

Hypergeometric Hodge modules

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Abstract

We consider mixed Hodge module structures on GKZ-hypergeometric differential systems. We show that the Hodge filtration on these \mathcal{D} -modules is given by the order filtration, up to suitable shift. As an application, we prove a conjecture on the existence of non-commutative Hodge structures on the reduced quantum \mathcal{D} -module of a nef complete intersection inside a toric variety.

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1 Introduction

In a series of paper Gel'fand, Graev, Kapranov and Zelevinskiĭ [GfGZ87], [GfZK89] introduced a system of differential equations which generalize the classical differential systems satisfied by the hypergeometric functions of Gauß, Appell, Bessel and others. These generalized systems are nowadays called GKZ-systems. The initial data of a GKZ system is a $d \times n$ integer matrix and a parameter vector β . Although the definition of a GKZ system has a combinatorial flavor it was early realized that at least for non-resonant parameter vectors β GKZ-systems come from geometry [GKZ90], i.e. they are isomorphic to a direct image of some twisted structure sheaf on an algebraic variety. In [Rei14], the first named author has shown that certain GKZ-systems actually carry a much richer structure, namely, they underlie

mixed Hodge modules in the sense of M. Saito (see [Sai90]). One of the main goals of this paper is the explicit calculation of the corresponding Hodge filtration on these modules.

A very important application of GKZ systems is mirror symmetry for complete intersections in toric varieties. We have shown in our previous papers [RS15, RS17] how to express variants of the mirror correspondence as an equivalence of differential systems of “GKZ-type”. However, an important point was left open in these articles: The mirror statements given there actually involve differential systems (i.e., holonomic \mathcal{D} -modules) with some additional data, sometimes called *lattices*. These are constructed by a variant of the Fourier-Laplace transformation from regular holonomic *filtered* \mathcal{D} -modules. The filtration in question is the Hodge filtration on these modules, but a concrete description of it is missing in [RS15, RS17]. As a consequence, the most important Hodge theoretic property of the differential system entering in the mirror correspondence was formulated only as a conjecture in [RS17] (conjecture 6.13): the so-called *reduced quantum \mathcal{D} -module*, which governs certain Gromov-Witten invariants of nef complete intersections in toric varieties conjecturally underlies a *variation of non-commutative Hodge structures*. We prove this conjecture here (see Theorem 5.6), it appears as a consequence of the main result of the present paper, which determines the Hodge filtration on the GKZ-systems. More precisely, as GKZ-systems are defined as cyclic quotients of the Weyl algebra, we obtain (Theorem 4.35) that this Hodge filtration is given by the filtration induced from the order of differential operators up to a suitable shift.

Let us give a short overview on the content of this article. The main result is obtained in two major steps, which occupy the sections two and three. First we study embeddings of tori into affine spaces given by a monomial map $h_A : T = (\mathbb{C}^*)^d \hookrightarrow \mathbb{C}^n; (t_1, \dots, t_d) \mapsto (\underline{t}^{\underline{a}_1}, \dots, \underline{t}^{\underline{a}_n})$, where $\underline{t}^{\underline{a}_i} = \prod_{k=1}^d t_j^{a_{ki}}$ and where the matrix of columns $A = (a_{ij})_{i=1, \dots, n}$ satisfies certain combinatorial properties related to the geometry of the semi-group ring $\mathbb{C}[\text{NA}]$. We consider the direct image $\mathcal{H}^0(h_{A*}^p \mathbb{C}_T^{H, \beta})$ in the category of complex mixed Hodge modules, and calculate its Hodge filtration (Theorem 3.16). If the matrix A we started with satisfy an homogeneity property, then the underlying \mathcal{D} -module of this mixed Hodge module is a (monodromic) Fourier-Laplace transformation of the GKZ-system we are interested in. It should be noticed that Theorem 3.16 is of independent interest, its statement is related to the description of the Hodge filtration on various cohomology groups associated to singular toric varieties. We plan to discuss this question in a subsequent work. The main point in Theorem 3.16 is to determine the canonical V -filtration on the direct image module along the boundary divisor $\overline{\text{im}}(h_A) \setminus \text{im}(h_A)$, i.e., the calculation of some Bernstein polynomials.

The second step, carried out in section three consists in studying the behavior of a twisted structure sheaf on a torus under a certain integral transformation which generalizes the Radon transformation in [Rei14]. It is well-known (see [Bry86] and [DE03]) that there is a close relation between the Fourier-Laplace transformation and the Radon transformation for holonomic \mathcal{D} -modules, however, the former one does not a priori preserve the category of mixed Hodge modules whereas the latter does. This fact is one of the main points in the prove of the existence of a mixed Hodge module structure on GKZ-systems in [Rei14]. We calculate the behaviour of the Hodge filtration under the various functors entering into the integral transformation functor, an essential tool for these calculations is the so called *Euler-Koszul-complex* (or some variants of it) as introduced in [MMW05]. The second last part of section three deals with the Hodge module structure on the holonomic dual GKZ-system (which is, under the assumptions on the initial data, also a GKZ-system). In section four we explain the above mentioned conjecture from [RS17] and show how its proof can be deduced from our main result.

While we were working on this paper, a preprint of T. Mochizuki ([Moc15]) appeared where [RS17, Conjecture 6.13] is shown with rather different methods. However, his approach does not seem to give control on the Hodge filtration of the GKZ-systems for non-zero parameter vectors β .

To finish this introduction, we will introduce some notation and conventions used throughout the paper. Let X be a smooth algebraic variety over \mathbb{C} of dimension d_X . We denote by $M(\mathcal{D}_X)$ the abelian category of algebraic left \mathcal{D}_X -modules on X and the abelian subcategory of (regular) holonomic \mathcal{D}_X -modules by $M_h(\mathcal{D}_X)$ (resp. $(M_{rh}(\mathcal{D}_X))$). The full triangulated subcategory in $D^b(\mathcal{D}_X)$, consisting of objects with (regular) holonomic cohomology, is denoted by $D_h^b(\mathcal{D}_X)$ (resp. $D_{rh}^b(\mathcal{D}_X)$).

Let $f : X \rightarrow Y$ be a map between smooth algebraic varieties. Let $M \in D^b(\mathcal{D}_X)$ and $N \in D^b(\mathcal{D}_Y)$, then we denote by

$$f_+M := Rf_*(\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes} M) \quad \text{resp.} \quad f^+M := \mathcal{D}_{X \rightarrow Y} \overset{L}{\otimes} f^{-1}M[d_X - d_Y]$$

the direct resp. inverse image for \mathcal{D} -modules. Recall that the functors f_+, f^+ preserve (regular) holonomicity (see e.g., [HTT08, Theorem 3.2.3]). We denote by $\mathbb{D} : D_h^b(\mathcal{D}_X) \rightarrow (D_h^b(\mathcal{D}_X))^{opp}$ the holonomic duality functor. Recall that for a single holonomic \mathcal{D}_X -module M , the holonomic dual is also a single holonomic \mathcal{D}_X -module ([HTT08, Proposition 3.2.1]) and that holonomic duality preserves regular holonomicity ([HTT08, Theorem 6.1.10]).

For a morphism $f : X \rightarrow Y$ between smooth algebraic varieties we additionally define the functors $f_{\dagger} := \mathbb{D} \circ f_+ \circ \mathbb{D}$ and $f^{\dagger} := \mathbb{D} \circ f^+ \circ \mathbb{D}$.

Let $MF(\mathcal{D}_X)$ be the category of filtered \mathcal{D}_X -modules (M, F) where the ascending filtration F_{\bullet} satisfies

1. $F_p M = 0$ for $p \ll 0$
2. $\bigcup_p F_p M = M$
3. $(F_p \mathcal{D}_X) F_q M \subset F_{p+q} M$ for $p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}$

where $F_{\bullet} \mathcal{D}_X$ is the filtration by the order of the differential operator.

We denote by $MHM(X)$ the abelian category of algebraic mixed Hodge modules and by $D^b MHM(X)$ the corresponding bounded derived category. The forgetful functor to the bounded derived category of regular holonomic \mathcal{D} -modules is denoted by

$$Dmod : D^b MHM(X) \longrightarrow D_{rh}^b(\mathcal{D}_X).$$

For each morphism $f : X \rightarrow Y$ between complex algebraic varieties, there are induced functors

$$f_*, f_{\dagger} : D^b MHM(X) \longrightarrow D^b MHM(Y)$$

and

$$f^*, f^{\dagger} : D^b MHM(Y) \rightarrow D^b MHM(X),$$

which are interchanged by \mathbb{D} . The functors $f_*, f_{\dagger}, f^*, f^{\dagger}$ lift the analogous functors $f_+, f_{\dagger}, f^+, f^{\dagger}$ on $D_{rh}^b(\mathcal{D}_X)$. Let \mathbb{Q}_{pt}^H be the unique mixed Hodge structure with $Gr_i^W = Gr_i^F = 0$ for $i \neq 0$ and underlying vector space \mathbb{Q} . Denote by $a_X : X \rightarrow \{pt\}$ the map to the point and set

$$\mathbb{Q}_X^H := a_X^* \mathbb{Q}_{pt}^H.$$

The shifted object ${}^p \mathbb{Q}_X^H := \mathbb{Q}_X^H[d_X]$ lies in $MHM(X)$ and is equal to $(\mathcal{O}_X, F, \mathbb{Q}_X[d_X], W)$ with $Gr_p^F = 0$ for $p \neq 0$ and $Gr_i^W = 0$ for $i \neq d_X$. We have $\mathbb{D} \mathbb{Q}_X^H \simeq a_X^! \mathbb{Q}_{pt}^H$ and, since X is smooth, the isomorphism

$$\mathbb{D} \mathbb{Q}_X^H \simeq \mathbb{Q}_X^H(d_X)[2d_X]. \tag{1}$$

Here (d_X) denotes the Tate twist (see e.g., [Sai90, page 257]).

Similarly, if X is a complex variety we consider the category $MHM(X, \mathbb{C})$ of complex mixed Hodge modules. Let $T = (\mathbb{C}^*)^d$ be a torus with coordinates t_1, \dots, t_d and $\beta \in \mathbb{R}^d$. We denote by \mathcal{O}_T^{β} the \mathcal{D}_T -module

$$\mathcal{O}_T^{\beta} := \mathcal{D}_T / ((\partial_{t_i} t_i + \beta_i)_{i=1, \dots, d})$$

and by $\mathbb{C}_T^{H, \beta}$ the complex Hodge module $(\mathcal{O}_T^{\beta}, F, W)$ with $Gr_p^F \mathbb{C}_T^{H, \beta} = 0$ for $p \neq 0$ and $Gr_i^W \mathbb{C}_T^{H, \beta} = 0$ for $i \neq d$. Finally we set ${}^p \mathbb{C}_T^{H, \beta} := \mathbb{C}_T^{H, \beta}[d_T]$.

2 GKZ-systems and Fourier-Laplace transform

2.1 GKZ-systems and strict resolutions

Given a $d \times n$ integer matrix $A = (a_{ki})$ we denote by $\underline{a}_1, \dots, \underline{a}_n$ its columns. We define

$$\mathbb{N}A := \sum_{i=1}^n \mathbb{N}\underline{a}_i$$

and similarly for $\mathbb{Z}A$ and $\mathbb{R}_{\geq 0}A$. Throughout the paper we assume that the matrix A satisfies

$$\mathbb{Z}A = \mathbb{Z}^d.$$

Definition 2.1. Let $A = (a_{ki})$ be a $d \times n$ integer matrix and $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{C}^d$. Write \mathbb{L}_A for the \mathbb{Z} -module of integer relations among the columns of A and write $\mathcal{D}_{\mathbb{C}^n}$ for the sheaf of rings of differential operators on \mathbb{C}^n (with coordinates $\lambda_1, \dots, \lambda_n$). Define

$$\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{C}^n} / \mathcal{I}_A,$$

where \mathcal{I}_A is the sheaf of left ideals generated by

$$\square_{\underline{l}} := \prod_{i:l_i < 0} \partial_{\lambda_i}^{-l_i} - \prod_{i:l_i > 0} \partial_{\lambda_i}^{l_i}$$

for all $\underline{l} \in \mathbb{L}_A$ and

$$E_k - \beta_k := \sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i} - \beta_k$$

for $i = 1, \dots, d$.

Since GKZ-systems are defined on the affine space \mathbb{C}^n , we will often work with the D -modules of global sections $M_A^\beta := \Gamma(\mathbb{C}^n, \mathcal{M}_A^\beta)$ rather than with the sheaves themselves.

We will now discuss filtrations on GKZ-systems given by a weight vector $(u, v) \in \mathbb{Z}^{2n}$. This weight vector induces an increasing filtration on D_V given by

$$F_p^{(u,v)} D_V = \left\{ \sum_{\substack{\sum_i u_i \gamma_i + v_i \delta_i \leq p \\ \text{finite}}} c_{\gamma\delta} \lambda^\gamma \partial_\lambda^\delta \mid \gamma, \delta \in \mathbb{Z}_{\geq 0}^n \right\}$$

where we set $\lambda^\gamma := \prod_{i=1}^n \lambda_i^{\gamma_i}$ etc. . For an element $P = \sum c_{\gamma\delta} \lambda^\gamma \partial_\lambda^\delta$ we define $\text{ord}_{(u,v)}(P) := \max\{\sum_i u_i \gamma_i + v_i \delta_i \mid c_{\gamma\delta} \neq 0\}$. The associated graded ring $\text{Gr}_\bullet^{(u,v)} D_V$ is given by $\bigoplus_p F_p^{(u,v)} D_V / F_{p-1}^{(u,v)} D_V$.

In order to construct a strictly filtered resolution of \mathcal{M}_A^β , we use the theory of Euler-Koszul complexes as introduced in [MMW05]. We will work on the level of global sections. We briefly recall the definition of the Euler-Koszul complex $(K_\bullet, E - \beta)$ from [MMW05, Definition 4.2] (we where it is called $\mathcal{K}_\bullet(E - \beta; \mathbb{C}[\mathbb{N}A])$ and placed in positive homological degrees). Its terms are given by

$$K^{-l} = \bigoplus_{0 \leq i_1 < \dots < i_l \leq n} (D_V / J_A) e_{i_1 \dots i_l},$$

where the left ideal $J_A \subset \mathbb{C}[\partial] := \mathbb{C}[\partial_{\lambda_1}, \dots, \partial_{\lambda_n}]$ is generated by

$$\square_{\underline{l}} := \prod_{i:l_i < 0} \partial_{\lambda_i}^{-l_i} - \prod_{i:l_i > 0} \partial_{\lambda_i}^{l_i}$$

A simple computation using the fact that $\sum_{i=1}^n l_i a_{ki} = 0$ shows that the maps

$$\begin{aligned} D_V / D_V J_A &\longrightarrow D_V / D_V J_A \\ P &\mapsto P \cdot E_k - \beta_k \quad \text{for } k = 1, \dots, d \end{aligned} \tag{2}$$

are well defined. Moreover, we have $[E_{k_1} - \beta_{k_1}, E_{k_2} - \beta_{k_2}] = 0$ for $k_1, k_2 \in \{1, \dots, d\}$, and hence we can build the Koszul complex

$$(K^\bullet, E - \beta) = (\dots \xrightarrow{d_{-2}} K^{-1} \xrightarrow{d_{-1}} K^0 \rightarrow 0) := \text{Kos}(D_V/D_V J_A, (E_k - \beta_k)_{k=0, \dots, d}).$$

with D_V -linear differential

$$d_{-l}(e_{i_1 \dots i_l}) := \sum_{k=1}^l (-1)^{l-1} E_{i_k} e_{i_1 \dots \hat{i}_k \dots i_l}$$

If we assume that the semigroup $\mathbb{N}A$ satisfies

$$\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0} A$$

then by a classical result due to Hochster ([Hoc72, theorem 1]) it follows that the semi group ring $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay. It was shown in [MMW05, Remark 6.4] that in this case $(K^\bullet, E - \beta)$ is a resolution of M_A^β for all $\beta \in \mathbb{C}^d$.

Notice that the filtration $F_\bullet^{(u,v)}$ on D_V induces a filtration on $D_V/D_V J_A$ which we denote by the same symbol. We define the following filtration on each term of the Koszul complex $(K^\bullet, E - \beta)$:

$$F_p^{(u,v)} K^{-l} := \bigoplus_{0 \leq i_1 < \dots < i_l \leq l} F_{p - \sum_{k=1}^l c_{i_k}}^{(u,v)} (D_V/D_V J_A) e_{i_1 \dots i_l},$$

where $c_i = \text{ord}_{(u,v)}(E_i - \beta_i)$. This shows that the complex $((K^\bullet, E - \beta), F_\bullet^{(u,v)})$ is filtered, i.e. that the differential d respects the filtration

$$d_{-l}(F_p^{(u,v)} K^{-l}) \subset F_p^{(u,v)} d_{-l}(K^{-l}) := \text{im}(d_{-l}) \cap F_p^{(u,v)} K^{-l+1}.$$

We will now recall a (well-known) criterion for when the complex is strictly filtered, which means

$$d_{-l}(F_p^{(u,v)} K^{-l}) = F_p^{(u,v)} d_{-l}(K^{-l}) = \text{im}(d_{-l}) \cap F_p^{(u,v)} K^{-l+1}$$

Lemma 2.2. *Let*

$$0 \longrightarrow (M_1, F) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} (M_n, F) \longrightarrow 0$$

be a sequence of filtered D -modules with bounded below filtration. The following is equivalent

1. *The map d_k is strict.*
2. *$H^k(F_p M_\bullet) \simeq F_p H^k(M_\bullet)$ for all p .*
3. *$H^k(\text{gr}_p^F M_\bullet) \simeq \text{gr}_p^F H^k(M_\bullet)$ for all p .*

Proof. First recall that the map d_k is strict iff $F_p \text{im } d_k = F_p M_k \cap \text{im } d_k = F_p \ker d_k \cap \text{im } d_k$ is equal to $d(F_p M_{k-1})$. The two commutative squares

$$\begin{array}{ccc} d_{k-1}(F_{p-1} M_{k-1}) & \longrightarrow & F_{p-1} \ker d_k \\ \downarrow & & \downarrow \\ d_{k-1}(F_p M_{k-1}) & \longrightarrow & F_p \ker d_k \end{array} \quad \begin{array}{ccc} F_{p-1} \text{im } d_{k-1} & \longrightarrow & F_{p-1} \ker d_k \\ \downarrow & & \downarrow \\ F_p \text{im } d_{k-1} & \longrightarrow & F_p \ker d_k \end{array}$$

can be extended by the "lemme des neuf" to the following diagrams with exact rows and columns

$$\begin{array}{ccccccc} d_{k-1}(F_{p-1} M_{k-1}) & \longrightarrow & F_{p-1} \ker d_k & \longrightarrow & H^k(F_{p-1} M_\bullet) & & F_{p-1} \text{im } d_{k-1} \longrightarrow F_{p-1} \ker d_k \longrightarrow F_{p-1} H^k(M_\bullet) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ d_{k-1}(F_p M_{k-1}) & \longrightarrow & F_p \ker d_k & \longrightarrow & H^k(F_p M_\bullet) & & F_p \text{im } d_{k-1} \longrightarrow F_p \ker d_k \longrightarrow F_p H^k(M_\bullet) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ d_{k-1}(\text{gr}_p^F M_{k-1}) & \longrightarrow & \text{gr}_p^F \ker d_k & \longrightarrow & H^k(\text{gr}_p^F M_\bullet) & & \text{gr}_p^F \text{im } d_{k-1} \longrightarrow \text{gr}_p^F \ker d_k \longrightarrow \text{gr}_p^F H^k(M_\bullet) \end{array}$$

Since the filtration is bounded below for all M_k and therefore also for $H^k(M_\bullet)$, this shows the claim. \square

Remark 2.3. *In order to prove that the filtered complex $((K^\bullet, E - \beta), F_\bullet^{(u,v)})$ is strict it is enough to show that $H^{-l}(Gr_\bullet^{(u,v)} K^\bullet) = 0$ for $l > 1$ and $H^0(Gr_\bullet^{(u,v)} K^\bullet) = Gr_\bullet^{(u,v)} M_A^\beta$, since we already know that $H^{-l}(K^\bullet) = 0$ for $l > 1$ and $H^0(K^\bullet) = M_A^\beta$.*

2.2 Fourier-Laplace transformed GKZ-systems

Let W be a finite-dimensional vector space over \mathbb{C} and denote by V its dual vectorspace. Let X be a smooth algebraic variety and $E = X \times W$ be a trivial vector bundle and $E' := X \times V$ its dual. We write $\langle, \rangle : W \times V \rightarrow \mathbb{C}$ for the canonical pairing which extends to a function $\langle, \rangle : E \times E' \rightarrow \mathbb{C}$.

