otimes\partial_{\lambda_0}\Big).$$

This yields a filtered complex, and so an induced filtration on the cohomology sheaves $\mathcal{H}^i\varphi_+^{\mathrm{red}}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}$. In particular, we obtain a filtration $F_\bullet$ on $M=\mathcal{H}^0\varphi_+^{\mathrm{red}}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}$.

On the other hand, it is well-known that $l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}$ carries a filtration $F^H l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}$ such that $\left(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}, F^H\right)$ underlies a pure polarizable complex Hodge module of weight $n=\dim(S_1)+\dim(S^{\mathrm{red}})$ (which is by definition a filtered $\mathcal{D}_P$-module which is a direct summand of a pure polarizable real Hodge module (cf., e.g., [DS13, Def. 3.2.1]). Since p_2 is projective (not just proper), the direct image $M_{\dagger+}=\mathcal{H}^0p_{2,+}l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}$ also carries a Hodge filtration $F^H\mathcal{H}^0p_{2,+}l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}$ such that $(M_{\dagger+}, F^H)$ underlies a pure polarisable complex Hodge module (see the first point of [Sai88, Thm. 1]). We consider the shifted filtration $F^{H_{sh}}_\bullet M_{\dagger+} := F^H_{\bullet-n}M_{\dagger+}$.

Lemma 3.16. *Consider again the morphism from Proposition 3.14 $\phi^{\mathrm{red}}: M_{\dagger+}\to M$ which after applying the functor $\mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}}$ induces an isomorphism of $\mathcal{D}_{\mathbb{A}^1_z\times S^{\mathrm{red}}}$-modules. Then ϕ^{red} is filtered with respect to the filtrations $F_\bullet$ on both sides, i.e., for every $k\in\mathbb{Z}$, we have the inclusion*

$$\phi\left(F^{H_{sh}}_k M_{\dagger+}\right)\subset F_kM.$$

Proof. Recall that $P^*=\mathbb{A}^n\times(\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}})$ is the the complement in P of the divisor $w_0=0$. Then the map l^{red} can be decomposed (in a different way than the map k) into a closed embedding $l^{\mathrm{red}}_1: S_1\times S^{\mathrm{red}}\hookrightarrow P^*$ and an open embedding $l^{\mathrm{red}}_2: P^*\hookrightarrow P$. More precisely, we have the following diagram

(15)

$$\begin{tikzcd}
S_1\times S^{\mathrm{red}} \arrow[r, hook, "i^{\mathrm{red}}_\varphi"] \arrow[rr, bend left, "l^{\mathrm{red}}_1"] \arrow[rrr, bend left=40, "l^{\mathrm{red}}"] \arrow[drr, "\varphi^{\mathrm{red}}"'] & S_1\times(\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}) \arrow[r, hook, "l^{\mathrm{red}}_0"] \arrow[dr, "{''p_2}"] & P^*=\mathbb{A}^n\times(\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}) \arrow[r, hook, "l^{\mathrm{red}}_2"] \arrow[d, "{'p_2}"] & P=\mathbb{P}^n\times(\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}) \arrow[dl, "p_2"] \\
 & & \mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}} &
\end{tikzcd}.$$

where p_2, $'p_2$ and $''p_2$ are all the projections to the last coordinates. Notice that under the hypotheses at the beginning of this section, we have that l^{red}_0 is a closed embedding, and so is l^{red}_1 (being the composition of two closed embeddings). Note further that the filtration $F^{H_{sh}}_\bullet M_{\dagger+}$ is induced from the filtration $F^{H_{sh}}_\bullet\mathrm{DR}^{\bullet+n}_{P/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}})$ on the relative de Rham complex, where

$$F^{H_{sh}}_k\mathrm{DR}^{\bullet+n}_{P/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}) := \Big(\ldots\longrightarrow\Omega^{n+l}_{P/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\otimes F^{H_{sh}}_{k+l}l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times S^{\mathrm{red}}}\longrightarrow\ldots\Big) \tag{16}$$

by the first point of [Sai88, Thm. 1]. From the natural morphism $l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}} \to l^{\mathrm{red}}_{+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}$, we obtain (by applying $p_{2,+}$) a morphism

(17)
$$\phi^{\mathrm{red}} : M_{\dagger+} = p_{2,+}l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}} \longrightarrow p_{2,+}l^{\mathrm{red}}_{+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}} \cong p_{2,+}l^{\mathrm{red}}_{2,+}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right) = {}'p_{2,+}l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}.$$

Notice that since l^{red}_1 is closed (so that $l^{\mathrm{red}}_{1,\dagger} = l^{\mathrm{red}}_{1,+} = l^{\mathrm{red}}_{1,\dagger+}$), we have $l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}} \cong (l^{\mathrm{red}}_2)^+(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}})$.

The mappings p_2 and ${}'p_2$ being projections, we can reformulate the morphism in (17) as

$$\begin{array}{rcl} Rp_{2,*}\mathrm{DR}^{n+\bullet}_{P/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right) & \xrightarrow{\phi} & R{}'p_{2,*}\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right) \\ & \cong & {}'p_{2,*}\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left((l^{\mathrm{red}}_2)^+l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right). \end{array}$$

The restriction morphism $(l^{\mathrm{red}}_2)^+$ is compatible with the Hodge filtrations on $l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}$ and $l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}$ as well as with their shifted versions. Hence ϕ is filtered with respect to the filtration on the left-hand side induced from (16) and the filtration on the right-hand side defined by

$$F_k\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right) := \left(\ldots \to \Omega^{n+l}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}} \otimes_{\mathcal{O}^*_P} F^{H_{sh}}_{k+l}\, l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}} \to \ldots\right).$$

Notice that the filtration on $Rp_{2,*}\mathrm{DR}^{n+\bullet}_{P/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right)$ induces (up to a shift) the Hodge filtration on each cohomology module. In particular $Rp_{2,*}\mathrm{DR}^{n+\bullet}_{P/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{\dagger+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right)$ is strictly filtered, but this is not (a priori) the case for the filtration on ${}'p_{2,*}\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right)$, since ${}'p_{2,*}$ is not projective.

In order to conclude, it remains to show the isomorphism of filtered complexes

$$\left({}'p_{2,*}\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right), F_\bullet\right) \cong (M, F_\bullet)$$

The equality of complexes of $\mathcal{D}_{\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}$-modules

$${}'p_{2,*}\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right) \cong M$$

is obvious from diagram (15). Recall that the complex $\varphi^{\mathrm{red}}_{+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}$ is given by $(\varphi^{\mathrm{red}}_{*}\Omega^{\bullet+d}_{S_1\times S^{\mathrm{red}}/S^{\mathrm{red}}}[\partial_{\lambda_0}], d-\kappa(\beta)\wedge -(df\wedge)\otimes\partial_{\lambda_0})$ with filtration given by formula (14). Now we use the fact that l^{red}_1 is a closed embedding, hence, the filtration $F^{H_{sh}}l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}$ can be explicitly written down, and one checks that the induced filtration on the complex ${}'p_{2,*}\mathrm{DR}^{n+\bullet}_{P^*/\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}\left(l^{\mathrm{red}}_{1,+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}\right)$ corresponds to that of $\varphi^{\mathrm{red}}_{+}\mathcal{O}^{\beta}_{S_1\times S^{\mathrm{red}}}$, as required.

□

In general, these two filtrations on $M_{\dagger+}$ and M are not equal, simply because the underlying $\mathcal{D}_{\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}$-modules are not equal. They become equal after localized partial Fourier transformation as we have shown in Proposition 3.12. First we will explain that these transformations can be performed at the filtered level and how the last result can be interpreted in this context.

We will use a general procedure which produces from a filtered $\mathcal{D}_{\mathbb{A}^1_{\lambda_0}\times S^{\mathrm{red}}}$-module $(N, F_\bullet)$ a lattice $G^{F_\bullet}_0$ inside $\mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}}N$, i.e., an $\mathcal{O}_{\mathbb{A}^1_z\times S^{\mathrm{red}}}$-module which generates $\mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}}N$ over $\mathcal{D}_{\mathbb{A}^1_z\times S^{\mathrm{red}}}$.

Definition 3.17. (cf. [Sab08, §1.d], [SY15, §A.1]) Let X be a smooth affine variety and let $(N, F_\bullet)$ be a filtered $\mathcal{D}_{\mathbb{A}^1_s\times X}$-module, which we identify with its module of global sections. Consider the algebraic microlocalization

$$N[\partial_s^{-1}] := \mathbb{C}[s]\langle\partial_s, \partial_s^{-1}\rangle \otimes_{\mathbb{C}[s]\langle\partial_s\rangle} N.$$

By letting act τ as $-\partial_s$ and ∂_τ as s, we can envision $N[\partial_s^{-1}]$ as a $\mathcal{D}_X[\tau]\langle\partial_\tau,\partial_\tau^{-1}\rangle$-module (which actually coincides with $\mathrm{FL}_X^{\mathrm{loc}} N$). Let now $\widehat{\mathrm{loc}}$ be the natural localization morphism $\widehat{\mathrm{loc}} : N \to N[\partial_s^{-1}]$. Then we define

$$G_0^{F_\bullet}\,\mathrm{FL}_X^{\mathrm{loc}}\, N := \sum_{j\geq 0} \partial_s^{-j}\widehat{\mathrm{loc}}(F_j N), \tag{18}$$

notice that then $G_0^{F_\bullet}\,\mathrm{FL}_X^{\mathrm{loc}}\, N$ has naturally the structure of a $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_s\times X}$-module. We also put for any $k\in\mathbb{Z}$

$$G_k^{F_\bullet}\,\mathrm{FL}_X^{\mathrm{loc}}\, N := z^k\cdot G_0^{F_\bullet}\,\mathrm{FL}_X^{\mathrm{loc}}\, N = \sum_{j\geq 0}\partial_s^{-(j+k)}\widehat{\mathrm{loc}}(F_j N) = \sum_{j\geq 0}\partial_s^{-j}\widehat{\mathrm{loc}}(F_{j+k} N).$$

There is an interpretation of this construction as a Fourier-Laplace transformation for $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_s\times X}$-modules as explained in [SY15, Rem. A.3]. Using this interpretation, one can show the following fact.

Lemma 3.18. *Let $(N, F_\bullet)$ be a filtered $\mathcal{D}_{\mathbb{A}^1_s\times X}$-module underlying an element in $\mathrm{MHM}(\mathbb{A}^1_s\times X,\mathbb{C})$ (the abelian category of complex mixed Hodge modules). Then $G_0^{F_\bullet}\,\mathrm{FL}_X^{\mathrm{loc}}(N)$ underlies an element of $\mathrm{IrrMHM}(\mathbb{A}^1_\tau\times X)$.*

Proof. First we identify N with a $\mathcal{D}_{\mathbb{P}^1_s\times X}(*\{\infty\}\times X)$-module, so that $(N,F_\bullet)$ underlies a Hodge module on $\mathbb{P}^1_s\times X$. We consider the projection $p:\mathbb{P}^1_s\times X\times\mathbb{A}^1_\tau\to\mathbb{P}^1_s\times X$, which is clearly a smooth morphism, so $p^+(M,F_\bullet)$ still underlies an element in $\mathrm{MHM}(\mathbb{P}^1_s\times X\times\mathbb{A}^1_\tau,\mathbb{C})$. Its Rees module $\mathcal{R}_F M$ than underlies an element in $\mathrm{MTM}^{\mathrm{int}}(\mathbb{P}^1_s\times X\times\mathbb{A}^1_\tau)$. Now the construction of $G_0^{F_\bullet} FL_X^{\mathrm{loc}}(N)$ can be rephrased as the composition of the exponential twist functor $(\mathcal{T}^{s\cdot\tau}\otimes -) = (\mathcal{T}^{s/z}\otimes -)$ applied to $\mathcal{R}_F M$ followed by the direct image $\mathcal{H}^0 q_+$ by the second projection $q:\mathbb{P}^1_s\times X\times\mathbb{A}^1_\tau\to X\times\mathbb{A}^1_\tau$. From [Sab15, Thm. 0.2(2)], we know that the exponential twist sends $\mathrm{MHM}(\mathbb{P}^1_s\times X\times\mathbb{A}^1_\tau,\mathbb{C})$ to $\mathrm{IrrMHM}(\mathbb{P}^1_s\times X\times\mathbb{A}^1_\tau)$, and the projection $\mathcal{H}^0 q_+$ preserves IrrMHM according to [Sab15, Thm. 0.2(1)] since q is projective. □

With these definitions at hand, we have the following consequence of Lemma 3.16.