Definition 2.4. Define $\mathcal{L} := \mathcal{O}_{E \times_X E'} e^{-\langle, \rangle}$ which is by definition the free rank one module with differential given by the product rule. Denote by $p_1 : E \times_X E' \rightarrow E$, $p_2 : E \times_X E' \rightarrow E'$ the canonical projections. For $\mathcal{M} \in D_h^b(\mathcal{D}_E)$ the Fourier-Laplace transformation is then defined by

$$\mathrm{FL}_X(\mathcal{M}) := p_{2+}(p_1^+ \mathcal{M} \otimes^L \mathcal{L})[-n-1]$$

Definition 2.5. Let $A = (a_{ki})$ be a $d \times n$ integer matrix. Let $\beta \in \mathbb{C}^d$. Write \mathbb{L}_A for the \mathbb{Z} -module of relations among the columns of A and write \mathcal{D}_W for the sheaf of rings of algebraic differential operators on W . Define

$$\check{\mathcal{M}}_A^\beta := \mathcal{D}_W / ((\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_A}, (\check{E}_k + \beta_k)_{k=1, \dots, d}),$$

where

$$\begin{aligned} \check{E}_k &:= \sum_{i=1}^n a_{ki} \partial_{w_i} w_i \quad \text{for } k = 1, \dots, d \\ \check{\square}_{\underline{m} \in \mathbb{L}_A} &:= \prod_{m_i > 0} w_i^{m_i} - \prod_{m_i < 0} w_i^{-m_i}. \end{aligned} \tag{3}$$

We will often work with the D_W -module of global sections

$$\check{M}_A^\beta := \Gamma(W, \check{\mathcal{M}}_A^\beta)$$

of the \mathcal{D}_W -module $\check{\mathcal{M}}_A^\beta$. Sometimes we will be interested in the case $\beta = 0$ and will write

$$\check{\mathcal{M}}_A := \check{\mathcal{M}}_A^0 \quad \text{and} \quad M_A := \Gamma(W, \check{\mathcal{M}}_A).$$

Remark 2.6. Notice that $\check{\mathcal{M}}_A^\beta$ is just a Fourier-Laplace transformation (in all variables) of the GKZ-system \mathcal{M}_A^β (cf. Definition 2.1).

The semigroup ring associated to the matrix A is

$$\mathbb{C}[\mathbb{N}A] \simeq \mathbb{C}[\underline{w}] / ((\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_A}),$$

where $\mathbb{C}[\underline{w}]$ is the commutative ring $\mathbb{C}[w_1, \dots, w_n]$ and the isomorphism follows from [MS05, Theorem 7.3]. The rings $\mathbb{C}[\underline{w}]$ and $\mathbb{C}[\mathbb{N}A]$ are naturally \mathbb{Z}^d -graded if we define $\deg(w_j) = \underline{a}_j$ for $j = 1, \dots, n$. This is compatible with the \mathbb{Z}^d -grading of the Weyl algebra D_W given by $\deg(\partial_{w_j}) = -\underline{a}_j$ and $\deg(w_j) = \underline{a}_j$.

Definition 2.7 ([MMW05, Definition 5.2]). Let N be a finitely generated \mathbb{Z}^d -graded $\mathbb{C}[\underline{w}]$ -module. An element $\alpha \in \mathbb{Z}^d$ is called a true degree of N if N_α is non-zero. A vector $\alpha \in \mathbb{C}^d$ is called a quasi-degree of N , written $\alpha \in \mathrm{qdeg}(N)$, if α lies in the complex Zariski closure $\mathrm{qdeg}(N)$ of the true degrees of N via the natural embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$.

Schulze and Walther now define the following set of parameters:

Definition 2.8 ([SW09]). The set

$$s\mathrm{Res}(A) := \bigcup_{j=1}^n s\mathrm{Res}_j(A),$$

where

$$s\mathrm{Res}_j(A) := \{\beta \in \mathbb{C}^d \mid \beta \in -(\mathbb{N}+1)\underline{a}_j + \mathrm{qdeg}(\mathbb{C}[\mathbb{N}A]/(w_j))\}$$

is called the set of strongly resonant parameters of A .

Notice that Schulze and Walther [SW09] use the GKZ-system \mathcal{M}_A^β and the convention $\deg(\partial_{\lambda_j}) = \underline{a}_j$. We will use $\check{\mathcal{M}}_A^\beta$ and $\deg(w_j) = \underline{a}_j$ instead.

The matrix A is called pointed if 0 is the only unit in $\mathbb{N}A$. The matrix A gives rise to a map from a torus $T = (\mathbb{C}^*)^d$ with coordinates (t_1, \dots, t_d) into the affine space $W = \mathbb{C}^n$ with coordinates w_1, \dots, w_n :

$$\begin{aligned} h_A : T &\longrightarrow W \\ (t_1, \dots, t_d) &\mapsto (\underline{t}^{\underline{a}_1}, \dots, \underline{t}^{\underline{a}_n}), \end{aligned}$$

where $\underline{t}^{\underline{a}_i} := \prod_{k=1}^d t_k^{a_{ki}}$. Notice that the map h_A is affine and a locally closed embedding, hence the direct image functor for \mathcal{D}_T -modules $(h_A)_+$ is exact.

For a pointed matrix A Schulze and Walther computed the direct image of the twisted structure sheaf

$$\mathcal{O}_T^\beta := \mathcal{D}_T / \mathcal{D}_T \cdot (\partial_{t_1} t_1 + \beta_1, \dots, \partial_{t_d} t_d + \beta_d)$$

under the morphism h_A .

Theorem 2.9 ([SW09] Theorem 3.6, Corollary 3.7). *Let A a pointed $(d \times n)$ integer matrix satisfying $\mathbb{Z}A = \mathbb{Z}^d$, then the following statements are equivalent*

1. $\beta \notin sRes(A)$.
2. $\check{\mathcal{M}}_A^\beta \simeq (h_A)_+ \mathcal{O}_T^\beta$.
3. Left multiplication with w_i is invertible on \check{M}_A^β for $i = 1, \dots, n$.

In this section we want to generalize the implication 1. \Rightarrow 2. to the case of a non-pointed matrix A .

For this we set $\underline{a}_0 := 0$. We will associate to the matrix A the homogenized $(d+1 \times n+1)$ matrix \tilde{A} with columns $\tilde{\underline{a}}_i := (1, \underline{a}_i)$ for $i = 0, \dots, n$. Notice that $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ holds and that the matrix \tilde{A} is pointed in any case. Consider now the augmented map

$$\begin{aligned} h_{\tilde{A}} : \tilde{T} &\longrightarrow \tilde{W} \\ (t_0, \dots, t_d) &\mapsto (t_0 \underline{t}^{\underline{a}_0}, t_0 \underline{t}^{\underline{a}_1}, \dots, t_0 \underline{t}^{\underline{a}_n}), \end{aligned} \tag{4}$$

where $\tilde{T} = (\mathbb{C}^*)^{d+1}$ and $\tilde{W} = \mathbb{C}^{n+1}$ with coordinates w_0, \dots, w_n . Let \tilde{W}_0 be the subvariety of \tilde{W} given by $w_0 \neq 0$ and denote by $k_0 : \tilde{W}_0 \rightarrow \tilde{W}$ the canonical embedding. The map $h_{\tilde{A}}$ factors through \tilde{W}_0 which gives rise to a map h_0 with $h_{\tilde{A}} = k_0 \circ h_0$. We get the following commutative diagram

$$\begin{array}{ccccc} & & h_{\tilde{A}} & & \\ & \curvearrowright & & \curvearrowleft & \\ \tilde{T} & \xrightarrow{h_0} & \tilde{W}_0 & \xrightarrow{k_0} & \tilde{W} \\ \downarrow \pi & & \downarrow \pi_0 & & \\ T & \xrightarrow{h_A} & W & & \end{array} \tag{5}$$

where π is the projection which forgets the first coordinate and π_0 is given by

$$\begin{aligned} \pi_0 : \tilde{W}_0 &\longrightarrow W \\ (w_0, w_1, \dots, w_n) &\mapsto (w_1/w_0, \dots, w_n/w_0). \end{aligned}$$

Lemma 2.10. *For each $\beta_0 \in \mathbb{Z}$ we have an isomorphism:*

$$\mathcal{H}^0 \left((h_A)_+ \mathcal{O}_T^\beta \right) \simeq \mathcal{H}^0 \left((\pi_0)_+ k_0^+ \left((h_{\tilde{A}})_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \right) \right).$$

Proof. We show the claim by using the following isomorphisms

$$\begin{aligned}\mathcal{H}^0 h_{A+} \mathcal{O}_T^\beta &\simeq \mathcal{H}^0 h_{A+} \mathcal{H}^0 \pi_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \simeq \mathcal{H}^0 (h_A)_+ \pi_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \simeq \mathcal{H}^0 (\pi_0)_+ (h_0)_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \\ &\simeq \mathcal{H}^0 (\pi_0)_+ k_0^+ (k_0)_+ (h_0)_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \simeq \mathcal{H}^0 (\pi_0)_+ k_0^+ (h_{\tilde{A}})_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)}.\end{aligned}$$

The first isomorphism follows from the fact that π is a projection with fiber \mathbb{C}^* , the second isomorphism follows from the exactness of $(h_A)_+$ and the fourth by the fact that $k_0^+(k_0)_+ \simeq id_{\tilde{W}_0}$. \square

The following proposition is the generalization of 2.9 to the non-pointed case.

Proposition 2.11. *Let A be a $d \times n$ integer matrix satisfying $\mathbb{Z}A = \mathbb{Z}^d$ and let $\beta \in \mathbb{C}^d$ with $\beta \notin sRes(A)$, then $\mathcal{H}^0 \left((h_A)_+ \mathcal{O}_T^\beta \right)$ is isomorphic to $\check{\mathcal{M}}_A^\beta$.*

Proof. The proof relies on Lemma 2.10 and the theorem of Schulze and Walther in the pointed case. Notice that we can find a $\beta_0 \in \mathbb{Z}$ with $\beta_0 \gg 0$ such that $(\beta_0, \beta) \notin sRes(\tilde{A})$ by [Rei14, Lemma 1.16] (in loc. cit. the statement is formulated for $\beta \in \mathbb{Q}^d$ but the proof carries over almost word for word in this more general case).

Consider the following map on \tilde{W}_0 :

$$\begin{aligned}f : \tilde{W}_0 &\longrightarrow W \times \mathbb{C}_{w_0}^* \\ (w_0, \dots, w_n) &\mapsto ((w_0, w_1/w_0, \dots, w_n/w_0)\end{aligned}$$

together with the canonical projection $p : W \times \mathbb{C}_{w_0}^* \rightarrow W$ which forgets the last coordinate. This factorizes $\pi_0 = p \circ f$, which gives

$$\begin{aligned}\mathcal{H}^0 \left((h_A)_+ \mathcal{O}_T^\beta \right) &\simeq \mathcal{H}^0 \left((\pi_0)_+ \left((h_{\tilde{A}})_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \right) \Big|_{\tilde{W}_0} \right) \simeq \mathcal{H}^0 \left(p_+ f_+ \left((h_{\tilde{A}})_+ \mathcal{O}_{\tilde{T}}^{(\beta_0, \beta)} \right) \Big|_{\tilde{W}_0} \right) \\ &\simeq \mathcal{H}^0 \left(p_+ f_+ (\check{\mathcal{M}}_{\tilde{A}}^{(\beta_0, \beta)}) \Big|_{\tilde{W}_0} \right).\end{aligned}$$

The \mathcal{D} -module $\mathcal{H}^0 f_+ (\check{\mathcal{M}}_{\tilde{A}}^{(\beta_0, \beta)}) \Big|_{\tilde{W}_0}$ is isomorphic to $\mathcal{D}_{W \times \mathbb{C}_{w_0}^*} / \mathcal{I}'_0$ where \mathcal{I}'_0 is generated by

$$\check{\square}_{\underline{m} \in \mathbb{L}_A} = \prod_{i: m_i > 0} w_i^{m_i} - \prod_{i: m_i < 0} w_i^{-m_i}$$

and

$$Z_0 = \partial_{w_0} w_0 + \beta_0 \quad \text{and} \quad E_k = \sum_{i=1}^n b_{ki} \partial_{w_i} w_i + \beta_k$$

Hence $\mathcal{H}^0 f_+ (\check{\mathcal{M}}_{\tilde{A}}^{(\beta_0, \beta)}) \Big|_{\tilde{W}_0}$ is isomorphic to $\check{\mathcal{M}}_A^\beta \boxtimes \mathcal{D}_{\mathbb{C}_{w_0}^*} / (\partial_{w_0} w_0 + \beta_0)$ as a \mathcal{D} -module. We therefore have

$$\mathcal{H}^0 \left(p_+ f_+ (\check{\mathcal{M}}_{\tilde{A}}^{(\beta_0, \beta)}) \Big|_{\tilde{W}_0} \right) \simeq \mathcal{H}^0 \left(p_+ \mathcal{H}^0 f_+ (\check{\mathcal{M}}_{\tilde{A}}^{(\beta_0, \beta)}) \Big|_{\tilde{W}_0} \right) \simeq \mathcal{H}^0 p_+ \left(\check{\mathcal{M}}_A^\beta \boxtimes \mathcal{D}_{\mathbb{C}_{w_0}^*} / (\partial_{w_0} w_0 + \beta_0) \right) \simeq \check{\mathcal{M}}_A^\beta.$$

\square

3 Hodge filtration on torus embeddings

The aim of this section is to compute explicitly the Hodge filtration of $(h_A)_+ \mathcal{O}_T^\beta$ as a mixed Hodge module for certain values of β (cf. Theorem 3.16). We will use this result in section 4 where the behavior of mixed Hodge modules obtained by such torus embeddings under the twisted Radon transformation is studied.

3.1 V-filtration

As above let A be a $d \times n$ integer matrix s.t. $\mathbb{Z}A = \mathbb{Z}^d$. In this section we additionally assume that the matrix A satisfies the following conditions:

$$\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A \quad \text{and} \quad \mathbb{N}A \neq \mathbb{Z}^d \quad (6)$$

where $\mathbb{R}_{\geq 0}A$ is the cone generated by the columns of A . The first condition is equivalent to the fact that the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is normal (see, e.g., [BH93, Section 6.1]).

We will again consider the locally closed embedding

$$\begin{aligned} h_A : T &\longrightarrow W \\ (t_1, \dots, t_d) &\mapsto (\underline{t}^{a_1}, \dots, \underline{t}^{a_n}). \end{aligned}$$

Put $D := \{w_1 \cdot \dots \cdot w_n = 0\} \subset W$, $W^* := W \setminus D$, and consider the decomposition $h_A = l_A \circ k_A$, where

$$\begin{aligned} k_A : T &\longrightarrow W^* \\ (t_1, \dots, t_d) &\mapsto (\underline{t}^{a_1}, \dots, \underline{t}^{a_n}). \end{aligned}$$

and where $l_A : W^* \rightarrow W$ is the canonical open embedding.

Lemma 3.1. *The morphism $k_A : T \rightarrow W^*$ is a closed embedding.*

Proof. This is clear, as the image of k_A is precisely the vanishing locus of $(\check{\square}_m)_{m \in \mathbb{L}_A} \subset \Gamma(W^*, \mathcal{O}_{W^*})$. \square

The aim of this section is to compute parts of the canonical (descending) V -filtration of $\check{\mathcal{M}}_A^\beta \simeq h_{A+} \mathcal{O}_T^\beta$ (or Kashiwara-Malgrange filtration) along the normal crossing divisor D for certain values of β .