Corollary 3.19. *In the above situation, we have*

$$G_0^{F_\bullet^{H_{sh}}} M_{\dagger+}\subset G_0^{F_\bullet}\,\mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}} M$$

Proof. This is a direct consequence of the definition in formula (18), taking into account the last lemma and the fact that the filtered morphism ϕ^{red} induces an isomorphism of $\mathcal{D}_{\mathbb{A}^1_z\times S^{\mathrm{red}}}$-modules by applying the functor $\mathrm{FL}^{\mathrm{loc}}_{S^{\mathrm{red}}}$. □

From now on, we will specify the above situation to our main example, where the matrix A is given as

$$A=\begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0\\ 1 & 0 & -1 & 0 & \dots & 0\\ \vdots & \dots & & & & \\ 1 & 0 & 0 & 0 & \dots & -1\end{pmatrix}$$

In particular, we have $d=n-1$, and $S^{\mathrm{red}}=\mathbb{G}_{m,t}$. We still write $M_{\dagger+}=\mathcal{H}^0 p_{2,+} l^{\mathrm{red}}_{\dagger+}\mathcal{O}^\beta_{S_1\times\mathbb{G}_{m,t}}$ and $M=\mathcal{H}^0\varphi^{\mathrm{red}}_+\mathcal{O}^\beta_{S_1\times\mathbb{G}_{m,t}}$. We seek to improve the inclusion of the last corollary to an equality of lattices inside the $\mathcal{D}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$-modules $\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}(M_{\dagger+})\cong\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}(M)$. We will follow an argument from the proof of [Sab08, Lem. 4.7]. In order to do so, we have to make more explicit the structure of the module $\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}(M)$, which is done by the next lemma.

Lemma 3.20. *(1) The singular locus $\Sigma := Sing(M) = Sing(\mathcal{H}^0\varphi_+\mathcal{O}^\beta_{S_1\times\mathbb{G}_{m,t}})$ is given by*

$$\Sigma := \bigcup_{\xi\in\mu_n}\{(n\cdot\xi\cdot t', t)\}\subset\mathbb{A}^1_{\lambda_0}\times\mathbb{G}_{m,t}$$

where $(t')^n = t$ *(here we chose once and for all an* n*-th root of* t*, the choice does not matter as a direct sum over all values* $n\xi t'$ *is taken).*

(2) *Write* $D := \{0\} \times \mathbb{G}_{m,t} \subset \mathbb{A}^1_z \times \mathbb{G}_{m,t}$. *Then* $\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}(M)$ *is* $\mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}(*D)$*-locally free of rank* n.

(3) *Consider the sheaf* $\widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$ *which is the formal completion of* $\mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$ *along the divisor* $\{0\} \times \mathbb{G}_{m,t}$. *Then we have a decomposition (as sheaves on* $\mathbb{G}_{m,t}$*)*

$$\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}(M) \otimes_{\mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}(*D)} \widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}(*D) \cong \bigoplus_{\xi\in\mu_n} \widehat{\mathcal{E}}_\xi \otimes \widehat{\mathcal{N}}_{\alpha_\xi}$$

where $\widehat{\mathcal{E}}_\xi := (\widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}(*D), d - d(n\cdot\xi\cdot t'/z))$ *and* $\widehat{\mathcal{N}}_{\alpha_\xi} = (\widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}(*D), d + \alpha_\xi dz/z)$*, with* $\alpha_\xi \in \mathbb{C}$.

Proof. (1) It is well known that the singular locus of a Gauss-Manin system, i.e., of the top-cohomology of the direct image complex $\varphi_+\mathcal{M}$ is nothing but the discriminant of the morphism φ provided that the module $\mathcal{M}$ is smooth (which is the case here, since $\mathcal{M} = \mathcal{O}^\beta_{S_1\times\mathbb{G}_{m,t}}$). Recall (see formula (13)) that $\varphi^{\mathrm{red}}(y_1,\ldots,y_{n-1},t) = \left(\frac{1}{y_1}+\ldots+\frac{1}{y_{n-1}} + t\cdot y_1\cdot\ldots\cdot y_{n-1}, t\right)$. One easily checks (see, e.g., [DS04, §1.B.]) that a point $(y_1,\ldots,y_{n-1}) \in S_1$ is critical if and only if $y_1 = \ldots = y_{n-1} =: y$ and $y^n\cdot t = 1$ (and that all critical points are Morse). Then the critical values are as indicated.

(2) This is a direct consequence of the second point of [DS03, Thm. 1.11], since by the discussion above the singular locus Σ satisfies the assumption (NC) of loc. cit. It is also known that the rank of $\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}(M)$ equals the global Milnor number of φ, i.e., the numbers of critical points, which is n.

(3) This follows from [Sab02, Ch. III, Thm. 5.7], since the critical values of φ, i.e., the eigenvalues of the pole part of $z^2\nabla_z$, are distinct for any $t \in \mathbb{G}_{m,t}$.

□

With these preparations, we can state the next result.

Theorem 3.21. *In the above situation, we have*

$$G_0^{F_\bullet^{H_{sh}}} \mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}} M_{\dagger+} = G_0^{F_\bullet} \mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}} M.$$

Proof. We have already proved the inclusion

$$G_0^{F_\bullet^{H_{sh}}} \mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}} M_{\dagger+} \subset G_0^{F_\bullet} \mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}} M$$

of $\mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$-modules. Since both sheaves coincide outside the divisor $D = \{0\}\times\mathbb{G}_{m,t}$, and since $\widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$ is $\mathcal{O}_{(\mathbb{A}^1_z\times\mathbb{G}_{m,t},D)}$-flat, it is therefore sufficient to show that

$$G_0^{F_\bullet^{H_{sh}}} \mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}} M_{\dagger+} \otimes \widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}} = G_0^{F_\bullet} \mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}} M \otimes \widehat{\mathcal{O}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}.$$

This follows as in the proof of [Sab08, Lem. 4.7]: Using the formal decomposition result from the last Lemma, both modules can be interpreted as filtered direct images under p_2 of two modules which coincide on P^*. In these microlocal direct images, the contributions from $P\backslash P^*$ vanish by Lemma 3.9 (notice that inside $\mathbb{P}^n \times \mathbb{A}^1_{\lambda_0} \times S_2$, we have $P^* \cap \Gamma X \cong S_1 \times S^{\mathrm{red}}$ where we see S^{red} as a subspace of S_2 via the embedding ι), and therefore both modules are equal. □

Our final result in this section is the following.