We review very briefly some facts about the V -filtration for differential modules. Let $X = \text{Spec}(R)$ be a smooth affine variety and $Y = \text{div}(t)$ be a smooth reduced principal divisor. Denote by $I = (t)$ the corresponding ideal. Let as before $D_X = \Gamma(X, \mathcal{D}_X)$ be the ring of algebraic differential operators on X , then the V -filtration on D_X is defined by

$$V^k D_X = \{P \in D_X \mid P I^j \subset I^{j+k} \text{ for any } j \in \mathbb{Z}\},$$

where $I^j = R$ for $j \leq 0$. One has

$$\begin{aligned} V^k D_X &= t^k V^0 D_X, \\ V^{-k} D_X &= \sum_{0 \leq j \leq k} \partial_t^j V^0 D_X. \end{aligned}$$

Choose a total ordering $<$ on \mathbb{C} such that, for any $\alpha, \beta \in \mathbb{C}$, the following conditions hold:

1. $\alpha < \alpha + 1$,
2. $\alpha < \beta$ if and only if $\alpha + 1 < \beta + 1$,
3. $\alpha < \beta + m$ for some $m \in \mathbb{Z}$.

We recall the definition of the canonical V -filtration (see, e.g., [Sai93, Section 1]).

Definition 3.2. *Let N be a coherent D_X -module. The canonical V -filtration (or Kashiwara-Malgrange filtration) is an exhaustive filtration on N indexed discretely by \mathbb{C} with total order as above and is uniquely determined by the following conditions*

1. $(V^k D_X)(V^\alpha N) \subset V^{\alpha+k} N$ for all k, α
2. $V^\alpha N$ is coherent over $V^0 D_X$ for any α
3. $t(V^\alpha N) = V^{\alpha+1} N$ for $\alpha \gg 0$
4. the action of $\partial_t t - \alpha$ on $\text{Gr}_V^\alpha N = V^\alpha N / V^{>\alpha} N$ is nilpotent

where $V^{>\alpha}N := \bigcup_{\beta>\alpha} V^\beta$.

The canonical V -filtration is unique if it exists. Its existence is guaranteed if N is D_X -holonomic.

We reduce the computation of the V -filtration on \check{M}_A^β along the possibly singular divisor D to the computation of a V -filtration along a smooth divisor by considering the following graph embedding:

$$\begin{aligned} i_g : W &\longrightarrow W \times \mathbb{C}_t \\ (w_1, \dots, w_n) &\mapsto (w_1, \dots, w_n, w_1 \cdot \dots \cdot w_n). \end{aligned}$$

Instead of computing the V -filtration on \check{M}_A^β , we will compute it on $\Gamma(W \times \mathbb{C}_t, \mathcal{H}^0(i_{g+}\check{\mathcal{M}}_A^\beta))$ along $t = 0$ (notice that i_g is an affine embedding hence i_{g+} is exact). In order to compute the direct image we consider the composed map

$$\begin{aligned} i_g \circ h_A : T &\longrightarrow W \times \mathbb{C}_t \\ (t_1, \dots, t_d) &\mapsto (\underline{t}^{a_1}, \dots, \underline{t}^{a_n}, \underline{t}^{a_1+\dots+a_n}). \end{aligned} \quad (7)$$

Notice that the matrix A' , which is built from the columns $\underline{a}_1, \dots, \underline{a}_n, \underline{a}_1 + \dots + \underline{a}_n$, gives a saturated semigroup $\mathbb{N}A' = \mathbb{N}A$. Hence we can apply again Proposition 2.11 to compute

$$\check{\mathcal{M}}_{A'}^\beta \simeq \mathcal{H}^0 i_{g+}\check{\mathcal{M}}_A^\beta \simeq \mathcal{H}^0(i_g \circ h_A)_+ \mathcal{O}_T^\beta.$$

This means that $\mathcal{H}^0 i_{g+}\check{\mathcal{M}}_A^\beta$ is a cyclic $\mathcal{D}_{W \times \mathbb{C}_t}$ -module $\mathcal{D}_{W \times \mathbb{C}_t}/\mathcal{I}'$, where \mathcal{I}' is generated by

$$\check{E}'_k := \sum_{i=1}^n a_{ki} \partial_{w_i} w_i + c_k \partial_t t + \beta_k \quad \text{for } k = 1, \dots, d, \quad (8)$$

where $c_k = a_{k1} + \dots + a_{kn}$ is the k -th component of $c \in \mathbb{Z}^d$ and

$$\check{\square}_{\underline{m} \in \mathbb{L}_{A'}} := \begin{cases} \prod_{m_i > 0} w_i^{m_i} t^{m_{n+1}} - \prod_{m_i < 0} w_i^{-m_i} & \text{for } m_{n+1} \geq 0 \\ \prod_{m_i > 0} w_i^{m_i} - \prod_{m_i < 0} w_i^{-m_i} t^{-m_{n+1}} & \text{for } m_{n+1} < 0 \end{cases} \quad (9)$$

where $\mathbb{L}_{A'}$ is the \mathbb{Z} -module of relations among the columns of A' .

We are going to use the following characterization of the canonical V -filtration along $t = 0$.

Proposition 3.3. [MM04, Definition 4.3-3, Proposition 4.3-9] Let $n \in N$ and set $E := \partial_t t$. The Bernstein-Sato polynomial of n is the unitary polynomial of smallest degree, satisfying

$$b(E)n \in V^1(D_X)n.$$

We denote it by $b_n(x) \in \mathbb{C}[x]$ and the set of roots of $b_n(x)$ by $\text{ord}(n)$. The canonical V -filtration on N is then given by

$$V^\alpha N = \{n \in N \mid \text{ord}(n) \subset [\alpha, \infty)\}.$$

We will use this characterization to compute the canonical V -filtration on $\check{M}_{A'}^\beta$ along $t = 0$ for certain $\beta \in \mathbb{R}^d$.

Let $\underline{c} := \underline{a}_1 + \dots + \underline{a}_n$. For all facets F of $\mathbb{R}_{\geq 0}A' = \mathbb{R}_{\geq 0}A$ let $0 \neq \underline{n}_F \in \mathbb{Z}^d$ be the uniquely determined primitive, inward-pointing, normal vector of F , i.e. \underline{n}_F satisfies $\langle \underline{n}_F, F \rangle = 0$, $\langle \underline{n}_F, \mathbb{N}A \rangle \subset \mathbb{Z}_{\geq 0}$ and $\lambda \cdot \underline{n}_F \notin \mathbb{Z}^d$ for $\lambda \in [0, 1)$ (where $\langle \cdot, \cdot \rangle$ is the euclidean pairing). Set

$$e_F := \langle \underline{n}_F, \underline{c} \rangle \in \mathbb{Z}_{\geq 0}.$$

We show that e_F is always positive. We have $\underline{c} \neq 0$ since otherwise $0 = -\underline{a}_1 - \dots - \underline{a}_n \in \mathbb{N}A$ and therefore $-\underline{a}_i \in \mathbb{N}A$ for all $i \in \{1, \dots, n\}$ which contradicts the assumption $\mathbb{N}A \neq \mathbb{Z}^d$. Furthermore \underline{c} lies in the interior of $\mathbb{R}_{\geq 0}A'$. In order to see this assume to the contrary that \underline{c} lies on some facet F of

$\mathbb{R}_{\geq 0}A'$. Then $\langle \underline{n}_F, \underline{c} \rangle = 0$ holds. For $\underline{a}_i \notin F$ we have on the one hand $\underline{c} - \underline{a}_i \in \mathbb{N}A$ and on the other hand $\langle \underline{c} - \underline{a}_i, \underline{n}_F \rangle < 0$ which is a contradiction. Hence \underline{c} is in the interior of $\mathbb{R}_{\geq 0}A$, which shows $e_F \in \mathbb{Z}_{>0}$. We define the following set of admissible parameters β :

$$\mathfrak{A}_A := \bigcap_{F: F \text{ facet}} \left\{ \mathbb{R} \cdot F - \left[0, \frac{1}{e_F}\right] \cdot \underline{c} \right\} \quad (10)$$

Lemma 3.4. *Suppose as above that $\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A$. Consider the cyclic $D_{W \times \mathbb{C}^t}$ -module $\check{M}_{A'}^\beta$, and its generator $[1] \in M_{A'}$. Then we have $\text{ord}([1]) \subset [0, 1)$ if $\beta \in \mathfrak{A}_A$.*

Proof. It was shown in [RSW17, Theorem 2.4] that the roots of $b_{[1]}(x)$ for $[1] \in \check{M}_A^\beta$ are contained in the set $\{\epsilon \in \mathbb{C} \mid \epsilon \cdot \underline{c} \in \text{qdeg}(\mathbb{C}[\mathbb{N}A']/(t)) - \beta\}$ which is discrete since \underline{c} lies in the interior of $\mathbb{R}_{\geq 0}A' = \mathbb{R}_{\geq 0}A$ and $\text{qdeg}(\mathbb{C}[\mathbb{N}A'])$ is a finite union of parallel translates of the complex span of faces of $\mathbb{R}_{\geq 0}A'$ (cf. [MMW05]). We will now compute an estimate of the quasi-degrees $\text{qdeg}(\mathbb{C}[\mathbb{N}A']/(t))$. For this we remark that $0 = [P] \in \mathbb{C}[\mathbb{N}A']/(t)$ for $P \in \mathbb{C}[\mathbb{N}A']$ iff $\exists P' \in \mathbb{C}[\mathbb{N}A']$ with $P = P' \cdot t$. In this case we have $\text{deg}(P) \in \mathbb{N}A + \underline{c}$.

Set $L_F := \left\{ \frac{k}{e_F} \cdot \underline{c} + \mathbb{C} \cdot F \mid k = 0, \dots, e_F - 1 \right\}$ and $L_F = \emptyset$ otherwise. Then $L = \bigcup_{F: \text{facet}} L_F$ is Zariski closed and we will show that the set $\text{deg}(\mathbb{C}[\mathbb{N}A']/(t))$ is contained in L . Let $P \in \mathbb{C}[\mathbb{N}A']$ with $0 \neq [P] \in \mathbb{C}[\mathbb{N}A']/(t)$ and set $\underline{p} := \text{deg}(P) \in \mathbb{N}A$. Since $-\underline{c} \notin \mathbb{R}_{\geq 0}A$ there exists a facet F and some $\lambda \in [0, 1)$ such that $\underline{p} - \lambda \underline{c} \in F$, i.e. $\underline{p} = \lambda \underline{c} + \underline{f}$ for some $\underline{f} \in F$. We have $\lambda \cdot e_F = \langle \lambda \underline{c} + \underline{f}, \underline{n}_F \rangle = \langle \underline{p}, \underline{n}_F \rangle \in \mathbb{Z}_{\geq 0}$. Hence $\underline{p} \in L_F \subset L$.

Since $\text{qdeg}(\mathbb{C}[\mathbb{N}A']/(t))$ is by definition the Zariski closure of $\text{deg}(\mathbb{C}[\mathbb{N}A']/(t))$ the former set is contained in L . In particular this shows that the roots of $b_{[1]}(x)$ are contained in the set $\{\epsilon \in \mathbb{C} \mid \epsilon \cdot \underline{c} \in L - \beta\}$. Since L is a union of hypersurfaces which are defined over \mathbb{R} , $\underline{c} \in \mathbb{Z}^d$ and $\beta \in \mathbb{R}^d$, this set is equal to $\{\epsilon \in \mathbb{R} \mid \epsilon \cdot \underline{c} \in L - \beta\}$. Hence for $\beta \in \bigcap_{F: \text{facet}} \left\{ \mathbb{R} \cdot F - \left[0, \frac{1}{e_F}\right] \cdot \underline{c} \right\}$ we can guarantee that the roots of $b_{[1]}(x)$ are contained in $[0, 1)$. \square

We will prove a basic lemma on the set \mathfrak{A}_A which will be of importance later.

Lemma 3.5. *Suppose $\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}$. Then $\mathfrak{A}_A \cap \text{sRes}(A) = \emptyset$.*

Proof. Recall that $\text{sRes}(A) = \bigcup_{j=1}^n \text{sRes}_j(A) = \bigcup_{j=1}^n -(\mathbb{N} + 1)\underline{a}_j + \text{qdeg}(\mathbb{C}[\mathbb{N}A]/((w_j)))$. Therefore it is enough to show that

$$\mathfrak{A}_A \cap \left\{ -(\mathbb{N} + 1)\underline{a}_j + \text{qdeg}(\mathbb{C}[\mathbb{N}A]/((w_j))) \right\} = \emptyset$$

holds. The following estimation of the quasi-degrees of $\mathbb{C}[\mathbb{N}A]/((w_j))$ can be shown similarly as in the proof of the lemma above

$$\text{qdeg}(\mathbb{C}[\mathbb{N}A]/((w_j))) \subset L_j := \bigcup_{F: \underline{a}_j \notin F} \left\{ \frac{k}{e_{F,j}} \cdot \underline{b}_j + \mathbb{C} \cdot F \mid k = 0, \dots, e_{F,j} - 1 \right\}$$

where $e_{F,j} := \langle \underline{n}_F, \underline{b}_j \rangle$. Hence it is enough to show that for each $j \in \{1, \dots, n\}$ and each facet F with $\underline{a}_j \notin F$ the following holds

$$\left\{ \mathbb{R} \cdot F - \left[0, \frac{1}{e_F}\right] \cdot \underline{c} \right\} \cap \left\{ -(\mathbb{N} + 1)\underline{a}_j + \bigcup_{k=0}^{e_{F,j}-1} \frac{k}{e_{F,j}} \cdot \underline{a}_j + \mathbb{R} \cdot F \right\} = \emptyset \quad (11)$$

Since F has codimension one in \mathbb{R}^d and $\underline{a}_j, \underline{c} \notin \mathbb{R} \cdot F$ we can write $\underline{c} = \lambda \underline{a}_j + \underline{f}$ for some $\underline{f} \in \mathbb{R} \cdot F$. We get $e_F = \lambda e_{F,j}$. We conclude that (11) is equivalent to

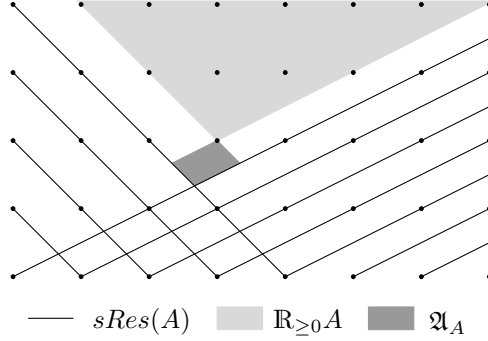
$$\left\{ \mathbb{R} \cdot F - \left[0, \frac{1}{e_{F,j}}\right] \cdot \underline{a}_j \right\} \cap \left\{ -(\mathbb{N} + 1)\underline{a}_j + \bigcup_{k=0}^{e_{F,j}-1} \frac{k}{e_{F,j}} \cdot \underline{a}_j + \mathbb{R} \cdot F \right\} = \emptyset.$$

But this holds since $\left(-\frac{1}{e_{F,j}}, 0\right] \cap \left\{ -(\mathbb{N} + 1) + \left\{0, \frac{1}{e_{F,j}}, \dots, \frac{e_{F,j}-1}{e_{F,j}}\right\} \right\} = \emptyset$. \square

Example 3.6. Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

the sets $sRes(A)$ and \mathfrak{A}_A are sketched below.



Next we draw a consequence for the canonical V -filtration with respect to $t = 0$ on $\mathcal{H}^0 i_{g+} \check{\mathcal{M}}_A^\beta$. We will not compute all of its filtration steps, but those corresponding to integer indices, which is sufficient for our purpose. For this consider the induced V -filtration on $\check{M}_{A'}^\beta = \Gamma(W \times \mathbb{C}_t, \mathcal{H}^0 i_{g+} \check{\mathcal{M}}_A^\beta)$

$$V_{ind}^\alpha \check{M}_{A'}^\beta := \{[P] \in \check{M}_{A'}^\beta \mid P \in V^\alpha D_{W \times \mathbb{C}_t}\}.$$

It is readily checked that $V_{ind}^\alpha \check{M}_{A'}^\beta$ is a good V -filtration on $\check{M}_{A'}^\beta$. As $\check{M}_{A'}$ is holonomic, hence specializable along any smooth hypersurface, it admits a Bernstein polynomial $b_{V_{ind}^\alpha}(x)$ in the sense of [MM04, Définition 4.2-3]. On the other hand, for any section $\sigma : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}$ of the canonical projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$, there is a *unique* good (\mathbb{Z} -indexed) V -filtration $V_\sigma^\bullet \check{M}_{A'}^\beta$, on $\check{M}_{A'}^\beta$, such that the roots of $b_{V_\sigma^\bullet}(x)$ lie in $Im(\sigma)$ (see loc.cit., Proposition 4.2-6). From this we deduce the following result.

Proposition 3.7. If $\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A$ and $\beta \in \mathfrak{A}_A$, then for any $k \in \mathbb{Z}$, we have the following equality

$$V^k \check{M}_{A'}^\beta = V_{ind}^k \check{M}_{A'}^\beta.$$

Notice that by definition we have $Gr_{V_{ind}^\alpha} \check{M}_{A'}^\beta = 0$ for all $\alpha \notin \mathbb{Z}$, but in general there may be some $\alpha \notin \mathbb{Z}$ with $Gr_V^\alpha \check{M}_{A'}^\beta \neq 0$. This is of course no contradiction to the above statement.

Proof. Recall (see [MM04, Proposition 4.3-5]) that we have $V^{\alpha+k} \check{M}_{A'}^\beta = V_{\sigma_\alpha}^k \check{M}_{A'}^\beta$ for any $\alpha \in \mathbb{C}, k \in \mathbb{Z}$, where $\sigma_\alpha : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}$ is the section of $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$ with image equal to $[\alpha, \alpha + 1)$. Hence, in order to prove the proposition it is enough to show that $V_{\sigma_0}^k \check{M}_{A'}^\beta = V_{ind}^k \check{M}_{A'}^\beta$. Using loc. cit., Proposition 4.2-6 it remains to show that the roots of the Bernstein polynomial $b_{V_{ind}^k}(x)$ are contained in $[0, 1)$.