Theorem 3.22. *Let* $\alpha_1,\ldots,\alpha_n$ *be real numbers. Let* P *and* H *be the operators*

$$P = z^2\partial_z + ntz\partial_t + \gamma z \quad\text{and}\quad H = \prod_{i=1}^n z(t\partial_t - \alpha_i) - t,$$

where $\gamma = n\alpha_1 - \sum_{i=2}^n \alpha_i$. *Consider the* $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z\times\mathbb{G}_m}$*-module*

$$\widehat{\mathcal{H}} := \mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_m}\langle z^2\partial_z, zt\partial_t\rangle/(P,H).$$

Then, $\widehat{\mathcal{H}}$ underlies an object of $\mathrm{IrrMHM}(\mathbb{G}_m)$*, whose associated $\mathcal{D}_{\mathbb{G}_{m,t}}$-module is $\mathcal{H}(\alpha_i;\emptyset)$. Moreover, $\widehat{\mathcal{H}}$ can be extended in a unique way to an irreducible $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1\times\mathbb{P}^1}$-module, $\widehat{\mathcal{H}}_{pr}$, such that it underlies an object of* $\mathrm{IrrMHM}\left(\mathbb{P}^1\right)$*.*

Proof. Let us assume first that $\alpha_1 = 0$, denote by α the vector $(\alpha_2,\ldots,\alpha_n)$ and put $\alpha_0 := 0$. By Proposition 2.7 we know that $\mathcal{H}(\alpha_1,\ldots,\alpha_n;\emptyset)$ is the inverse image $\iota^+\mathcal{M}^\alpha_A$, where we recall that ι is the morphism $\mathbb{G}_{m,t}\to\mathbb{A}^n$ given by $t\mapsto(t,1\ldots,1)$ and A is the $(n-1)\times n$-matrix of the beginning of this section. As a matter of fact, we have the same isomorphism with the twistor versions of both objects. Indeed, consider now $\iota : \mathbb{G}_{m,t}\to\mathbb{G}^n_m = S_2$. The inverse image functor in the category of twistor $\mathcal{D}$-modules is induced by the usual inverse image functor of $\mathcal{O}$-modules, $(\mathrm{id}_{\mathbb{A}^1_z}\times\iota)^*$ in this case (cf. [Moc15a, §2.1.6.2]). Then it is easy to see that

$$\widehat{\mathcal{H}}\cong\iota^+\widehat{\mathcal{M}}^{(\alpha_0,\alpha)}_A.$$

Namely, we replace each $z\lambda_i\partial_{\lambda_i}$ by $z\lambda_1\partial_{\lambda_1}-z\alpha_i$, so that we can present $\widehat{\mathcal{M}}^{(\alpha_0,\alpha)}_A$ as the $\mathcal{O}_{S_2}$-module $\mathcal{O}_{S_2}\langle z^2\partial_z, z\lambda_1\partial_{\lambda_1}\rangle/\mathcal{J}$, where $\mathcal{J}$ is generated by

$$\prod_{i=1}^n z(\lambda_1\partial_{\lambda_1}-\alpha_i)-\lambda_1\cdot\ldots\cdot\lambda_n \text{ and } z^2\partial_z+nz\lambda_1\partial_{\lambda_1}+\gamma z.$$

Now the inverse image amounts simply to set $\lambda_1 = t$ and $\lambda_i = 1$ for $i = 2,\ldots,n$ in the generators of the ideal, from which the desired isomorphism follows.

From Proposition 3.6 we conclude that

$$\widehat{\mathcal{H}}\cong\iota^+\mathcal{H}^0\left(\pi_{2,*}\Omega^{\bullet+d}_{S_1\times S_2/S_2}[z], z\left(d-\kappa(\alpha)\wedge\right)-df\wedge\right)\cong\mathcal{H}^0\left(\pi_{2,*}\Omega^{\bullet+d}_{S_1\times\mathbb{G}_{m,t}/\mathbb{G}_{m,t}}[z], z\left(d-\kappa(\alpha)\wedge\right)-df\wedge\right),$$

and it is easy to see from Definition 3.17 that we have

$$\mathcal{H}^0\left(\pi_{2,*}\Omega^{\bullet+d}_{S_1\times\mathbb{G}_{m,t}/\mathbb{G}_{m,t}}[z], z\left(d-\kappa(\alpha)\wedge\right)-df\wedge\right)\cong G_0^{F_\bullet}\,\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}\,\mathcal{H}^0\varphi^{\mathrm{red}}_+\mathcal{O}^\beta_{S_1\times\mathbb{G}_{m,t}} = G_0^{F_\bullet}\,\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}\,M.$$

Since by Theorem 3.21 we can further conclude

$$\widehat{\mathcal{H}}\cong G_0^{F_\bullet^{H_{sh}}}\,\mathrm{FL}^{\mathrm{loc}}_{\mathbb{G}_{m,t}}\,M_{\dagger+},$$

we obtain that $\widehat{\mathcal{H}}$ underlies an element of $\mathrm{IrrMHM}(\mathbb{G}_{m,t})$ by Lemma 3.18 (recall that $M_{\dagger+}$ underlies a pure polarizable complex Hodge module). Restricting $\widehat{\mathcal{H}}$ to $z = 1$ we get the original $\mathcal{D}_{\mathbb{G}_{m,t}}$-module $\mathcal{H}(\alpha_i;\emptyset)$.

Assume now that $\alpha_1\neq 0$. The tensor product of $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$-modules $\widehat{\mathcal{H}}\otimes_{\mathcal{O}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}}\widehat{\mathcal{K}}_{-\alpha_1}$ gives rise to the corresponding tensor product of twistor $\mathcal{D}_{\mathbb{G}_{m,t}}$-modules, which, by the discussion above when $\alpha_1 = 0$, is known to be an irregular mixed Hodge module of exponential-Hodge origin. By loc. cit. again, since $\widehat{\mathcal{K}}_{\alpha_1}$ is the faithful image of a mixed Hodge module on $\mathbb{G}_{m,t}$, the tensor product with it preserves the condition of being in $\mathrm{IrrMHM}(\mathbb{G}_{m,t})$, and so is the case of our original $\widehat{\mathcal{H}}$.

We will call $\widehat{\mathcal{H}}$ a classical hypergeometric $\mathcal{R}^{\mathrm{int}}_{\mathbb{A}^1_z\times\mathbb{G}_{m,t}}$-module, underlying a classical hypergeometric integrable twistor $\mathcal{D}_{\mathbb{G}_{m,t}}$-module.

Let now $j : \mathbb{G}_{m,t}\hookrightarrow\mathbb{P}^1$ be the canonical inclusion and consider the $\mathcal{D}_{\mathbb{P}^1}$-module $\mathcal{H}_{pr} := j_{\dagger+}\mathcal{H}$. It is an irreducible holonomic $\mathcal{D}_{\mathbb{P}^1}$-module, because so is $\mathcal{H}$ by Proposition 2.3. Then it gives rise to a unique pure integrable twistor $\mathcal{D}_{\mathbb{P}^1}$-module $\widehat{\mathcal{H}}_{pr}$ by [Moc11, Thm. 1.4.4] and [Sab15, Rem. 1.40]. In addition, its underlying $\mathcal{D}_{\mathbb{P}^1}$-module $\mathcal{H}_{pr}$ is rigid by virtue of Proposition 2.4. As a consequence, we can invoke [Sab15, Thm. 0.7] and claim that such twistor $\mathcal{D}_{\mathbb{P}^1}$-module is in fact an object of $\mathrm{IrrMHM}(\mathbb{P}^1)$. Take now $\widehat{\mathcal{H}}' := j^+\widehat{\mathcal{H}}_{pr}$, which is an irregular mixed Hodge module whose underlying $\mathcal{D}_{\mathbb{G}_{m,t}}$-module is $\mathcal{H}$, by [Moc15a, Prop. 14.1.24]. Since the functor Ξ_{DR} is faithful by [op. cit., Rem. 7.2.9], we have an injection of Hom groups

$$\mathrm{Hom}_{\mathrm{IrrMHM}(\mathbb{G}_{m,t})}(\widehat{\mathcal{H}},\widehat{\mathcal{H}}')\hookrightarrow\mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_{m,t}}}(\mathcal{H},\mathcal{H}),$$

but since $\mathcal{H}$ is irreducible its only endomorphism is the identity, so $j^+\widehat{\mathcal{H}}_{pr} = \widehat{\mathcal{H}}$ and we are done. □

4. The irregular Hodge filtration

We will use the following conventions and notations: From now on, X will mean the torus $\mathbb{G}_m = \operatorname{Spec}\mathbb{C}[t^{\pm}]$. We will write $\mathcal{X} := \mathbb{A}^1_z \times X$, ${}^\theta\mathcal{X} = \mathcal{X} \times \mathbb{G}_{m,\theta}$, ${}^\tau\mathcal{X} = \mathcal{X} \times \mathbb{A}^1_\tau$ and ${}^\tau X_0 = X \times \{\tau = 0\}$, where $\theta = 1/\tau$. We will fix again n to be a positive integer number, and $\alpha_1, \dots, \alpha_n$ to be n real numbers, now increasingly ordered and assumed to belong to the interval $[0, 1)$. With that notation, we recover from the end of the previous section the classical hypergeometric $\mathcal{D}_X$-module $\mathcal{H} = \mathcal{H}(\alpha_i; \emptyset)$ and its associated twistor $\mathcal{D}_X$-module $\widehat{\mathcal{H}}$. We will denote the underlying $\mathcal{R}^{\mathrm{int}}_{\mathcal{X}}$-module by the same symbol. From what we shown in Theorem 3.22 and [Sab15, Thm. 0.7] we know that there exist a unique irregular Hodge filtration of $\mathcal{H}$. In this section we are going to calculate it.

Proposition 4.1. *Recall that we could write $\widehat{\mathcal{H}}$ as the $\mathcal{R}^{\mathrm{int}}_{\mathcal{X}}$-module $\mathcal{R}^{\mathrm{int}}_{\mathcal{X}}/(P, H)$, where P and H were, respectively,*

$$z^2\partial_z + nzt\partial_t + \gamma z \ \text{ and } \ \prod_{i=1}^n z(t\partial_t - \alpha_i) - t,$$

for certain value of γ. Then, ${}^\theta\widehat{\mathcal{H}} = \mathcal{R}^{\mathrm{int}}_{{}^\theta\mathcal{X}}/(P, R, {}^\theta H)$, with P as before and

$${}^\theta R = z^2\partial_z - z\theta\partial_\theta \ \text{ and } \ {}^\theta H = \prod_{i=1}^n z\theta(t\partial_t - \alpha_i) - t.$$

Proof. Note that ${}^\theta\widehat{\mathcal{H}}$ is just the inverse image $\mathcal{O}_{{}^\theta\mathcal{X}}$-module $\mu^*\widehat{\mathcal{H}}$, endowed with a natural action of $\mathcal{R}^{\mathrm{int}}_{{}^\theta\mathcal{X}}$ as depicted in [Sab15, 2.4], where μ is the morphism

$$\mu : \begin{array}{ccc} {}^\theta\mathcal{X} & \to & \mathcal{X} \\ (z, t, \theta) & \mapsto & (z\theta, t) \end{array}.$$

Evidently, μ can be decomposed as $p \circ \phi$, where p is just the canonical projection from ${}^\theta\mathcal{X}$ to $\mathcal{X}$ and ϕ is the automorphism of ${}^\theta\mathcal{X}$ given by $(z, t, \theta) \mapsto (z\theta, t, \theta)$. Then, in the category of $\mathcal{O}_{{}^\theta\mathcal{X}}$-modules we will have that

$$\mu^*\widehat{\mathcal{H}} \cong \phi^*p^*\widehat{\mathcal{H}} \cong \phi^*\mathcal{R}^{\mathrm{int}}_{\mathcal{X}}\langle\theta\partial_\theta\rangle/(P, H(z, t, \partial_t)) \cong \mathcal{R}^{\mathrm{int}}_{\mathcal{X}}\langle z\theta\partial_\theta\rangle/(P, z^2\partial_z - z\theta\partial_\theta, H(z\theta, t, \partial_t)),$$

where the last isomorphism follows easily by looking at the stalks.