An element $[P]$ of $V_{ind}^k \check{M}_{A'}^\beta$ for $k \geq 0$ can be written as

$$[P] = \left[\sum_{i=0}^l t^k (\partial_t t)^i P_i \right] + [R],$$

where $[R] \in V_{ind}^{k+1} \check{M}_{A'}^\beta$ and $P_i \in \mathbb{C}[w_1, \dots, w_n] \langle \partial_{w_1}, \dots, \partial_{w_n} \rangle$. We have

$$\begin{aligned} b_{[1]}(\partial_t t - k) \cdot [P] &= \left[\sum_{i=0}^l t^k (\partial_t t)^i P_i \cdot b_{[1]}(\partial_t t) \right] + b_{[1]}(\partial_t t - k) \cdot [R] \\ &= \sum_{i=0}^l t^k (\partial_t t)^i P_i \cdot b_{[1]}(\partial_t t) \cdot [1] + b_{[1]}(\partial_t t - k) \cdot [R]. \end{aligned}$$

But $\sum_{i=0}^l t^k (\partial_t t)^i P_i \cdot b_{[1]}(\partial_t t) \cdot [1] \in V_{ind}^{k+1} \check{M}_{A'}^\beta$, because $\sum_{i=0}^l t^k (\partial_t t)^i P_i \in V^k D$ and $b_{[1]}(\partial_t t) \cdot [1] \in V_{ind}^1 \check{M}_{A'}^\beta$. Therefore

$$b_{[1]}(\partial_t t - k) \cdot [P] \in V_{ind}^{k+1} \check{M}_{A'}^\beta.$$

Now let $[P] \in V_{ind}^{-k} \check{M}_{A'}^\beta$ with $k > 0$. It can be written as

$$[P] = \left[\sum_{i=0}^l \partial_t^k (\partial_t t)^i P_i \right] + [R],$$

where $[R] \in V_{ind}^{k+1} \check{M}_{A'}^\beta$. By a similar argument we have

$$b_{[1]}(\partial_t t + k) \cdot [P] \in V_{ind}^{-k+1} \check{M}_{A'}^\beta.$$

This shows $b_{V_{ind}}(x) \mid b_{[1]}(x)$. Because of Lemma 3.4 the roots of $b_{V_{ind}}(x)$ are contained in $[0, 1)$, the claim follows. \square

3.2 Compatibility of filtrations

In this subsection we are going to show a compatibility result between two filtrations on the module $\check{M}_{A'}^\beta$. For notational convenience, we put $W' := W \times \mathbb{C}_t$, and we rename the coordinate t on W' to be w_{n+1} , that is $\Gamma(W', \mathcal{O}_{W'}) = \mathbb{C}[w_1, \dots, w_n, w_{n+1}]$. The two filtrations in question are the one induced from the filtration by the order of differential operators from the ring $D_{W'}$ and the Kashiwara-Malgrange filtration considered in the last subsection. The main result is Proposition 3.12. Its proof relies on the very specific structure of the hypergeometric ideal $\check{I}_{A'} = ((\check{\square}_m)_{m \in \mathbb{L}_{A'}} + (\check{E}_k + \beta_k)_{k=1, \dots, d}) \subset D_{W'}$ and uses non-commutative Gröbner basis techniques, a good reference for results needed is [SST00]. We will recall the main definitions for the readers convenience.

We work in the Weyl algebra $D_{W'} = \mathbb{C}[w_1, \dots, w_{n+1}] \langle \partial_{w_1}, \dots, \partial_{w_{n+1}} \rangle$. Any operator $P \in D_{W'}$ has the so-called normally ordered expression $P = \sum_{(\gamma, \delta)} c_{\gamma\delta} w^\gamma \partial_w^\delta \in D_{W'}$, where the sum runs over all pairs (γ, δ) in some finite subset of $\mathbb{N}^{2(n+1)}$.

First we define partial orders on the set of monomials in $D_{W'}$ resp. $\mathbb{C}[w] := \mathbb{C}[w_1, \dots, w_{n+1}]$ resp. $\mathbb{C}[w, \xi] := \mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]$ by choosing the weight vectors $(u, v) \in \mathbb{Z}^{2(n+1)}$ with $u_i + v_i \geq 0$ resp. $u \in \mathbb{Z}^{n+1}$. This means that the variables w_i have weight u_i and the partial differentials ∂_{w_i} resp. ξ_i have weight v_i . The associated partial order in $D_{W'}$ is defined as follows: If for two monomials $w^\gamma \partial_w^\delta, w^c \partial_w^d$ we have $\sum_i u_i c_i + v_i d_i < \sum_i u_i \gamma_i + v_i \delta_i$, then by definition $w^\gamma \partial_w^\delta$ is larger than $w^c \partial_w^d$, we write $w^c \partial_w^d \prec_{(u,v)} w^\gamma \partial_w^\delta$ and similarly for $\mathbb{C}[w]$ and $\mathbb{C}[w, \xi]$. The weight vector (u, v) induces an increasing resp. decreasing filtrations on $D_{W'}$ given by

$$F_p^{(u,v)} D_{W'} = \left\{ \sum_{\sum_i u_i \gamma_i + v_i \delta_i \leq p} c_{\gamma\delta} w^\gamma \partial_w^\delta \right\} \quad \text{resp.} \quad F_{(u,v)}^p D_{W'} = \left\{ \sum_{\sum_i u_i \gamma_i + v_i \delta_i \geq p} c_{\gamma\delta} w^\gamma \partial_w^\delta \right\}$$

We define the graded ring $Gr_\bullet^{(u,v)} D_{W'} := \bigoplus_p F_p^{(u,v)} D_{W'} / F_{p-1}^{(u,v)} D_{W'}$ associated to the weight (u, v) . Notice that for $(u, v) = (0, \dots, 0, 1, \dots, 1)$ (i.e. the w_i have weight 0 and the ∂_{w_i} have weight 1) the ascending filtration $F_\bullet^{(u,v)} D_{W'}$ is the order filtration $F_\bullet D_{W'}$ and for $(u, v) = (0, \dots, 0, -1, 0, \dots, 0, 1)$ the descending filtration $F_{(u,v)}^\bullet D_{W'}$ is the V -filtration with respect to w_{n+1} .

We get well-defined maps

$$\begin{aligned} in_{(u,v)} : D_{W'} &\longrightarrow Gr_\bullet^{(u,v)} D_{W'} = \mathbb{C}[w, \xi] \\ P = \sum_{\gamma, \delta} c_{\gamma\delta} w^\gamma \partial_w^\delta &\longmapsto in_{(u,v)}(P) := \sum_{\sum_i u_i \gamma_i + v_i \delta_i = m} c_{\gamma\delta} w^\gamma \xi^\delta \end{aligned}$$

where $m := \text{ord}_{(u,v)}(P) := \max\{\sum_i u_i \gamma_i + v_i \delta_i \mid c_{\gamma\delta} \neq 0\}$ and

$$\begin{aligned} \text{in}_u : \mathbb{C}[w] &\longrightarrow Gr_{\bullet}^u \mathbb{C}[w] = \mathbb{C}[w] \\ Q = \sum_{\gamma} c_{\gamma} w^{\gamma} &\longmapsto \text{in}_u(Q) := \sum_{\sum_i u_i \gamma_i = m} c_{\delta} w^{\delta} \end{aligned}$$

where $m = \max\{\sum_u u_i \gamma_i \mid c_{\gamma} \neq 0\}$ and

$$\begin{aligned} \text{in}_{(u,v)} : \mathbb{C}[w, \xi] &\longrightarrow Gr_{\bullet}^{(u,v)} \mathbb{C}[w, \xi] = \mathbb{C}[w, \xi] \\ R = \sum_{\gamma, \delta} c_{\gamma} w^{\gamma} \xi^{\delta} &\longmapsto \text{in}_{(u,v)}(R) := \sum_{\sum_i u_i \gamma_i + v_i \delta_i = m} c_{\delta} w^{\delta} \end{aligned}$$

where $m := \text{ord}_{(u,v)}(R) := \max\{\sum_i u_i \gamma_i + v_i \delta_i \mid c_{\gamma\delta} \neq 0\}$. Notice that, in contrast to the case of a total ordering, the initial terms $\text{in}_{(u,v)}$ resp. in_u are not monomials.

Let $I' \subset D_{W'}$ be a left ideal. The set $\text{in}_{(u,v)}(I')$ is an ideal in $Gr_{\bullet}^{(u,v)} D_{W'}$ and is called initial ideal of I' with respect to the weight vector (u, v) . A finite subset G of $D_{W'}$ is a Gröbner basis of I' with respect to (u, v) if I' is generated by G and $\text{in}_{(u,v)}(I')$ is generated by $\text{in}_{(u,v)}(G)$. Similarly, let $J' \subset \mathbb{C}[w]$ resp. $K' \subset \mathbb{C}[w, \xi]$ be an ideal. The set $\text{in}_u(J')$ resp. $\text{in}_{(u,v)}(K')$ is an ideal in $Gr_{\bullet}^u \mathbb{C}[w]$ resp. $Gr_{\bullet}^{(u,v)} \mathbb{C}[w, \xi]$ and is called initial ideal of J' resp. K' with respect to the weight vector u resp. (u, v) . The definition of a Gröbner basis is parallel to the definition above.

Let $\check{I}_{A'} = \left((\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_{A'}} + (\check{E}'_k + \beta_k)_{k=1, \dots, d} \right)$ be the hypergeometric ideal. The fake initial ideal $\text{fin}_{(u,v)}(\check{I}_{A'})$ is the following ideal in $Gr_{(u,v)} D_{W'}$:

$$\text{fin}_{(u,v)}(\check{I}_{A'}) := Gr_{(u,v)} D_{W'} \cdot \text{in}_u(\check{J}_{A'}) + \sum_{k=1}^d Gr_{(u,v)} D_{W'} \cdot \text{in}_{(u,v)}(\check{E}'_k + \beta_k)$$

where $\check{J}_{A'} \subset \mathbb{C}[w]$ is the ideal generated by $(\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_{A'}}$.

Consider the Koszul complex

$$\dots \xrightarrow{d_{-2}} K^{-1}(Gr_{\bullet}^{(u,v)}(D_{W'}/D_{W'} \check{J}_{A'})) \xrightarrow{d_{-1}} K^0(Gr_{\bullet}^{(u,v)}(D_{W'}/D_{W'} \check{J}_{A'})) \longrightarrow 0$$

where

$$K^{-p}(Gr_{\bullet}^{(u,v)}(D_{W'}/D_{W'} \check{J}_{A'})) = \bigoplus_{1 \leq i_1 < \dots < i_p \leq s+1} Gr_{\bullet}^{(u,v)}(D_{W'}/D_{W'} \check{J}_{A'}) e_{i_1 \dots i_p}$$

and

$$d_{-p}(e_{i_1 \dots i_p}) = \sum_{k=1}^p (-1)^{k-1} \text{in}_{(u,v)}(\check{E}'_k + \beta_k) e_{i_1 \dots \widehat{i}_k \dots i_p}$$

The following statement is an easy adaption of [SST00, Theorem 4.3.5]

Proposition 3.8. *If the cohomology $H^{-1}(K^{\bullet}(Gr_{\bullet}^{(u,v)}(D_{W'}/D_{W'} \check{J}_{A'})))$ vanishes, then the initial ideal satisfies $\text{in}_{(u,v)}(\check{I}_{A'}) = \text{fin}_{(u,v)}(\check{I}_{A'})$.*

Proof. After a Fourier-Laplace transform $w_i \rightarrow \partial_{x_i}$ and $\partial_{w_i} \rightarrow -x_i$ the proof carries over word for word from loc. cit. (Notice that in Chapter 4 of loc. cit. the homogeneity of A' assumed, however the proof of this statement does not need this requirement). \square

Recall that A' is a matrix built from the matrix A by adding a column which is the sum over all columns of A . Let $\check{J}_A \subset \mathbb{C}[w_1, \dots, w_n]$ be the ideal generated by $(\check{\square}_l)_{l \in \mathbb{L}_A}$. We choose generators $g_1, \dots, g_{\ell-1}$ of \check{J}_A . Notice that $g_1, \dots, g_{\ell-1}, g_\ell := w_{n+1} - w_1 \cdot \dots \cdot w_n$ is a basis of $\check{J}_{A'} \subset \mathbb{C}[w_1, \dots, w_{n+1}]$.

Lemma 3.9. *The elements g_1, \dots, g_ℓ form a Gröbner basis of $\check{J}_{A'}$ with respect to the weight vector $(0, \dots, 0, -e)$ with $e > 0$.*

Proof. We have already seen that g_1, \dots, g_ℓ is a basis of $\check{J}_{A'}$. It remains to prove that $in_{(0, \dots, 0, -e)}(g_1) = g_1, \dots, in_{(0, \dots, 0, -e)}(g_{\ell-1}) = g_{\ell-1}, in_{(0, \dots, 0, -e)}(g_\ell) = w_1 \cdot \dots \cdot w_n$ is a basis of $in_{(0, \dots, 0, -e)}(\check{J}_{A'})$. Let

$$x = \sum_{i=1}^{\ell} x_i g_i \quad (12)$$

and $-e \cdot N := \max\{\text{ord}_{(0, \dots, 0, -e)}(x_i g_i) \mid i = 1, \dots, \ell\}$. Assume that $\text{ord}_{(0, \dots, 0, -e)}(x) < -e \cdot N$, then the maximal w_{n+1} -degree component of the equation (12) is given by

$$0 = \sum_{i=1}^{\ell-1} w_{n+1}^N p_i g_i + w_{n+1}^N p_\ell \cdot (w_1 \cdot \dots \cdot w_n)$$

for polynomials $p_i \in \mathbb{C}[w_1, \dots, w_n]$. Since $\check{J}_A = (g_1, \dots, g_{\ell-1})$ is a prime ideal and $w_1 \cdot \dots \cdot w_n \notin \check{J}_A$ we conclude that $p_\ell \in \check{J}_A$. Hence there exist polynomial $q_i \in \mathbb{C}[w_1, \dots, w_n]$ such that $p_\ell = \sum_{i=1}^{\ell-1} q_i g_i$. We get

$$x = \sum_{i=1}^{\ell} x_i g_i - \sum_{i=1}^{\ell-1} w_{n+1}^N p_i g_i - w_{n+1}^N p_\ell \cdot g_\ell + w_{n+1}^{N+1} \left(\sum_{i=1}^{\ell-1} q_i g_i \right) = \sum_{i=1}^{\ell} x'_i g_i$$

for $x'_i \in \mathbb{C}[w_1, \dots, w_{n+1}]$ with $\max\{\text{ord}_{(0, \dots, 0, -e)}(x'_i g_i) \mid i = 1, \dots, \ell\} < -e \cdot N$. By induction we can reduce to the case $\text{ord}_{(0, \dots, 0, -e)}(x) = -e \cdot N$. In this case we get for the maximal w_{n+1} -degree component

$$in_{(0, \dots, 0, -e)}(x) = \sum w_{n+1}^N p'_i g_i + w_{n+1}^N p'_\ell \cdot (w_1 \cdot \dots \cdot w_n) = \sum w_{n+1}^N p'_i in_{(0, \dots, 0, -e)}(g_i) + w_{n+1}^N p'_\ell in_{(0, \dots, 0, -e)}(g_\ell)$$

for polynomials $p'_i \in \mathbb{C}[w_1, \dots, w_n]$. This shows the claim. \square

Proposition 3.10. *Let A be a $d \times n$ integer matrix such that $\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ and $\mathbb{N}A \neq \mathbb{Z}^d$. Let A' the matrix built from A by adding a column which is the sum over all columns of A . Then*

$$fin_{(u,v)}(\check{I}_{A'}) = in_{(u,v)}(\check{I}_{A'})$$

if

1. $(u, v) = (0, 0, \dots, 0, 1, 1, \dots, 1)$
2. $(u, v) = (0, \dots, 0, -e, 1, \dots, 1, 1 + e)$ for $0 < e < 1$.

Proof. The first case was proven in [SST00, Corollary 4.36] for homogeneous A . In order to prove the statement for $(u, v) = (0, 0, \dots, 0, 1, 1, \dots, 1)$ in the general case we first observe that $Gr_{(v,u)}(D_{W'}/\check{J}_{A'})$ is isomorphic to

$$\mathbb{C}[\xi_1, \dots, \xi_{n+1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A']$$

which is Cohen-Macaulay by the assumption $\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ and the fact that $\mathbb{N}A = \mathbb{N}A'$ as well as $\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}A'$. It follows from [BZGM15, Theorem 1.2] that the elements $in_{(u,v)}(\check{E}_k + \beta_k)$ are part of a system of parameters in $\mathbb{C}[\xi_1, \dots, \xi_{n+1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A']$ and since this ring is Cohen-Macaulay they also form a regular sequence. Therefore $H^{-1}(K^\bullet(Gr_{(u,v)}(D_{W'}/D_{W'}\check{J}_{A'}))) = 0$ and the claim follows from Proposition 3.8.

We prove the second claim. Since $\mathbb{N}A \neq \mathbb{Z}^d$ holds the last column of A' , which is the sum of the columns of A , is non-zero (this was shown above Lemma 3.4). Hence we can assume (by elementary row

manipulations of A' , which do not change the ideal $\check{I}_{A'}$) that the last column of A' is zero except for the entry in the first row. Set

$$\check{e}_k := \sum_{i=1}^n a_{ki} w_i \xi_i \quad \text{for } k = 1, \dots, d.$$

We will use the generators $g_1, \dots, g_{\ell-1}$ of \check{J}_A from Lemma 3.9. It follows from [BZGM15, Theorem 1.2] that $\check{e}_1, \dots, \check{e}_d$ is part of a system of parameters for

$$\begin{aligned} \mathbb{C}[\xi_1, \dots, \xi_n] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A] &\simeq \mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] / \mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] \check{J}_A \\ &\simeq \mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] / (g_1, \dots, g_{\ell-1}) \end{aligned}$$

where $\mathbb{C}[\mathbb{N}A]$ has Krull dimension d . Therefore

$$\mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] / (g_1, \dots, g_{\ell-1}, \check{e}_1, \dots, \check{e}_r)$$

has Krull dimension n .

We will show that the Krull-dimension of

$$\mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] / (g_1, \dots, g_{\ell-1}, w_1 \cdots w_n, \check{e}_2, \dots, \check{e}_r) \quad (13)$$

is also n (notice that we omitted \check{e}_1). The variety corresponding to $\mathbb{C}[\xi_1, \dots, \xi_n] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A]$ is

$$\mathbb{C}^n \times X_A \subset \mathbb{C}^n \times \mathbb{C}^n$$

where $X_A := \text{Spec } \mathbb{C}[\mathbb{N}A]$. The toric variety X_A is a finite disjoint union of torus orbits where the big dense torus lies in $\{w_1 \cdots w_n \neq 0\}$ and the smaller dimensional tori lie in $\{w_1 \cdots w_n = 0\}$. Hence

$$\mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] / (g_1, \dots, g_{\ell-1}, w_1 \cdots w_n) \quad (14)$$

has Krull dimension $n + d - 1$. The torus orbits of X_A correspond to the faces of the cone $\mathbb{R}_{\geq 0}A$ where the big dense torus corresponds to $\mathbb{R}_{\geq 0}A$ itself. For a face $\tau \subsetneq \mathbb{R}_{\geq 0}A$ the torus orbit $Orb(\tau)$ is given by $Orb(\tau) = X_A \cap (\mathbb{C}_w^*)^\tau$ where $(\mathbb{C}_w^*)^\tau = \{w \in \mathbb{C}^n \mid w_i = 0 \text{ for } a_i \notin \tau, w_j \neq 0 \text{ for } a_j \in \tau\}$. Hence it suffices to prove that $(\mathbb{C}^n \times Orb(\tau)) \cap V((\check{e}_2, \dots, \check{e}_d))$ has dimension n , where $V((\check{e}_2, \dots, \check{e}_d))$ is the vanishing locus of the ideal generated by $\check{e}_2, \dots, \check{e}_d$. Set $\mathbb{C}_\xi^\tau = \{\xi \in \mathbb{C}^n \mid \xi_i = 0 \text{ for } a_i \notin \tau\}$. It is enough to show that $\mathbb{C}_\xi^\tau \times Orb(\tau) \cap V((\check{e}_2^\tau, \dots, \check{e}_d^\tau))$ has dimension at most $\#\{i \mid a_i \in \tau\}$, where $\check{e}_k^\tau := \sum_{i: a_i \in \tau} a_{ki} w_i \xi_i$. The codimension of $V((\check{e}_2^\tau, \dots, \check{e}_d^\tau))$ is $\dim(\tau)$ since $(1, 0, \dots, 0) = \frac{1}{c}(a_1 + \dots + a_n)$ (for a suitable $c \in \mathbb{Z} \setminus \{0\}$) lies in the interior of $\mathbb{R}_{\geq 0}A$, hence not in τ and therefore the matrix $(a_{ki})_{k \geq 2, i: a_i \in \tau}$ has rank $\dim(\tau)$. By [BZGM15, Lemma 1.1] the intersection of $\mathbb{C}_\xi^\tau \times Orb(\tau)$ with $V((\check{e}_2^\tau, \dots, \check{e}_d^\tau))$ is transverse. Since the codimension of $V((\check{e}_2^\tau, \dots, \check{e}_d^\tau))$ is $\dim Orb(\tau) = \dim(\tau)$ the intersection has dimension $\#\{i \mid a_i \in \tau\}$. This shows that the Krull dimension of (13) is n .

Let

$$\begin{aligned} \check{e}'_1 &:= \check{e}_1 + \left(\sum_{i=1}^n a_{1i} x_{n+1} \xi_{n+1} \right) = in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(\check{E}'_1 + \beta_1) \\ \check{e}'_k &:= \check{e}_k = in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(\check{E}'_k + \beta_k) \quad \text{for } k = 2, \dots, d. \end{aligned}$$

and

$$\begin{aligned} \bar{g}_i &:= in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(g_i) = g_i \quad \text{for } i = 1, \dots, \ell - 1 \\ \bar{g}_\ell &:= in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(g_\ell) = w_1 \cdots w_n \end{aligned}$$

□

Since $\bar{g}_1, \dots, \bar{g}_\ell$ and $\check{e}'_2, \dots, \check{e}'_d$ are independent of w_{n+1}, ξ_{n+1} and $\check{e}'_1 = \check{e}_1 + (\sum_{i=1}^n a_{1i} x_{n+1} \xi_{n+1})$ is (for degree reasons) a non-zerodivisor on

$$\mathbb{C}[w_{n+1}, \xi_{n+1}] \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \dots, \xi_n, w_1, \dots, w_n] / (\bar{g}_1, \dots, \bar{g}_\ell, \check{e}_2, \dots, \check{e}_d),$$

one easily sees that

$$\mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]/(\bar{g}_1, \dots, \bar{g}_\ell, \check{e}'_1, \dots, \check{e}'_d)$$

has Krull dimension $n + 1$. It follows from (14) that $\mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]/(\bar{g}_1, \dots, \bar{g}_\ell)$ has Krull dimension $(n + d + 1)$, hence $\check{e}'_1, \dots, \check{e}'_d$ is part of a system of parameters. By the assumption on A the ring

$$\mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]/(\bar{g}_1, \dots, \bar{g}_{\ell-1}) \simeq \mathbb{C}[w_{n+1}, \xi_1, \dots, \xi_{n+1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A]$$

is Cohen-Macaulay. Since $\bar{g}_\ell = w_1 \dots w_n$ is not a zero-divisor in the ring above (because $\mathbb{C}[\mathbb{N}A]$ has no non-zero zero-divisors), we see that the ring

$$\mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]/(\bar{g}_1, \dots, \bar{g}_\ell)$$

is also Cohen-Macaulay and therefore $\check{e}'_1, \dots, \check{e}'_d$ is a regular sequence in $\mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]/(\bar{g}_1, \dots, \bar{g}_\ell)$.

Since

$$Gr_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(D_{W'} / D_{W'} \check{J}_{A'}) \simeq \mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]/(\bar{g}_1, \dots, \bar{g}_\ell)$$

and $\check{e}'_k = in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(\check{E}'_k + \beta_k)$ for $k = 1, \dots, d$, we have

$$H^{-1}(K^\bullet(Gr_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(D_{W'} / D_{W'} \check{J}_{A'}))) = 0.$$

Using again Proposition 3.8, this shows the second claim.

Corollary 3.11. *Let $g_1, \dots, g_\ell \in \mathbb{C}[w_1, \dots, w_{n+1}]$ be the generators of $\check{J}_{A'}$ defined above Lemma 3.9.*

1. *The $(g_i)_{i=1, \dots, \ell}$ together with $(\check{E}'_k + \beta_k)_{k=1, \dots, d}$ form a Gröbner basis of $\check{J}_{A'}$ with respect to the weight vector $(u, v) = (0, \dots, 0, 1, \dots, 1)$.*
2. *Let $(u, v) = (0, \dots, 0, 1, \dots, 1)$ and set $\tilde{g}_i := in_{(u, v)}(g_i)$ and $\tilde{E}'_k = in_{(u, v)}(\check{E}'_k + \beta_k)$. The elements $(\tilde{g}_i)_{i=1, \dots, \ell}$ and $(\tilde{E}'_k)_{k=1, \dots, d}$ form a Gröbner-basis of*

$$\begin{aligned} in_{(u, v)}(\check{J}_{A'}) &= in_{(u, v)}((g_1, \dots, g_\ell, \check{E}'_1 + \beta_1, \dots, \check{E}'_d + \beta_d)) \\ &= (\tilde{g}_1, \dots, \tilde{g}_\ell, \tilde{E}'_1, \dots, \tilde{E}'_d) \subset \mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}] \end{aligned}$$

with respect to the weight vector $(0, \dots, 0, -1, 0, \dots, 0, 1)$ (i.e. w_{n+1} has weight -1 and ξ_{n+1} has weight $+1$).

Proof. 1.) The set $(g_i)_{i=1, \dots, \ell}$ is a Gröbner basis for $\check{J}_{A'}$. Therefore the elements $in_u(g_i) = g_i$ generate $in_u(\check{J}_{A'}) = \check{J}_{A'}$. The elements $(g_i)_{i=1, \dots, \ell}$ and $(\check{E}'_k + \beta_k)_{k=1, \dots, d}$ generate $\check{J}_{A'}$ and the elements $(in_{u, v}(g_i))_{i=1, \dots, \ell}$ and $(in_{(u, v)}(\check{E}'_k + \beta_k))_{k=1, \dots, d}$ generate $fin_{(u, v)}$ by definition. The claim follows now from Proposition 3.10 1. .

2.) It follows from the first point that the $\tilde{g}_i = in_{(u, v)}(g_i)$ and the $\tilde{E}'_k = in_{(u, v)}(\check{E}'_k + \beta_k)$ generate $in_{(u, v)}(\check{J}_{A'})$. We have to show that the $in_{(0, \dots, 0, -1, 0, \dots, 0, 1)}(\tilde{g}_i)$ for $i = 1, \dots, \ell$ and the $in_{(0, \dots, 0, -1, 0, \dots, 0, 1)}(\tilde{E}'_k)$ generate $in_{(0, \dots, 0, -1, 0, \dots, 0, 1)}(in_{(u, v)}(\check{J}_{A'}))$. But this follows from (cf. [SST00, Lemma 2.1.6 (2)])

$$\begin{aligned} in_{(0, \dots, 0, -1, 0, \dots, 0, 1)}(\tilde{g}_i) &= in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(g_i) && \text{for } i = 1, \dots, \ell \\ in_{(0, \dots, 0, -1, 0, \dots, 0, 1)}(\tilde{E}'_k) &= in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(\check{E}'_k + \beta_k) && \text{for } k = 1, \dots, d \\ in_{(0, \dots, 0, -1, 0, \dots, 0, 1)}(in_{(u, v)}(\check{J}_{A'})) &= in_{(0, \dots, 0, -e, 1, \dots, 1, 1+e)}(\check{J}_{A'}) \end{aligned}$$

for $0 < e \ll 1$ and Proposition 3.10 2. . □

The second notion we are going to introduce relates the order filtration F_\bullet on $D_{W'}$ with the V -filtration that already occurred in the last subsection. Here we consider the descending V -filtration on $D_{W'}$ with respect to $w_{n+1} = 0$, which we denote again by $V^\bullet D_{W'}$. We have

$$V^0 D_{W'} = \sum_{i, k \geq 0} (\partial_{w_{n+1}} w_{n+1})^i (w_{n+1})^k P_i$$

for $P_i \in \mathbb{C}[w_1, \dots, w_n] \langle \partial_{w_1}, \dots, \partial_{w_n} \rangle$ and

$$V^k D_{W'} = w_{n+1}^k V^0 D_{W'} \quad \text{and} \quad V^{-k} D_{W'} = \sum_{j \geq 0} \partial_{w_{n+1}}^j V^0 D_{W'} \quad (15)$$

for $k > 0$.

Recall the left ideal $\check{I}_{A'} \subset D_{W'}$ and the left $D_{W'}$ -modules $\check{M}_{A'}^\beta := D_{W'} / \check{I}_{A'}$ from above. We define filtrations V_{ind}^\bullet and F_p^{ord} on $\check{M}_{A'}^\beta$ by:

$$V_{ind}^k \check{M}_{A'}^\beta := \frac{V^k D_{W'} + \check{I}_{A'}}{\check{I}_{A'}} \quad \text{and} \quad F_p^{ord} \check{M}_{A'}^\beta := \frac{F_p D_{W'} + \check{I}_{A'}}{\check{I}_{A'}}.$$

The main result of this section is a compatibility between these filtrations.

Proposition 3.12. *Let $\mathbb{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0} A$, $\mathbb{N}A \neq \mathbb{Z}^d$ and $\check{M}_{A'}^\beta = D_{W'} / ((\check{\square}_m)_{m \in \mathbb{L}_{A'}} + (\check{E}'_k + \beta_k)_{k=1, \dots, d})$, then the map*

$$V^k D_{W'} \cap F_p D_{W'} \longrightarrow V_{ind}^k \check{M}_{A'}^\beta \cap F_p^{ord} \check{M}_{A'}^\beta.$$

is surjective.

Proof. Let $m \in V_{ind}^k \check{M}_{A'}^\beta \cap F_p^{ord} \check{M}_{A'}^\beta$. We can find $P, Q \in D_{W'}$ such that $P \in F_p D_{W'}$, $Q \in V^k D_{W'}$ and $[P] = m = [Q]$, i.e. $P = Q - i$ for some $i \in \check{I}_{A'}$. We have to find a Q' with $Q' \in V^k D_{W'} \cap F_p D_{W'}$ with $P = Q' - i'$ for $i' \in \check{I}_{A'}$. We will construct this element Q' by decreasing induction on the order of Q by killing its leading term in each step. For this we will use the special Gröbner basis of $\check{I}_{A'}$ which we constructed in Corollary 3.11 above.

Recall that the weight vector $(u, v) := (0, \dots, 0, 1, \dots, 1)$ induces the order filtration $F_\bullet^{(u, v)} = F_\bullet^{ord}$ on $D_{W'}$. If $R \in D_{W'}$ and $k := ord_{(u, v)}(R)$ we define the symbol of R by $\sigma_k(R) = in_{(u, v)}(R)$ and set $\sigma_q(R) = 0$ for $q \neq k$. We define a second weight vector $(u', v') := (0, \dots, 0, -1, 0, \dots, 0, 1)$ which induces the descending V -filtration from (15) on $D_{W'}$. The V -filtration and F -filtration also induce filtrations \tilde{V} and \tilde{F} on $Gr_\bullet^{(u, v)} D_{W'} = Gr_\bullet^F D_{W'} = \mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]$.

Let $t_Q := ord_{(u, v)} Q$, $t_i := ord_{(u, v)} i$ and set $t := \max(t_Q, t_i)$. Obviously we have $t \geq p$. If $t = p$ we are done. Hence, we assume $t > p$, thus we have

$$0 = \sigma_t(P) = \sigma_t(Q - i)$$

and therefore $t = t_Q = t_i$ which implies $\sigma_t(Q) = \sigma_t(i) \neq 0$. Set $k_Q := ord_{(u', v')}(\sigma_t(Q))$, then we have $\sigma_t(i) = \sigma_t(Q) \in \tilde{V}^{k_Q}$.

Recall from Corollary 3.11 that $\check{I}_{A'}$ is generated by $\{G_1, \dots, G_m\} := \{g_1, \dots, g_\ell, \check{E}'_1 + \beta_1, \dots, \check{E}'_d + \beta_d\}$ and these elements form a Gröbner basis with respect to weight vector (u, v) and their initial forms

$$\{\tilde{G}_1, \dots, \tilde{G}_m\} := \{in_{(u, v)}(G_1), \dots, in_{(u, v)}(G_m)\}$$

are a Gröbner basis of $in_{(u, v)}(\check{I}_{A'})$ with respect to the weight vector (u', v') . Therefore we can write

$$\sigma_t(i) = \sum_{l=1}^m \tilde{i}_l \tilde{G}_l$$

with $\tilde{i}_l \in \mathbb{C}[w_1, \dots, w_{n+1}, \xi_1, \dots, \xi_{n+1}]$. Using a commutative version of [SST00, Theorem 1.2.10] we can assume that $\tilde{i}_l \in \tilde{V}^{k_Q - k_l}$ where $k_l = ord_{(u', v')}(\tilde{G}_l)$. Since the elements \tilde{G}_l are homogeneous with respect to the variables ξ_1, \dots, ξ_{n+1} we can also assume that $\tilde{i}_l \in \tilde{F}_{t-t_l} Gr_\bullet^F D_{W'}$ where $t_l = ord_{(u, v)}(\tilde{G}_l)$. Let $i_l \in D_{W'}$ be the normally ordered element which we obtain from \tilde{i}_l by replacing ξ_i with ∂_{w_i} . One sees easily that $i_l G_l \in F_t D_{W'} \cap V^{k_Q} D_{W'}$. Therefore the element $i' := \sum_{l=1}^m i_l G_l$ has the following two properties

$$\sigma_t(i') = \sigma_t(i) = \sigma_t(Q) \quad \text{and} \quad i' \in V^{k_Q} D_{W'}$$

where the second property follows from $\text{ord}_{(u',v')}(G_I) = \text{ord}_{(u',v')}(\tilde{G}_I)$. We therefore have

$$P = Q - \tilde{i} - (i - \tilde{i})$$

with $Q - \tilde{i} \in F_{t-1}D_{W'} \cap V^{k_Q}D_{W'}$. The claim follows now by descending induction on the order t . \square

3.3 Calculation of the Hodge filtration

In this section we want to compute the Hodge filtration on the mixed Hodge module

$$\mathcal{H}^0(h_{A*} {}^p\mathbf{C}_T^{\beta,H}),$$

recall from section 1 that ${}^p\mathbf{C}_T^{\beta,H} = \mathbf{C}_T^{\beta,H}[d] \in \text{MHM}(T)$. Recall also that ${}^p\mathbf{C}_T^{\beta,H}$ has the underlying filtered \mathcal{D} -module $(\mathcal{O}_T^\beta, F_\bullet^H \mathcal{O}_T^\beta)$, where the Hodge filtration is given by

$$F_p^H \mathcal{O}_T^\beta = \begin{cases} \mathcal{O}_T^\beta & \text{for } p \geq 0 \\ 0 & \text{else.} \end{cases}$$

We will use several different presentations of \mathcal{O}_T^β as a \mathcal{D}_T -module, namely, for each $\alpha = (\alpha_k)_{k=1,\dots,d} \in \mathbb{Z}^d$ we have a \mathcal{D}_T -linear isomorphism

$$\Gamma(T, \mathcal{O}_T^\beta) \simeq \mathcal{D}_T / (\partial_{t_k} t_k + \beta_k + \alpha_k)_{k=1,\dots,d}$$

such that the Hodge filtration is simply the order filtration on the right hand side.

As we have seen in Lemma 3.1, the morphism h_A can be decomposed into the closed embedding $k_A : T \rightarrow W^* = W \setminus D$ and the canonical open embedding $l_A : W^* \rightarrow W$. We have to determine the Hodge filtration on the direct image modules for both mappings. The former is (after some coordinate change) a rather direct calculation, and will be carried out in Lemma 3.13 below. However, understanding the behaviour of the Hodge filtration under the direct image of an open embedding of the complement of a divisor (like the map l_A) is more subtle and at the heart of the theory of mixed Hodge modules (see, e.g., [Sai90, Section (2.b)]). More precisely, since the steps of the Hodge filtration of a mixed Hodge module are coherent modules over the structure sheaf of the underlying variety, the usual direct image functors are not suitable for the case of an open embedding as they do not preserve coherence. In order to circumvent this difficulty, one uses the canonical V -filtration along the boundary divisor, as computed in subsection 3.1 above. Let us give an overview of the strategy to be used below. The actual calculation will be finished only in Theorem 3.16, the main step being Proposition 3.15.