What remains now is to prove the compatibilities of loc. cit. among the actions of $\mathcal{R}^{\mathrm{int}}_{{}^\theta\mathcal{X}}$ on $\mathcal{R}^{\mathrm{int}}_{{}^\theta\mathcal{X}}/(P, {}^\theta R, {}^\theta H)$, seen as $\mu^*\widehat{\mathcal{H}} = \mu^{-1}\widehat{\mathcal{H}} \otimes_{\mu^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{{}^\theta\mathcal{X}}$. They are just a consequence of the presence of $z^2\partial_z - z\theta\partial_\theta$ in the ideal with which we take the quotient in $\mu^*\widehat{\mathcal{H}}$ and how z, ∂_i or ∂_z act at both sides of the tensor product. For instance, if we multiply by z at the right is the same as if we multiply by $z\theta$ at the left. □

Remark 4.2. Let $i_{\tau=z}$ be the inclusion $\mathcal{X}^0 := \mathbb{G}_{m,z} \times X \hookrightarrow {}^\theta\mathcal{X}$ given by $(z, t) \mapsto (z, t, \tau)$. Note that, according to the fourth point of [Sab15, Lem. 2.5], we must have $i^*_{\tau=z}{}^\theta\widehat{\mathcal{H}} \cong \pi^{0,+}\mathcal{H}$ as $\mathcal{R}_{\mathcal{X}^0}$-modules, with π^0 being the projection $\mathcal{X}^0 \to X$. Indeed, $i^*_{\tau=z}{}^\theta\widehat{\mathcal{H}}$ is just

$$\mathcal{R}^{\mathrm{int}}_{{}^\theta\mathcal{X}}/(P, {}^\theta R, {}^\theta H, \theta z - 1) = \mathcal{O}_{\mathbb{G}^2_m}\langle zt\partial_t\rangle/(H_1) \cong \mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{G}_{m,z}} \cong \pi^{0,+}\mathcal{H},$$

where H_1 is the value of H at $z = 1$.

In order to continue, we must pass from ${}^\theta\mathcal{X}$ to ${}^\tau\mathcal{X}$ to work in the context of [Sab15, §2.3]. Therefore, we can invert θ and extend τ to the affine line to get a $\mathcal{R}^{\mathrm{int}}_{{}^\tau\mathcal{X}}(*{}^\tau X_0)$-module. In other words, call $\mathrm{inv} : \mathbb{G}_{m,\theta} \to \mathbb{G}_{m,\tau}$ the inversion operator $\theta \mapsto \tau$ and $j : \mathbb{G}_{m,\tau} \hookrightarrow \mathbb{A}^1_\tau$ the canonical inclusion. From now on, we will denote by ${}^\tau\widehat{\mathcal{H}}$ the $\mathcal{R}^{\mathrm{int}}_{{}^\tau\mathcal{X}}(*{}^\tau X_0)$-module $(\mathrm{id}_X \times j \circ \mathrm{inv})_*{}^\theta\widehat{\mathcal{H}}$. By virtue of Proposition 4.1 we can write ${}^\tau\widehat{\mathcal{H}}$ as the $\mathcal{R}^{\mathrm{int}}_{{}^\tau\mathcal{X}}(*{}^\tau X_0)$-module ${}^\tau\widehat{\mathcal{H}} = \mathcal{R}^{\mathrm{int}}_{{}^\tau\mathcal{X}}(*{}^\tau X_0)/(P, {}^\tau R, {}^\tau H)$, with P as before and

$${}^\tau R = z^2\partial_z + z\tau\partial_\tau \ \text{ and } \ {}^\tau H = \prod_{i=1}^n \frac{z}{\tau}(t\partial_t - \alpha_i) - t.$$

Lemma 4.3. *For each $k = 0, \dots, n-1$, let Q_k be the operator*

$$Q_k = (-n)^k \prod_{j=1}^{k} \frac{z}{\tau}(t\partial_t - \alpha_j),$$

where the empty product must be understood as one. Then, the Q_k form a basis of ${}^\tau\widehat{\mathcal{H}}$ as a $\mathcal{O}_{{}^\tau\mathcal{X}}$-module. The integrable connection arising from the $\mathcal{R}^{\mathrm{int}}_{{}^\tau\mathcal{X}}(*{}^\tau X_0)$-module structure associated with ${}^\tau\widehat{\mathcal{H}}$ has the following matrix expression with respect to that basis:

$$\nabla \underline{Q} = \underline{Q}\left((\tau A_0 + zA_\infty)\frac{dz}{z^2} + \left(-\tau A_0 + zA'_\infty\right)\frac{dt}{nzt} - (\tau A_0 + zA_\infty)\frac{d\tau}{z\tau} \right),$$

where, respectively, A_0, A'_∞ and A_∞ are the matrices

$$\begin{pmatrix} 0 & & & (-n)^n t \\ 1 & \ddots & & 0 \\ & \ddots & 0 & \vdots \\ & & 1 & 0 \end{pmatrix}, \ \mathrm{diag}(n\alpha_1, \dots, n\alpha_n) \ \textit{and} \ \mathrm{diag}(0, 1, \dots, n-1) - \gamma I_n - A'_\infty.$$

Proof. It is clear that ${}^\tau\widehat{\mathcal{H}}$ is generated by the Q_k. Replacing $z^2\partial_z$ and $z\tau\partial_\tau$ in terms of $zt\partial_t$ and dividing by ${}^\tau H$ we can express every element of ${}^\tau\widehat{\mathcal{H}}$ as a sum $a = \sum_{i=0}^{n-1} a(z,t,\tau)(zt\partial_t)^i$. Since $\deg_{\partial_t} Q_k = k$, they must be linearly independent over $\mathcal{O}_{{}^\tau\mathcal{X}}$ and so a basis.

Let now $k < n-1$. Then from the relation $-nz/\tau(t\partial_t - \alpha_{k+1})Q_k = Q_{k+1}$ we can write that $nzt\partial_t Q_k = -\tau Q_{k+1} + nz\alpha_{k+1}Q_k$. Now if $k = n-1$, then

$$-nz/\tau(t\partial_t - \alpha_n)Q_{n-1} = (-n)^n \prod_{j=1}^{n} z/\tau(t\partial_t - \alpha_j) = (-n)^n t.$$

This gives us the second summand of the formula above in the statement.

The first one is a consequence of the fact that $z^2\partial_z + z\tau\partial_\tau = 0$ and the last one; let us show the expression for the latter.

Take again $k < n-1$. Then,

$$z\tau\partial_\tau Q_k = (-n)^k z\tau\partial_\tau \tau^{-k} \prod_{j=1}^{k} z(t\partial_t - \alpha_j) = (-n)^k z(-k\tau^{-k} + \tau^{-k+1}\partial_\tau) \prod_{j=1}^{k} z(t\partial_t - \alpha_j) =$$

$$= -kzQ_k + Q_k z\tau\partial_\tau = -kzQ_k + Q_k z(nt\partial_t + \gamma) = z(nt\partial_t + \gamma - k)Q_k = -\tau Q_{k+1} + z(n\alpha_{k+1} + \gamma - k)Q_k.$$

The analogous calculation for $k = n-1$ gives us that $z\tau\partial_\tau Q_{n-1} = -\tau(-n)^n t + z(n\alpha_n + \gamma - (n-1))Q_{n-1}$. □

Definition 4.4. For each $\alpha \in \mathbb{R}$, let us define the following subsets of ${}^\tau\widehat{\mathcal{H}}$:

$${}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}} := \left\{ \sum_{k=0}^{n-1} f_k \tau^{\nu_{jk}} Q_k \ : \ f_k \in \mathcal{O}_{\mathcal{X}}[\tau] \ , \ \max(k - n\alpha_{k+1} - \gamma - \nu_k) \le \alpha \right\},$$

$${}^\tau\mathcal{U}_{<\alpha} {}^\tau\widehat{\mathcal{H}} := \left\{ \sum_{k=0}^{n-1} f_k \tau^{\nu_k} Q_k \ : \ f_k \in \mathcal{O}_{\mathcal{X}}[\tau] \ , \ \max(k - n\alpha_{k+1} - \gamma - \nu_k) < \alpha \right\}.$$

Remark 4.5. Note that the ${}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}}$ form an increasing filtration of ${}^\tau\widehat{\mathcal{H}}$, indexed by the real numbers but with a discrete set of jumping numbers, such that $\tau {}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}} = {}^\tau\mathcal{U}_{\alpha-1} {}^\tau\widehat{\mathcal{H}}$ and $z\partial_\tau {}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}} \subseteq {}^\tau\mathcal{U}_{\alpha+1} {}^\tau\widehat{\mathcal{H}}$ for any α. The graded piece associated with α is $\mathrm{Gr}_\alpha^{{}^\tau\mathcal{U}} {}^\tau\widehat{\mathcal{H}} = {}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}} / {}^\tau\mathcal{U}_{<\alpha} {}^\tau\widehat{\mathcal{H}}$. We can define the ${}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}}$ in an alternative way, as the free $\mathcal{O}_{\mathcal{X}}[\tau]$-modules of finite rank

$${}^\tau\mathcal{U}_\alpha {}^\tau\widehat{\mathcal{H}} = \bigoplus_{k=0}^{n-1} \mathcal{O}_{\mathcal{X}}[\tau] \cdot \tau^{\nu_\alpha(k)} Q_k,$$

where $\nu_\alpha(k) = \lceil -\alpha + k - \gamma - n\alpha_{k+1} \rceil$.

With this expression it is easy to see that we also have ${}^{\tau}\!U_{\alpha}{}^{\tau}\widehat{\mathcal{H}}/(\tau-z){}^{\tau}\!U_{\alpha}{}^{\tau}\widehat{\mathcal{H}}$ is the z-graded free $\mathcal{O}_{\mathcal{X}}$-module $\bigoplus_k \mathcal{O}_{\mathcal{X}} z^{\nu_\alpha(k)}\bar{Q}_k$, where

$$\bar{Q}_k = (-n)^k \prod_{j=1}^{k}(t\partial_t - \alpha_j).$$

Proposition 4.6. *${}^{\tau}\widehat{\mathcal{H}}$ is strictly $\mathbb{R}$-specializable along ${}^{\tau}X_0$.*

Proof. First of all we will see that ${}^{\tau}\!U_{\alpha}{}^{\tau}\widehat{\mathcal{H}}$ is the ${}^{\tau}V$-filtration of ${}^{\tau}\widehat{\mathcal{H}}$, following [Moc15a, §§2.1.2.1, 2.1.2.2]. Apart from what we already shown at the remark above, from the definition of the filtration we can easily see that the graded pieces are $\mathcal{O}_{\mathbb{A}^1_z}$-flat and thus strict. What remains then is showing point (v) of [op. cit., §2.1.2.1]. Let us consider then the mappings $\mathfrak{p}_z, \mathfrak{e}_z$ given by

$$\begin{array}{rcl}(\mathfrak{p}_z,\mathfrak{e}_z):\mathbb{R}\times\mathbb{C} & \longrightarrow & \mathbb{R}\times\mathbb{C} \\ (\beta,\omega) & \longmapsto & (\beta+2\Re(z\bar{\omega}), -\beta z+\omega-\bar{\omega}z^2)\end{array}.$$

We ought to see now that the operator $z\tau\partial_\tau - \mathfrak{e}_z(\beta,\omega)$ is nilpotent on the graded pieces $\mathrm{Gr}^{{}^{\tau}U}_{\alpha}\,{}^{\tau}\widehat{\mathcal{H}}$ only for a finite amount of $(\beta,\omega)\in\mathcal{K}:=\{\mathfrak{p}_1(\beta,\omega)=\alpha\}$. Moreover, ${}^{\tau}\widehat{\mathcal{H}}$ will be strictly $\mathbb{R}$-specializable if those (β,ω) are in fact contained in $\mathbb{R}\times\{0\}$ (cf. [Sab15, §1.3.a]).

Take then $(\beta,\omega)\in\mathcal{K}$ and $f\tau^{\nu}Q_k\in{}^{\tau}\!U_{\alpha}{}^{\tau}\widehat{\mathcal{H}}$, with $f\in\mathcal{O}_{\mathcal{X}}[\tau]$. Then we must have that $k-n\alpha_{k+1}-\gamma-\nu\le\alpha$. Assume that $k<n-1$. Thanks to Lemma 4.3 we know that

$$(z\tau\partial_\tau-\mathfrak{e}_1(\beta,\omega))f\tau^{\nu}Q_k = \big(z\tau\partial_\tau+(\nu+n\alpha_{k+1}+\gamma-k+\beta)z-\omega+\bar{\omega}z^2\big)(f)\tau^{\nu}Q_k-\tau^{\nu+1}Q_{k+1}.$$

Recall that the α_i are increasingly ordered. Thus $f\tau^{\nu+1}Q_{k+1}$ belongs to ${}^{\tau}\!U_{\alpha}{}^{\tau}\widehat{\mathcal{H}}$, for

$$k+1-n\alpha_{k+2}-\gamma-\nu-1\le((k+1)-n\alpha_{k+2}-\gamma)-(k-n\alpha_{k+1}-\gamma))-1+\alpha\le\alpha.$$

Now we should look at what happens to the class of $f\tau^{\nu+1}Q_{k+1}$ in the α-graded piece of ${}^{\tau}\widehat{\mathcal{H}}$.