We will need the following formula (copied from [Sai93, Proposition 4.2.]) which describes the extension of a mixed Hodge-module over a smooth hypersurface. Let X be a smooth variety, let t, x_1, \dots, x_n be local coordinates on X and $j : Y \hookrightarrow X$ be a smooth hypersurface given by $t = 0$. Let ${}^H\mathcal{M}$ be a mixed Hodge module on $X \setminus Y$ with underlying filtered \mathcal{D} -module $(\mathcal{M}, F_\bullet^H \mathcal{M})$, then

$$F_p^H \mathcal{H}^0 j_+ \mathcal{M} = \sum_{i \geq 0} \partial_t^i F_{p-i}^H V^0 \mathcal{H}^0 j_+ \mathcal{M}, \quad \text{where} \quad F_q^H V^0 \mathcal{H}^0 j_+ \mathcal{M} := V^0 \mathcal{H}^0 j_+ \mathcal{M} \cap j_*(F_q^H \mathcal{M}), \quad (16)$$

here $V^0 \mathcal{H}^0 j_+ \mathcal{M}$ is the canonical V -filtration on the \mathcal{D} -module $\mathcal{H}^0 j_+ \mathcal{M} \simeq j_* \mathcal{M}$, as introduced in Definition 3.2.

If Y is a non-smooth hypersurface locally given by $g = 0$, we consider (locally) the graph embedding

$$\begin{aligned} i_g : X &\longrightarrow X \times \mathbb{C}_t \\ x &\mapsto (x, g(x)) \end{aligned}$$

together with its restriction $i_g^\circ : X \setminus Y \rightarrow X \times \mathbb{C}_t^*$. Notice that i_g° is a closed embedding. Given a mixed Hodge module \mathcal{M} on $X \setminus Y$ we proceed as follows. We first extend the Hodge filtration of $(i_g^\circ)_+ \mathcal{M}$ over the smooth divisor given by $\{t = 0\}$ as explained above. Afterwards we restrict the mixed Hodge module

which we obtained to the smooth divisor given by $\{t = g\}$.

After these general remarks, we come back to the situation of the torus embedding $h_A : T \rightarrow W$ described at the beginning of this section. Consider the following commutative diagram

$$\begin{array}{ccccc}
& & h_A & & \\
& \curvearrowright & & \curvearrowleft & \\
T & \xrightarrow{k_A} & W^* & \xrightarrow{l_A} & W & \xrightarrow{i_g} & W \times \mathbb{C}_t \\
& & & \searrow i_g^\circ & & \uparrow j_t \\
& & & & & W \times \mathbb{C}_t^*
\end{array}$$

where $W^* := W \setminus D = W \setminus \{w_1 \dots w_n = 0\} \simeq (\mathbb{C}^*)^n$ and where i_g is the graph embedding

$$\begin{aligned}
i_g : W &\longrightarrow W \times \mathbb{C}_t \\
w &\mapsto (w, w_1 \cdot \dots \cdot w_n)
\end{aligned} \tag{17}$$

associated to the function $g : W \rightarrow \mathbb{C}_t, w \mapsto w_1 \cdot \dots \cdot w_n$. Notice that $i_g \circ l_A$ factors over $W \times \mathbb{C}_t^*$. We have the following isomorphisms

$$i_{g+} h_{A+} \mathcal{O}_T^\beta \simeq i_{g+} l_{A+} k_{A+} \mathcal{O}_T^\beta \simeq j_{t+} i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta.$$

Lemma 3.13. *The direct image $\mathcal{H}^0 k_{A+} \mathcal{O}_T^\beta$ is isomorphic to the cyclic \mathcal{D}_{W^*} -module*

$${}^* \check{\mathcal{M}}_{A'}^\beta := \mathcal{D}_{W^*} / \check{\mathcal{I}}_\beta^*$$

where $\check{\mathcal{I}}_\beta^*$ is the left ideal generated by $(\check{E}_k + \beta_k)_{k=1, \dots, d}$ for $\beta = (\beta_k)_{k=1, \dots, d} \in \mathbb{R}^d$ and $(\check{\square}_m)_{m \in \mathbb{L}_A}$. Furthermore, the Hodge-filtration on $\check{\mathcal{M}}_{A'}$ is equal to the induced order filtration, shifted by $n - d$, i.e.

$$F_p^H {}^* \check{\mathcal{M}}_{A'} = F_{p-(n-d)}^{ord} \mathcal{D}_{W^*} / \check{\mathcal{I}}_\beta^*.$$

Proof. We factorize the map k_A from above in the following way. Let $A = C \cdot E \cdot F$ be the Smith normal form of A , i.e. $C = (c_{pq}) \in GL(d, \mathbb{Z})$, $F = (f_{uv}) \in GL(n, \mathbb{Z})$ and $E = (I_d, 0_{d, n-d})$. This gives rise to the maps

$$\begin{aligned}
k_C : T &\longrightarrow T \\
(t_1, \dots, t_d) &\mapsto (\tilde{t}_1, \dots, \tilde{t}_d) = (\underline{t}^{c_1}, \dots, \underline{t}^{c_d}) \\
k_E : T &\longrightarrow (\mathbb{C}^*)^n \\
(\tilde{t}_1, \dots, \tilde{t}_d) &\mapsto (\tilde{w}_1, \dots, \tilde{w}_n) = (\tilde{t}_1, \dots, \tilde{t}_d, 1, \dots, 1) \\
k_F : (\mathbb{C}^*)^s &\longrightarrow W^* \\
(\tilde{w}_1, \dots, \tilde{w}_n) &\mapsto (w_1, \dots, w_n) = (\underline{\tilde{w}}^f, \dots, \underline{\tilde{w}}^f).
\end{aligned}$$

For $\gamma \in \mathbb{Z}^d$ we have

$$k_{A+} \mathcal{O}_T^\gamma \simeq (k_F \circ k_E \circ k_C)_+ \mathcal{O}_T^\gamma \simeq k_{F+} k_{E+} k_{C+} \mathcal{O}_T^\gamma.$$

Since all maps and all spaces involved are affine, we will work on the level of global sections. We have $\Gamma(T, \mathcal{O}_T^\gamma) = D_T / (\partial_{t_k} t_k + \gamma_k)_{k=1, \dots, d} = D_T / (t_k \partial_{t_k} + \gamma_k + 1)_{k=1, \dots, d}$. Notice that the Hodge filtration in this presentation is simply the order filtration. Since k_C is a change of coordinates we have $\Gamma(T, \mathcal{H}^0(k_C)_+ \mathcal{O}_T^\gamma) \simeq D_T / (\sum_{i=1}^d c_{ki} \tilde{t}_i \partial_{\tilde{t}_i} + \gamma_k + 1)_{k=1, \dots, d}$ and again the Hodge filtration is equal to the order filtration. We now calculate $\Gamma((\mathbb{C}^*)^n, k_{E+} k_{C+} \mathcal{O}_T^\gamma)$. We have

$$\begin{aligned}
&\Gamma((\mathbb{C}^*)^n, \mathcal{H}^0 k_{E+} k_{C+} \mathcal{O}_T^\gamma) \simeq \Gamma(T, \mathcal{H}^0 k_{C+} \mathcal{O}_T^\gamma) [\partial_{\tilde{w}_{d+1}}, \dots, \partial_{\tilde{w}_n}] \\
&\simeq D_{(\mathbb{C}^*)^n} / \left(\sum_{i=1}^d c_{ki} \tilde{w}_i \partial_{\tilde{w}_i} + \gamma_k + 1 \right)_{k=1, \dots, d}, (\tilde{w}_i - 1)_{i=d+1, \dots, n}.
\end{aligned} \tag{18}$$

The Hodge-filtration is (cf. [Sai93, Formula (1.8.6)])

$$\begin{aligned} F_{p+(n-d)}^H(\Gamma((\mathbb{C}^*)^n, \mathcal{H}^0 k_{E+k_{C+}} \mathcal{O}_T^\gamma)) &= \sum_{p_1+p_2=p} F_{p_1}^H \Gamma(T, \mathcal{H}^0 k_{C+} \mathcal{O}_T^\gamma) \otimes \underline{\partial}^{p_2} \\ &= \sum_{p_1+p_2=p} F_{p_1}^{ord} \Gamma(T, \mathcal{H}^0 k_{C+} \mathcal{O}_T^\gamma) \otimes \underline{\partial}^{p_2} = F_p^{ord}(\Gamma((\mathbb{C}^*)^n, \mathcal{H}^0 k_{E+k_{C+}} \mathcal{O}_T^\gamma)). \end{aligned} \quad (19)$$

Hence we see that the Hodge filtration on the presentation (18) shifted by $(n-d)$ is equal to the order filtration, i.e. $F_{p+(n-d)}^H = F_p^{ord}$.

The map k_F is again a change of coordinates, so we have

$$\begin{aligned} \Gamma((\mathbb{C}^*)^n, \mathcal{H}^0 k_{F+k_{E+k_{C+}}} \mathcal{O}_T^\gamma) &\simeq D_{(\mathbb{C}^*)^n} / \left(\left(\sum_{j=1}^n a_{kj} w_j \partial_{w_j} + \gamma_k + 1 \right)_{k=1, \dots, d}, (\underline{w}^{\underline{m}_i} - 1)_{i=d+1, \dots, n} \right) \\ &\simeq D_{(\mathbb{C}^*)^n} / \left(\left(\sum_{j=1}^n a_{kj} w_j \partial_{w_j} + \gamma_k + 1 \right)_{k=1, \dots, d}, (\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_A} \right), \end{aligned} \quad (20)$$

where \underline{m}_i are the columns of the inverse matrix $M = F^{-1}$. The first isomorphism follows from the equality $A = C \cdot E \cdot F$. The second isomorphism follows from the fact that an element $\underline{m} \in \mathbb{Z}^n$ is a relation between the columns of A if and only if it is a relation between the columns of $E \cdot F$. So the Hodge filtration on the presentation (20) shifted by $(n-d)$ is again the order filtration. We have

$$\sum_{j=1}^n a_{kj} w_j \partial_{w_j} + \gamma_k + 1 = \sum_{j=1}^n a_{kj} \partial_{w_j} w_j - \sum_{j=1}^n a_{kj} + \gamma_k + 1.$$

Setting $\gamma_k := \sum_{j=1}^n a_{kj} + \beta_k - 1$, shows that

$$*\check{\mathcal{M}}_{A'}^\beta = \mathcal{H}^0 k_{A+} \mathcal{O}_T^{\sum_{j=1}^n a_j + \beta - 1} \simeq \mathcal{H}^0 k_{A+} \mathcal{O}_T^\beta \quad (21)$$

where the last isomorphism is given by right multiplication with $\underline{t}^{-\sum_{j=1}^n a_j + 1}$ (here $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^d$). \square

The next step is to compute the Hodge filtration of $h_{A+} \mathcal{O}_T^\beta \simeq l_{A+} k_{A+} \mathcal{O}_T^\beta$ from that of $k_{A+} \mathcal{O}_T^\beta$. As the map l_A is an open embedding of the complement of a normal crossing divisor, we need to consider the graph embedding embedding i_g° with respect to the function $g = w_1 \cdots w_n$. We proceed as described in the beginning of this section, i.e., we first extend the module $i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta$ over the smooth divisor $\{t=0\}$.

Lemma 3.14. *The direct image $i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta$ is isomorphic to the cyclic $\mathcal{D}_{W \times \mathbb{C}_t^*}$ -module $\mathcal{D}_{W \times \mathbb{C}_t^*} / \mathcal{I}^\circ$ where \mathcal{I}° is the left ideal generated by $(\check{E}'_k + \beta_k)_{k=1, \dots, d}$ for $\beta = (\beta_k)_{k=1, \dots, d} \in \mathbb{R}^d$ and $(\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_{A'}}$. Recall that the vector fields \check{E}'_k have been defined in formula (8) as $\check{E}'_k := \sum_{i=1}^n b_{ki} \partial_{w_i} w_i + c_k \partial_t t$ for $k = 1, \dots, d$. Furthermore, the Hodge filtration shifted by $n-d+1$ is equal to the induced order filtration, i.e., we have*

$$F_p^H i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta \simeq F_{p-(n-d+1)}^{ord} \mathcal{D}_{W \times \mathbb{C}_t^*} / \mathcal{I}^\circ.$$

Proof. We define

$$\widetilde{W} := (W^* \times \mathbb{C}_t^*) \setminus \{\tilde{t} + g(\underline{w}) = 0\}$$

and factor the map i_g° in the following way. Set

$$\begin{aligned} l_1 : W^* &\longrightarrow \widetilde{W} \\ &\underline{w} \mapsto (\underline{w}, 0) \\ l_2 : \widetilde{W} &\longrightarrow W^* \times \mathbb{C}_t^* \\ &(\underline{w}, \tilde{t}) \mapsto (\underline{w}, \tilde{t} + g(\underline{w})) \end{aligned}$$

and let $l_3 : W^* \times \mathbb{C}_t^* \rightarrow W \times \mathbb{C}_t^*$ be the canonical inclusion. We have $i_g^\circ = l_3 \circ l_2 \circ l_1$. For the convenience of the reader, let us summarize these maps in the following diagram

$$\begin{array}{ccccc}
W^* & \xrightarrow{\substack{\text{open embedding} \\ \text{complement of NCD}}} & W & \xrightarrow{i_g} & W \times \mathbb{C}_t \\
\downarrow \substack{\text{closed} \\ l_1} & \searrow l_A & & \nearrow i_g^\circ & \uparrow \substack{\text{open embedding} \\ \text{complement of smooth divisor}} \\
\widetilde{W} & \xrightarrow[l_2]{\simeq} & W^* \times \mathbb{C}_t^* & \xrightarrow[l_3]{\text{closed on support}} & W \times \mathbb{C}_t^*
\end{array}$$

Notice again that all spaces involved are affine, hence we will work with the modules of global sections. Since l_1 is just the inclusion of a coordinate hyperplane we have

$$\Gamma(\widetilde{W}, \mathcal{H}^0 l_{1+} k_{A+} \mathcal{O}_T^\beta) \simeq \Gamma(W^*, \mathcal{H}^0 k_{A+} \mathcal{O}_T^\beta)[\partial_{\tilde{t}}].$$

The Hodge-filtration is given by

$$\Gamma(\widetilde{W}, F_{p+1}^H(\mathcal{H}^0 l_{1+} k_{A+} \mathcal{O}_T^\beta)) \simeq \sum_{p_1+p_2=p} \Gamma(W^*, F_{p_1}^H \mathcal{H}^0 k_{A+} \mathcal{O}_T^\beta) \otimes \partial_{\tilde{t}}^{p_2}. \quad (22)$$

Notice that $\Gamma(\widetilde{W}, \mathcal{H}^0 l_{1+} k_{A+} \mathcal{O}_T^\beta) \simeq D_{\widetilde{W}} / I_1^\circ$ where I_1° is the left ideal generated by $(\check{E}_k + \beta_k)_{k=1, \dots, d}$, $(\check{\square}_m)_{m \in \mathbb{L}_A}$ and \tilde{t} .

Under this isomorphism the Hodge filtration on $\Gamma(\widetilde{W}, \mathcal{H}^0 l_{1+} k_{A+} \mathcal{O}_T^\beta)$ shifted by $(n-d)+1$ is equal to the order filtration by Lemma 3.13 and (22). The map l_2 is just a change of coordinates, hence under the substitutions $\tilde{t} \mapsto t = \tilde{t} + g(\underline{w})$, $w_i \partial_{w_i} \mapsto w_i \partial_{w_i} + g(\underline{w}) \partial_t \equiv w_i \partial_{w_i} + \partial_t$ for $i = 1, \dots, n$ and by using the presentation of $\mathcal{H}^0(k_{A+} \mathcal{O}_T^\beta)$ as acyclic \mathcal{D} -module, we get that

$$\begin{aligned}
\Gamma(W^* \times \mathbb{C}_t^*, \mathcal{H}^0 l_{2+} l_{1+} k_{A+} \mathcal{O}_T^\beta) &\simeq D_{W^* \times \mathbb{C}_t^*} / ((\check{E}'_k + \beta_k)_{k=1, \dots, d} + (\check{\square}_m)_{m \in \mathbb{L}_A} + (t - w_1 \cdots w_n)) \\
&\simeq D_{W^* \times \mathbb{C}_t^*} / ((\check{E}'_k + \beta_k)_{k=1, \dots, d} + (\check{\square}_m)_{m \in \mathbb{L}_{A'}}), \quad (23)
\end{aligned}$$

where \check{E}'_k was defined in formula (8) and A' is the matrix defined just before that formula. Notice that the Hodge filtration shifted by $(n-d)+1$ is again equal to the order filtration.

Since the support of $\mathcal{H}^0 l_{2+} l_{1+} k_{A+} \mathcal{O}_T^\beta$ lies in the subvariety $\{t = g(\underline{w})\}$, the closure of the support in $W \times \mathbb{C}_t^*$ does not meet $D \times \mathbb{C}_t^*$. We conclude that

$$\begin{aligned}
\Gamma(W \times \mathbb{C}_t^*, \mathcal{H}^0 i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta) &\simeq \Gamma(W \times \mathbb{C}_t^*, \mathcal{H}^0 l_{3+} l_{2+} l_{1+} k_{A+} \mathcal{O}_T^\beta) \\
&\simeq \Gamma(W^* \times \mathbb{C}_t^*, \mathcal{H}^0 l_{2+} l_{1+} k_{A+} \mathcal{O}_T^\beta) \\
&\simeq D_{W^* \times \mathbb{C}_t^*} / ((\check{E}'_k + \beta_k)_{k=1, \dots, d} + (\check{\square}_m)_{m \in \mathbb{L}_{A'}}) \\
&\simeq D_{W \times \mathbb{C}_t^*} / ((E'_k + \beta_k)_{k=1, \dots, d} + (\check{\square}_m)_{m \in \mathbb{L}_{A'}}).
\end{aligned}$$

The Hodge filtration is then simply extended by using the following formula

$$F_p^H \mathcal{H}^0 i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta \simeq F_p^H \mathcal{H}^0 l_{3+} l_{2+} l_{1+} k_{A+} \mathcal{O}_T^\beta \simeq l_{3*} F_p^H \mathcal{H}^0 l_{2+} l_{1+} k_{A+} \mathcal{O}_T^\beta.$$

□

Proposition 3.15. *Let $\beta \in \mathbb{R}^d \setminus sRes(A)$. The direct image $\mathcal{H}^0 j_{t+} i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta$ is isomorphic to the quotient $\check{\mathcal{M}}_{A'}^\beta = D_{W \times \mathbb{C}_t} / \mathcal{I}'$, where \mathcal{I}' is the left ideal which is generated by $(\check{E}'_k + \beta_k)_{k=1, \dots, d}$ and $(\check{\square}_m)_{m \in \mathbb{L}_{A'}}$. Furthermore, if $\beta \in \mathfrak{A}_A$, then the Hodge-filtration on $D_{W \times \mathbb{C}_t} / \mathcal{I}'$ shifted by $(n-d)+1$ is equal to the induced order filtration, that is,*

$$F_p^H \mathcal{H}^0 j_{t+} i_{g+}^\circ k_{A+} \mathcal{O}_T^\beta = F_{p-(n-d+1)}^{ord} D_{W \times \mathbb{C}_t} / \mathcal{I}'.$$

Proof. First recall that we have an isomorphism $\mathcal{H}^0 j_{t+} i_{g+}^{\circ} k_{A+} \mathcal{O}_T^{\beta} \simeq \mathcal{H}^0 i_{g+} h_{A+} \mathcal{O}_T^{\beta}$. The composed map $i_g \circ h_A$ is a torus embedding given by the matrix A' . Hence, we have an isomorphism $\mathcal{H}^0 j_{t+} i_{g+}^{\circ} k_{A+} \mathcal{O}_T^{\beta} \simeq \check{\mathcal{M}}_{A'}^{\beta}$, for $\beta \in \mathbb{R}^d$ and $\beta \notin sRes(A') = sRes(A)$. This shows the first claim.