Note that $[f\tau^{\nu}Q_k]\neq 0$ if and only if $\nu+n\alpha_{k+1}+\gamma-k+\alpha=0$, so

$$(z\tau\partial_\tau-\mathfrak{e}_1(\beta,\omega))f\tau^{\nu}Q_k = \big(z\tau\partial_\tau+(\beta-\alpha)z-\omega+\bar{\omega}z^2\big)(f)\tau^{\nu}Q_k-\tau^{\nu+1}Q_{k+1} =$$

$$= \big(z\tau\partial_\tau+2\Re\omega z-\omega+\bar{\omega}z^2\big)(f)\tau^{\nu}Q_k-\tau^{\nu+1}Q_{k+1}.$$

Now notice that τ divides $\tau\partial_\tau(f)$, so in fact $z\tau\partial_\tau(f)\tau^{\nu}Q_k\in{}^{\tau}\!U_{\alpha-1}{}^{\tau}\widehat{\mathcal{H}}$ and then we can further reduce our expression to

$$(z\tau\partial_\tau-\mathfrak{e}_1(\beta,\omega))f\tau^{\nu}Q_k = (-\omega+2\Re\omega z+\bar{\omega}z^2)f\tau^{\nu}Q_k-\tau^{\nu+1}Q_{k+1}.$$

On the other hand, $\tau^{\nu+1}Q_{k+1}$ does not vanish either in $\mathrm{Gr}^{{}^{\tau}U}_{\alpha}\,{}^{\tau}\widehat{\mathcal{H}}$ if and only if $\alpha_{k+2}=\alpha_{k+1}$. Indeed, we know that $\nu+n\alpha_{k+1}+\gamma-k+\alpha=0$, so doing the same as before, $k+1-n\alpha_{k+2}-\gamma-\nu-1=\alpha+n(\alpha_{k+2}-\alpha_{k+1})$ and the claim follows. Furthermore, in order to $(z\tau\partial_\tau-\mathfrak{e}_1(\beta,\omega))$ to vanish, we should impose that $\omega=0$, just by looking at the coefficients of the powers of z in the expression for f.

Now if $k=n-1$, then everything would be the same as before except $-\tau^{\nu+1}Q_{k+1}$, which becomes $-(-n)^n t\tau^{\nu+1}$, whose class vanishes obviously in the graded piece under consideration.

In conclusion, $(z\tau\partial_\tau-\mathfrak{e}_1(\beta,\omega))^l f\tau^{\nu}Q_k$ can only vanish in $\mathrm{Gr}^{{}^{\tau}U}_{\alpha}\,{}^{\tau}\widehat{\mathcal{H}}$ if $\alpha=\beta$ (and then $\omega=0$), and does not do so until we get to an index $k+l$ such that α_{k+l} is strictly bigger than α_k. Since there is a finite set of indexes, $(z\tau\partial_\tau+\alpha z)$ is nilpotent, of nilpotency index n at most. Summing up, ${}^{\tau}\widehat{\mathcal{H}}$ is strictly $\mathbb{R}$-specializable along ${}^{\tau}X_0$. □

Theorem 4.7. *Let as before $\alpha_1,\dots,\alpha_n$ be real numbers in $[0,1)$ and put $\mathcal{H}=\mathcal{H}(\alpha_i,\emptyset)$. For each $j=1,\dots,n$, set $\rho(j)=-n\alpha_j+j$. Then the jumping numbers of the irregular Hodge filtration of $\mathcal{H}$ are, up to a real shift, the numbers $\rho(j)$. The irregular Hodge numbers are the multiplicities of those jumping numbers, or in other words, the nonzero values of $|\rho^{-1}(x)|$, for x real.*

Notice that this result has a similar shape to Proposition 2.5 which treats the case of regular hypergeometric systems. We hope that an extension of our methods will lead to a computation of irregular Hodge numbers for arbitrary irregular systems $\mathcal{H}(\alpha_i;\beta_j)$. Such a general formula, if it exists, could then certainly be compared to the result in the regular case.

Proof. Since we know that $\widehat{\mathcal{H}}$ underlies an object in $\mathrm{IrrMHM}(\mathbb{G}_{m,t})$ by Theorem 3.22, we conclude by [Sab15, Def. 2.52] that $\widehat{\mathcal{H}}$ is well-rescalable (as defined in [op. cit., Def. 2.19]) and so we apply [op. cit., Def. 2.22]. From Remark 4.5, we have

$$i^*_{\tau=z}{}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}} = {}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}}/(\tau-z){}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}} = \bigoplus_k \mathcal{O}_{\mathcal{X}} z^{\nu_\alpha(k)}\bar{Q}_k,$$

which is z-graded of finite rank, so the z-adic filtration on $\pi^*\mathcal{H}[z^{-1}]$ induces another filtration on $i^*_{\tau=z}{}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}}$, given by

$$F_r i^*_{\tau=z}{}^{\tau}V_\alpha{}^{\tau}\widehat{\mathcal{H}} := \bigoplus_{s\leq r}\left(\bigoplus_{k\,:\,s\geq\nu_\alpha(k)} \mathcal{O}_X\bar{Q}_k\right) z^s.$$

Then, $\mathrm{Gr}^F\left(i^*_{\tau=z}{}^{\tau}\mathcal{U}_\alpha{}^{\tau}\widehat{\mathcal{H}}\right)$ is the Rees module associated to a new good filtration $F^{\mathrm{irr}}_{\alpha+\bullet}$ on $\mathcal{H}$, for some $k=0,\dots,n-1$, which is the irregular Hodge filtration. More concretely, $F^{\mathrm{irr}}_{\bullet}\mathcal{H}$ is given by

$$F^{\mathrm{irr}}_{\alpha+j} = \bigoplus_{k\,:\,j\geq\nu_\alpha(k)} \mathcal{O}_X\bar{Q}_k.$$

Therefore, its jumping numbers are $-\gamma+j-1-n\alpha_j$ for $j=1,\dots,n$. Since the irregular Hodge filtration is defined up to an overall real shift, we can normalize the jumping numbers to $j-n\alpha_j$ and the irregular Hodge numbers will be their multiplicities. □

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