For the second statement, suppose that $\beta \in \mathfrak{A}_A$. The formula for extending the Hodge filtration over the smooth divisor $\{t = 0\}$ is

$$F_p^H \check{\mathcal{M}}_{A'}^{\beta} = \sum_{i \geq 0} \partial_t^i (V^0 \check{\mathcal{M}}_{A'}^{\beta} \cap j_{t*} j_t^{-1} F_{p-i}^H \check{\mathcal{M}}_{A'}^{\beta}). \quad (24)$$

On the level of global sections the adjunction morphism $\check{\mathcal{M}}_{A'}^{\beta} \rightarrow j_{t+} j_t^+ \check{\mathcal{M}}_{A'}^{\beta}$ is given by the inclusion $\check{\mathcal{M}}_{A'}^{\beta} \rightarrow {}^* \check{\mathcal{M}}_{A'}^{\beta}$, where ${}^* \check{\mathcal{M}}_{A'}^{\beta}$ is the $D_{W \times \mathbb{C}_t}$ -module from Lemma 3.14 seen as a $D_{W \times \mathbb{C}_t}$ -module. Hence on the level of global section formula (24) becomes

$$F_p^H \check{\mathcal{M}}_{A'}^{\beta} = \sum_{i \geq 0} \partial_t^i (V^0 \check{\mathcal{M}}_{A'}^{\beta} \cap F_{p-i}^H {}^* \check{\mathcal{M}}_{A'}^{\beta}).$$

Since we have $F_{p+(n-d+1)}^H {}^* \check{\mathcal{M}}_{A'}^{\beta} = F_p^{ord} {}^* \check{\mathcal{M}}_{A'}^{\beta}$ by the same lemma, we conclude that $F_{n-d}^H \check{\mathcal{M}}_{A'}^{\beta} = 0$. The element $1 \in \check{\mathcal{M}}_{A'}^{\beta}$ is in $V^0 \check{\mathcal{M}}_{A'}^{\beta}$ by Proposition 3.7 and $1 \in F_{(n-d)+1}^H {}^* \check{\mathcal{M}}_{A'}^{\beta} = F_0^{ord} {}^* \check{\mathcal{M}}_{A'}^{\beta}$ and therefore $1 \in F_{(n-d)+1}^H \check{\mathcal{M}}_{A'}^{\beta}$. Notice that both $(\check{\mathcal{M}}_{A'}^{\beta}, F_{\bullet}^H)$ and $(\check{\mathcal{M}}_{A'}^{\beta}, F_{\bullet}^{ord})$ are cyclic, well-filtered $D_{W \times \mathbb{C}_t}$ -modules (see e.g. [Sai88, Section 2.1.1]), therefore we can conclude

$$F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta} \subset F_{p+(n-d+1)}^H \check{\mathcal{M}}_{A'}^{\beta}.$$

In order to show the converse inclusion, we have to show

$$F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta} \supset F_{p+(n-d+1)}^H \check{\mathcal{M}}_{A'}^{\beta} = \sum_{i \geq 0} \partial_t^i (V^0 \check{\mathcal{M}}_{A'}^{\beta} \cap F_{p+(n-d+1)-i}^H {}^* \check{\mathcal{M}}_{A'}^{\beta}) = \sum_{i \geq 0} \partial_t^i (V_{ind}^0 \check{\mathcal{M}}_{A'}^{\beta} \cap F_{p-i}^{ord} {}^* \check{\mathcal{M}}_{A'}^{\beta})$$

for all $p \geq 0$, where the last equality follows from Proposition 3.7 and Lemma 3.14. Since we have

$$F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta} \supset \partial_t^i F_{p-i}^{ord} \check{\mathcal{M}}_{A'}^{\beta} \text{ for } 0 \leq i \leq p \text{ and } p \geq 0$$

it remains to show

$$F_{p-i}^{ord} \check{\mathcal{M}}_{A'}^{\beta} \supset V_{ind}^0 \check{\mathcal{M}}_{A'}^{\beta} \cap F_{p-i}^{ord} {}^* \check{\mathcal{M}}_{A'}^{\beta} \text{ for } 0 \leq i \leq p \text{ and } p \geq 0$$

resp.

$$F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta} \supset V_{ind}^0 \check{\mathcal{M}}_{A'}^{\beta} \cap F_p^{ord} {}^* \check{\mathcal{M}}_{A'}^{\beta} \text{ for } p \geq 0.$$

Now let $[P] \in V_{ind}^0 \check{\mathcal{M}}_{A'}^{\beta} \cap F_p^{ord} {}^* \check{\mathcal{M}}_{A'}^{\beta}$, then $P \in D_{W \times \mathbb{C}_t}$ can be written as

$$P = t^{-k} P_k + t^{-k+1} P_{k-1} + \dots$$

with $P_i \in \mathbb{C}[w_1, \dots, w_n] \langle \partial_t, \partial_{w_1}, \dots, \partial_{w_n} \rangle$ and $P_k \neq 0$. Since $t^k \cdot [P] \in V_{ind}^k \check{\mathcal{M}}_{A'}^{\beta} \cap F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta}$, it is enough to prove

$$t^k F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta} \supset V_{ind}^k \check{\mathcal{M}}_{A'}^{\beta} \cap F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta} \text{ for } p \geq 0.$$

Given an element $[Q] \in V_{ind}^k \check{\mathcal{M}}_{A'}^{\beta} \cap F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta}$, we can find, using Lemma 3.12, a $Q' \in V^k D_{W \times \mathbb{C}_t} \cap F_p D_{W \times \mathbb{C}_t}$ with $[Q] = [Q']$. But this element Q' can be written as a linear combination of $t^{l_0} w_1^{l_1} \dots w_n^{l_n} \partial_t^{p_0} \partial_{w_1}^{p_1} \dots \partial_{w_n}^{p_n}$ with $p_0 + \dots + p_n \leq p$ and $l_0 - p_0 \geq k$, hence $[Q'] \in t^k F_p^{ord} \check{\mathcal{M}}_{A'}^{\beta}$. This shows the statement in the case $\mathbb{N}A \neq \mathbb{Z}^d$.

In the case $\mathbb{N}B = \mathbb{Z}^d$ the support of $\check{\mathcal{M}}_{A'}^{\beta}$ is disjoint from the divisor $\{t = 0\}$, hence the extension of the Hodge filtration is simply given by $F_p^H \check{\mathcal{M}}_{A'}^{\beta} = j_{t*} F_p^H {}^* \check{\mathcal{M}}_{A'}^{\beta}$. Since j_t is an open embedding we have $\check{\mathcal{M}}_{A'}^{\beta} = {}^* \check{\mathcal{M}}_{A'}^{\beta}$ and therefore also $F_p^H \check{\mathcal{M}}_{A'}^{\beta} = F_p^H {}^* \check{\mathcal{M}}_{A'}^{\beta}$. This shows the claim. \square

Now we want to deduce the Hodge filtration on $h_{A+}\mathcal{O}_T^\beta$ from the proposition above.

Theorem 3.16. *Let $\mathbf{N}A = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A$ and $\beta \in \mathbb{R}^d \setminus sRes(A)$. The direct image $h_{A+}\mathcal{O}_T^\beta$ is isomorphic to the cyclic D_W -module $\check{\mathcal{M}}_A^\beta := \mathcal{D}_W/\check{\mathcal{I}}$, where $\check{\mathcal{I}}$ is the left ideal generated by $(\check{E}_k + \beta_k)_{k=1,\dots,d}$ and $(\check{\square}_m)_{m \in \mathbb{L}_A}$. For $\beta \in \mathfrak{A}_A$ the Hodge filtration on $\check{\mathcal{M}}_A^\beta$ is equal to the order filtration shifted by $n-d$, i.e.*

$$F_{p+(n-d)}^H \check{\mathcal{M}}_A^\beta = F_p^{ord} \check{\mathcal{M}}_A^\beta.$$

Proof. Recall that we have $j_t \circ i_g^\circ \circ k_A = i_g \circ h_A$ where i_f is the graph embedding from (17). The map i_f can be factored by

$$\begin{aligned} i_0 : W &\longrightarrow W \times \mathbb{C}_{\tilde{t}} \\ \underline{w} &\mapsto (\underline{w}, 0) \\ l_g : W \times \mathbb{C}_{\tilde{t}} &\longrightarrow W \times \mathbb{C}_t \\ (\underline{w}, \tilde{t}) &\mapsto (\underline{w}, \tilde{t} + g(\underline{w})). \end{aligned}$$

Once again, we summarize the relevant maps in the following diagram.

$$\begin{array}{ccccc} & & W \times \mathbb{C}_{\tilde{t}} & & \\ & & \uparrow \text{closed } i_0 & \searrow \simeq & \\ W^* & \xrightarrow{l} & W & \xrightarrow[\text{closed}]{i_g} & W \times \mathbb{C}_t \\ & \searrow i_g^\circ & & & \uparrow \text{open embedding} \\ \widetilde{W} & \xrightarrow[l_2]{\simeq} & W^* \times \mathbb{C}_t^* & \xrightarrow[\text{closed on support}]{l_3} & W \times \mathbb{C}_t^* \\ & & & & \uparrow \text{complement of smooth divisor } j_t \end{array}$$

We first compute $\mathcal{H}^0(l_g^{-1})_+ \check{\mathcal{M}}_{A'}^\beta$, with its corresponding Hodge filtration. Since $(l_g)^{-1}$ is just a coordinate change we get similarly to formula (23)

$$\Gamma(W \times \mathbb{C}_{\tilde{t}}, \mathcal{H}^0(l_g^{-1})_+ \check{\mathcal{M}}_{A'}^\beta) \simeq D_{W \times \mathbb{C}_{\tilde{t}}} / ((\check{E}_k + \beta_k)_{k=1,\dots,d}, (\check{\square}_m)_{m \in \mathbb{L}_A}, (\tilde{t})), \quad (25)$$

where the Hodge filtration on the right hand side is the induced order filtration shifted by $(n-d+1)$. Notice that the right hand side of (25) is simply $\check{\mathcal{M}}_A^\beta[\partial_{\tilde{t}}]$, hence the Hodge filtration on

$$\check{\mathcal{M}}_A^\beta = \Gamma(W, \check{\mathcal{M}}_A^\beta) = \Gamma(W \times \mathbb{C}_{\tilde{t}}, \mathcal{H}^0 i_0^+(l_g^{-1})_+ \check{\mathcal{M}}_{A'}^\beta)$$

is simply the order filtration shifted by $(n-d)$ by [Sai88, Proposition 3.2.2 (iii)]. \square

Remark 3.17. *Let $\mathcal{O}_{\mathbb{C}^*}^\beta = \mathcal{D}_{\mathbb{C}^*}/(\partial_t t + \beta)$ and $j_0 : \mathbb{C}^* \rightarrow \mathbb{C}$ be the inclusion. If $\beta \notin \{-1, -2, -3, \dots\}$ then*

$$j_{0+}\mathcal{O}_{\mathbb{C}^*}^\beta \simeq \mathcal{D}_{\mathbb{C}}/(\partial_t t + \beta)$$

as well as

$$j_{0\dagger}\mathcal{O}_{\mathbb{C}^*}^{-\beta-1} \simeq \mathbb{D}j_{0+}\mathbb{D}\mathcal{O}_{\mathbb{C}^*}^\beta \simeq \mathbb{D}j_{0+}\mathcal{D}_{\mathbb{C}^*}/(\partial_t t + \beta) \simeq \mathbb{D}\mathcal{D}_{\mathbb{C}}/(\partial_t t + \beta) \simeq \mathcal{D}_{\mathbb{C}}/(t\partial_t - \beta)$$

If additionally $\beta \in (-1, 0]$ holds then by Theorem 3.16

$$(j_{0+}\mathcal{O}_{\mathbb{C}^*}^\beta, F_{\bullet}^H) \simeq (\mathcal{D}_{\mathbb{C}}/(\partial_t t + \beta), F_{\bullet}^{ord})$$

In order to compute the Hodge filtration on $j_{0\dagger}\mathcal{O}_{\mathbb{C}^*}^{-\beta-1}$ we use that $(\mathbb{D}\mathcal{O}_{\mathbb{C}^*}^{-\beta-1}, F_p^H) = (\mathcal{D}_{\mathbb{C}^*}/(\partial_t t + \beta), F_{\bullet-1}^{ord})$ and therefore $(j_{0\dagger}\mathbb{D}\mathcal{O}_{\mathbb{C}^*}^{-\beta-1}, F_p^H) = (\mathcal{D}_{\mathbb{C}}/(\partial_t t + \beta), F_{\bullet-1}^{ord})$ holds, since we assumed $\beta \in (-1, 0]$. We use the filtered resolution

$$(\mathcal{D}_{\mathbb{C}^*}, F_{\bullet-2}^{ord}) \xrightarrow{\partial_t t + \beta} (\mathcal{D}_{\mathbb{C}^*}, F_{\bullet-1}^{ord})$$

to resolve $(\mathcal{D}_{\mathbb{C}}/(\partial_t t + \beta), F_{\bullet-1}^{ord})$ and apply $\mathcal{H}om(-, (\mathcal{D}_{\mathbb{C}}, F_{\bullet-2}^{ord}) \otimes \omega_X^{\vee})$ to compute the dual of $(\mathcal{D}_{\mathbb{C}}/(\partial_t t + \beta), F_{\bullet-1}^{ord})$ (cf. [Sai94, page 55] for the choice of filtration on $\mathcal{D}_{\mathbb{C}} \otimes \omega_X^{\vee}$). This gives

$$(j_{0\dagger} \mathcal{O}_{\mathbb{C}^*}^{-\beta-1}, F_p^H) \simeq (\mathcal{D}_{\mathbb{C}}/(t\partial_t - \beta), F_p^{ord})$$

for $\beta \in (-1, 0]$.

4 Integral transforms of torus embeddings

In this section we investigate how the Hodge filtration of $\mathbb{C}_T^{H,\beta}$ behaves under a certain integral transformation, which depends on a matrix \tilde{A} and a parameter β_0 . We show that the outcome of this transform is isomorphic to the GKZ-system $\mathcal{M}_{\tilde{A}}^{(\beta_0, \beta)}$ (cf. Proposition 4.4). In the special case of $\beta_0 \in \mathbb{Z}$ we will show that the integral transformation mentioned above is isomorphic to the Radon transform of a torus embedding.

The Radon transformation has been extensively used in the previous papers [Rei14] and [RS17] in order to study hyperplane sections of toric varieties, more precisely, fibres of Laurent polynomials and their compactifications.

We give in the first subsection below a brief reminder how certain GKZ-systems can be constructed using the Radon transformation. The second subsection introduces an integral transform how is able to produce all GKZ-systems $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$ with $\tilde{\beta} \notin sRes(\tilde{A})$ and homogeneous \tilde{A} .

The next subsections (until 4.9) constitute the main part of this section, where we study in detail how the various functors entering in the definition of this integral transformation act on the twisted structure sheaf. One can roughly divide the construction in two parts: In subsection 4.3 we calculate the push-forward of a tensor product between the twisted structure sheaf and a kernel to a partial compactification. Then one has to study the projection to the parameter space (i.e., the space on which the GKZ-system is defined). The calculation of the behavior of the filtration steps is non-trivial, as the higher direct images of these filtration steps, being coherent \mathcal{O} -modules, do not, a priori, vanish. However, we can show that this is actually the case in the current situation. We formulate this result in the language of \mathcal{R} -modules (i.e., using the Rees construction for filtered \mathcal{D} -modules), and make extensive use of (variants of the) Euler-Koszul complex of hypergeometric modules. All these intermediate steps are contained in the subsections 4.4 until 4.9. A very important technical result is the calculation of some local cohomology groups of a certain semi-group ring, contained in subsection 4.8. The culminating point is then Theorem 4.35 which gives a precise description of the Hodge filtration on certain GKZ-systems.

4.1 Hypergeometric modules, Gauß-Manin systems and the Radon transformation

Here we give a brief reminder on the relationship between GKZ-hypergeometric systems, Gauß-Manin systems of families of Laurent polynomials as developed in [Rei14].

As in [RS15, RS17], we will consider a homogenization of the above systems. Namely, given the matrix $A = (a_{ki})$, we consider the system $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$, where \tilde{A} is the $(d+1) \times (n+1)$ integer matrix

$$\tilde{A} := (\tilde{a}_0, \dots, \tilde{a}_n) := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{d1} & \dots & a_{dn} \end{pmatrix} \quad (26)$$

and $\tilde{\beta} \in \mathbb{C}^{d+1}$.

In order to show that such a homogenized GKZ-system comes from geometry we have to review briefly the so-called Radon transformation for \mathcal{D} -modules which was introduced by Brylinski [Bry86] and variants were later added by d'Agnolo and Eastwood [DE03].

Let W be the dual vector space of V with coordinates w_0, \dots, w_n and $\lambda_0, \dots, \lambda_n$, respectively. We will denote by $Z \subset \mathbb{P}(W) \times V$ the universal hyperplane given by $Z := \{\sum_{i=0}^n \lambda_i w_i = 0\}$ and by $U := (\mathbb{P}(W) \times V) \setminus Z$ its complement. Consider the following diagram

$$\begin{array}{ccccc}
& & U & & \\
& \swarrow^{\pi_1^U} & \downarrow^{j_U} & \searrow^{\pi_2^U} & \\
\mathbb{P}(W) & \xleftarrow{\pi_1} & \mathbb{P}(W) \times V & \xrightarrow{\pi_2} & V \\
& \swarrow_{\pi_1^Z} & \uparrow^{i_Z} & \searrow_{\pi_2^Z} & \\
& & Z & &
\end{array} \tag{27}$$

We will use in the sequel several variants of the so-called Radon transformation in the derived category of mixed Hodge modules. These are functors from $D^bMHM(\mathbb{P}(W))$ to $D^bMHM(\mathcal{D}_V)$ given by

$$\begin{aligned}
\mathcal{R}(M) &:= \pi_{2}^Z(\pi_1^Z)^* M \simeq \pi_{2*} i_{Z*} i_Z^* \pi_1^* M, \\
{}^l\mathcal{R}(M) &:= \mathbb{D} \circ *\mathcal{R} \circ \mathbb{D}(M) \simeq \pi_{2*}^Z(\pi_1^Z)^! M \simeq \pi_{2*} i_{Z*} i_Z^! \pi_1^! M \\
\mathcal{R}_{cst}(M) &:= \pi_{2} \pi_1^* M. \\
{}^l\mathcal{R}_{cst}(M) &:= \mathbb{D} \circ *\mathcal{R}_{cst} \circ \mathbb{D}(M) \simeq \pi_{2*} \pi_1^! M. \\
\mathcal{R}_c(M) &:= \pi_{2!}^U(\pi_1^U)^(M) \simeq \pi_{2*} j_{U!} j_U^* \pi_1^*(M) \\
{}^l\mathcal{R}^\circ(M) &:= \mathbb{D} \circ *\mathcal{R}_c^\circ \circ \mathbb{D}(M) \simeq \pi_{2*}^U(\pi_1^U)^!(M) \simeq \pi_{2*} j_{U*} j_U^! \pi_1^!(M).
\end{aligned}$$

The adjunction triangle corresponding to the open embedding j_U and the closed embedding i_Z gives rise to the following triangles of Radon transformations

$${}^l\mathcal{R}(M) \longrightarrow {}^l\mathcal{R}_{cst}(M) \longrightarrow {}^l\mathcal{R}^\circ(M) \xrightarrow{+1}, \tag{28}$$

$$*\mathcal{R}_c^\circ(M) \longrightarrow *\mathcal{R}_{cst}(M) \longrightarrow *\mathcal{R}(M) \xrightarrow{+1}, \tag{29}$$

where the second triangle is dual to the first.

We now introduce a family of Laurent polynomials defined on $T \times \Lambda := (\mathbb{C}^*)^d \times \mathbb{C}^n$ using the columns of the matrix A , more precisely, we put

$$\begin{aligned}
\varphi_A : T \times \Lambda &\longrightarrow V = \mathbb{C}_{\lambda_0} \times \Lambda, \\
(t_1, \dots, t_d, \lambda_1, \dots, \lambda_n) &\mapsto \left(-\sum_{i=1}^n \lambda_i t_i^{\underline{a}_i}, \lambda_1, \dots, \lambda_n\right).
\end{aligned} \tag{30}$$

The following theorem of [Rei14] constructs a morphism between the Gauß-Manin system $\mathcal{H}^0(\varphi_{A,+} \mathcal{O}_{T \times \Lambda})$ resp. its proper version $\mathcal{H}^0(\varphi_{A,\uparrow} \mathcal{O}_{T \times \Lambda})$ and certain GKZ-hypergeometric systems and identify both with a corresponding Radon transform.

For this we apply the triangle (28) to $M = g_! \mathbb{D}^p \mathbb{Q}_T^H$ and the triangle (29) to $M = g_*^p \mathbb{Q}_T^H$, where the map g was defined by

$$\begin{aligned}
g : T &\longrightarrow \mathbb{P}(W) \\
(t_1, \dots, t_d) &\mapsto (1 : \underline{t}^{\underline{a}_1} : \dots : \underline{t}^{\underline{a}_n}).
\end{aligned}$$

Theorem 4.1. [Rei14, Lemma 1.11, Proposition 3.4] *Let $A = (\underline{a}_1, \dots, \underline{a}_n) \in M(d \times n, \mathbb{Z})$ and $\tilde{A} = (\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n) \in M((d+1) \times (n+1), \mathbb{Z})$ as above and assume that the \tilde{A} satisfies*

1. $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$
2. $\mathbb{N}\tilde{A} = \mathbb{R}_{\geq 0}\tilde{A} \cap \mathbb{Z}^{d+1}$

Then for every $\tilde{\beta} \in \mathbb{N}\tilde{A}$ and every $\tilde{\beta}' \in \text{int}(\mathbb{N}\tilde{A})$, we have that

$$\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \simeq \text{DMod}(\mathcal{H}^{n+1}(*\mathcal{R}_c^\circ(g_*^p \mathbb{Q}_T^H))) \quad \text{and} \quad \mathcal{M}_{\tilde{A}}^{-\tilde{\beta}'} \simeq \text{DMod}(\mathcal{H}^{-n-1}({}^l\mathcal{R}^\circ(g_! \mathbb{D}^p \mathbb{Q}_T^H)))$$

If we define

$${}^H\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} := \mathcal{H}^{n+1}(*\mathcal{R}_c^\circ(g_*^p \mathbb{Q}_T^H)) \quad \text{and} \quad {}^H\mathcal{M}_{\tilde{A}}^{-\tilde{\beta}'} := \mathcal{H}^{-n-1}({}^l\mathcal{R}^\circ(g_! \mathbb{D}^p \mathbb{Q}_T^H))$$

the following sequences of mixed Hodge-modules are exact and dual to each other:

$$\begin{array}{ccccccc} H^{d-1}(T, \mathbb{C}) \otimes {}^p\mathbb{Q}_V^H & & \mathcal{H}^0(\varphi_{A^*} {}^p\mathbb{Q}_{T \times \Lambda}^H) & & {}^H\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} & & H^d(T, \mathbb{C}) \otimes {}^p\mathbb{Q}_V^H \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ 0 \longrightarrow \mathcal{H}^n(*\mathcal{R}_{cst}(g_*^p \mathbb{Q}_T^H)) & \longrightarrow & \mathcal{H}^n(*\mathcal{R}(g_*^p \mathbb{Q}_T^H)) & \longrightarrow & \mathcal{H}^{n+1}(*\mathcal{R}_c^\circ(g_*^p \mathbb{Q}_T^H)) & \longrightarrow & \mathcal{H}^{n+1}(*\mathcal{R}_{cst}(g_*^p \mathbb{Q}_T^H)) \longrightarrow 0 \\ \\ 0 \leftarrow \mathcal{H}^{-n}({}^l\mathcal{R}_{cst}(g_! \mathbb{D}^p \mathbb{Q}_T^H)) & \leftarrow & \mathcal{H}^{-n}({}^l\mathcal{R}(g_! \mathbb{D}^p \mathbb{Q}_T^H)) & \leftarrow & \mathcal{H}^{-n-1}({}^l\mathcal{R}^\circ(g_! \mathbb{D}^p \mathbb{Q}_T^H)) & \leftarrow & \mathcal{H}^{-n-1}({}^l\mathcal{R}_{cst}(g_! \mathbb{D}^p \mathbb{Q}_T^H)) \leftarrow 0. \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ H_{d+1}(T, \mathbb{C}) \otimes \mathbb{D}^p \mathbb{Q}_V^H & & \mathcal{H}^0(\varphi_{A^!} \mathbb{D}^p \mathbb{Q}_{T \times \Lambda}^H) & & {}^H\mathcal{M}_{\tilde{A}}^{-\tilde{\beta}'} & & H_d(T, \mathbb{C}) \otimes \mathbb{D}^p \mathbb{Q}_V^H \end{array}$$

Proposition 4.2. *Let $\tilde{\beta} \in \mathbb{N}\tilde{A}$ and $\tilde{\beta}' \in \text{int}(\mathbb{N}\tilde{A})$. There exists natural morphism of mixed Hodge modules between ${}^H\mathcal{M}_{\tilde{A}}^{-\tilde{\beta}'}$ ($-d-n$) and ${}^H\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$, which is (up to multiplication with a non-zero constant) given on the underlying \mathcal{D}_V -modules by*

$$\begin{aligned} \mathcal{M}_{\tilde{A}}^{-\tilde{\beta}'} &\longrightarrow \mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \\ P &\mapsto P \cdot \partial^{\tilde{\beta}+\tilde{\beta}'}, \end{aligned}$$

where $\partial^{\tilde{\beta}+\tilde{\beta}'} := \prod_{i=0}^n \partial_{\lambda_i}^{k_i}$ for any $\underline{k} = (k_0, \dots, k_n)$ with $\tilde{A} \cdot \underline{k} = \tilde{\beta} + \tilde{\beta}'$.

Proof. First notice that there is a natural morphism of mixed Hodge modules

$$\mathcal{H}^0(\varphi_{A^!} \mathbb{D}^p \mathbb{Q}_{T \times \Lambda}^H)(-d-n) \longrightarrow \mathcal{H}^0(\varphi_{A^*} {}^p\mathbb{Q}_{T \times \Lambda}^H)$$

which is induced by the morphism $\mathbb{D}^p \mathbb{Q}_{T \times \Lambda}^H(-d-n) \rightarrow {}^p\mathbb{Q}_{T \times \Lambda}^H$. Using the isomorphisms in the second column, this gives a morphism

$$\mathcal{H}^{-n}({}^l\mathcal{R}(g_! \mathbb{D}^p \mathbb{Q}_T^H))(-n-d) \longrightarrow \mathcal{H}^n(*\mathcal{R}(g_*^p \mathbb{Q}_T^H)).$$

Now we can concatenate this with the following morphisms

$$\begin{array}{ccc} \mathcal{H}^n(*\mathcal{R}(g_*^p \mathbb{Q}_T^H)) & \longrightarrow & \mathcal{H}^{n+1}(*\mathcal{R}_c^\circ(g_*^p \mathbb{Q}_T^H)) \\ \uparrow & & \uparrow \text{---} \hat{\text{---}} \text{---} \\ \mathcal{H}^{-n}({}^l\mathcal{R}(g_! \mathbb{D}^p \mathbb{Q}_T^H))(-n-d) & \longleftarrow & \mathcal{H}^{-n-1}({}^l\mathcal{R}^\circ(g_! \mathbb{D}^p \mathbb{Q}_T^H))(-n-d). \end{array}$$

This gives the desired morphism of mixed Hodge modules between ${}^H\mathcal{M}_{\tilde{A}}^{-\tilde{\beta}'}$ ($-d-n$) and ${}^H\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$. Then it follows from [RS17, Lemma 2.12] that the corresponding morphism of the underlying \mathcal{D}_V -modules is (up to multiplication by a non-zero constant) right multiplication with $\partial^{\tilde{\beta}+\tilde{\beta}'}$. \square

We will now prove a partial generalization of Theorem 4.1 for non-integer β .

Proposition 4.3. *With the notations as above, let $\tilde{\beta} = (\beta_0, \beta) \in (\mathbb{Z} \times \mathbb{R}^d) \setminus \text{sRes}(\tilde{A})$, then we have the following isomorphism*

$$\text{DMod}(\mathcal{H}^{n+1}(*\mathcal{R}_c^\circ(g_*^p \mathbb{C}_T^{\beta, H}))) \simeq \mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$$

This induces the structure of a complex mixed Hodge module on $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$ which we call ${}^H\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$.

Proof. Consider the following commutative diagram with cartesian square

$$\begin{array}{ccc}
& & W \\
& \nearrow h_{\tilde{A}} & \uparrow \bar{j} \\
\mathbb{C}^* \times T & \xrightarrow{\tilde{h}} & W \setminus \{0\} \\
\downarrow \pi_T & & \downarrow \pi \\
T & \xrightarrow{g} & \mathbb{P}(W)
\end{array}$$

where π_T is the projection to the first factor. We have

$$\bar{j}_+ \pi^+ g_+ \mathcal{O}_T^\beta \simeq \bar{j}_+ \tilde{h}_+ \pi_T^+ \mathcal{O}_T^\beta \simeq h_{\tilde{A}^+} \mathcal{O}_{\mathbb{C}^* \times T}^{(0, \beta)}[1] \simeq h_+ \mathcal{O}_{\mathbb{C}^* \times T}^{(\beta_0, \beta)}[1]$$

for every $\beta_0 \in \mathbb{Z}$. Let ${}^* \mathcal{R}_c^\circ : D_{rh}^b(\mathcal{D}_X) \rightarrow D_{rh}^b(\mathcal{D}_X)$ be the corresponding functor for \mathcal{D} -modules which is given by $M \mapsto \pi_{2+} j_{U^\dagger} j_U^\dagger \pi_1^\dagger M$. We have the following isomorphism

$${}^* \mathcal{R}_c^\circ(g_+ \mathcal{O}_T^\beta)[-n-1] \simeq \text{FL}(\bar{j}_+ \pi^+ g_+ \mathcal{O}_T^\beta[-1]) \simeq \text{FL}(h_{\tilde{A}^+} \mathcal{O}_{\mathbb{C}^* \times T}^{(\beta_0, \beta)}) \simeq \mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$$

where the first isomorphism follows from [DE03, Proposition 1]. Notice that the various shifts, occurring in the formulas above, stem from a different (shifted) definition of the (exceptional) inverse image for \mathcal{D} -modules in loc. cit. \square

4.2 Integral transforms of twisted structure sheaves

Unfortunately, the Radon transformation produces only GKZ-systems with $\beta_0 \in \mathbb{Z}$, as we can see in Proposition 4.3. To remedy this fact, we introduce a integral transformation which takes care of that by twisting with a kernel which depends on β_0 .

Let $T = (\mathbb{C}^*)^d$ resp. $\tilde{T} := (\mathbb{C}^*)^{d+1}$ be a tori with coordinates t_1, \dots, t_d resp. t_0, \dots, t_d and consider the torus embedding with respect to the matrix \tilde{A}

$$\begin{aligned}
h &:= h_{\tilde{A}} : \tilde{T} \longrightarrow \mathbb{C}^{n+1} \\
(t_0, \dots, t_d) &\mapsto (t_0, t_0 \underline{t}^{a_1}, \dots, t_0 \underline{t}^{a_n})
\end{aligned}$$

If $\tilde{\beta} \notin \text{sRes}(\tilde{A})$ the GKZ-system $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$ is given by $\text{FL}(h_+ \mathcal{O}_{\tilde{T}}^{\tilde{\beta}})$. Consider the maps

$$\begin{array}{ccccc}
\mathbb{C}^* & \xrightarrow{j} & \mathbb{C} & \xleftarrow{F} & T \times \mathbb{C}^{n+1} & \xrightarrow{q} & \mathbb{C}^{n+1} \\
& & & & \downarrow p & & \\
& & & & T & &
\end{array}$$

where F is given by $(t_1, \dots, t_d, \lambda_0, \dots, \lambda_n) \mapsto \lambda_0 + \sum_{i=1}^n \lambda_i \underline{t}^{a_i}$ and p_1 resp p_2 is the projection to the first resp. second factor and j is the inclusion. We have the following alternative description of the GKZ-system $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$.

Proposition 4.4. *Let $\tilde{\beta} = (\beta_0, \beta) \notin \text{sRes}(\tilde{A})$ then*

$$\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \simeq \text{FL}(h_+ \mathcal{O}_{\tilde{T}}^{\tilde{\beta}}) \simeq \mathcal{H}^{2n+d+1}(p_+(q^\dagger \mathcal{O}_T^\beta \otimes_{\mathcal{O}} F^\dagger(j_+ \mathcal{O}_{\mathbb{C}^*}^{-\beta_0-1})))$$

Proof. Notice that the morphism $h : \tilde{T} \longrightarrow \mathbb{C}^{n+1}$ factors by

$$\tilde{T} \xrightarrow{\tilde{j}} \mathbb{C} \times T \xrightarrow{k} W$$

where \tilde{j} is the canonical embedding and k is given by $(t_0, \dots, t_d) \mapsto (t_0, t_0 t_1^{\alpha_1}, \dots, t_0 t_d^{\alpha_n})$. Consider the diagram

$$\begin{array}{ccccc}
\mathbb{C} & \xleftarrow{l} & \mathbb{C} \times T & \xrightarrow{k} & W \\
\uparrow p_1 & & \uparrow p_{12} & & \uparrow q_1 \\
\mathbb{C} \times \mathbb{C} & \xleftarrow{id_{\mathbb{C}} \times F} & \mathbb{C} \times T \times V & \xrightarrow{k \times id_{\mathbb{C}^{n+1}}} & W \times V \\
\downarrow p_2 & & \downarrow p_{13} & & \downarrow q_2 \\
\mathbb{C} & \xleftarrow{F} & T \times V & \xrightarrow{p} & V
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C} \times T & & \\
\uparrow p_{12} & \searrow f & \\
\mathbb{C} \times T \times V & \xrightarrow{g} & T \\
\downarrow p_{13} & \nearrow q & \\
T \times V & &
\end{array}$$

where p_{ij} are the projections to the factors i and j , the maps l, q, p_1, q_1 are the projection to the first and the maps f, g, p, p_2, q_2 , are the projection to the second factor.

We have that $\tilde{j}_+ \mathcal{O}_{\tilde{T}}^{\tilde{\beta}} \simeq j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \boxtimes \mathcal{O}_T^{\beta} \simeq l^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0}[-d] \otimes f^+ \mathcal{O}_T^{\beta}[-1]$ hence we get the following isomorphisms

$$\begin{aligned}
\mathrm{FL}(h_+ \mathcal{O}_{\tilde{T}}^{\tilde{\beta}}) &\simeq \mathrm{FL}(k_+ \tilde{j}_+ \mathcal{O}_{\tilde{T}}^{\tilde{\beta}}) \\
&\simeq q_{2,+} (q_1^+ k_+ \tilde{j}_+ \mathcal{O}_{\tilde{T}}^{\tilde{\beta}} \otimes \mathcal{L})[-n-1] \\
&\simeq q_{2,+} (q_1^+ k_+ (l^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes f^+ \mathcal{O}_T^{\beta}) \otimes \mathcal{L})[-n-d-2] \\
&\simeq q_{2,+} ((k \times id)_+ p_{12}^+ (l^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes f^+ \mathcal{O}_T^{\beta}) \otimes \mathcal{L})[-n-d-2] \\
&\simeq q_{2,+} (k \times id)_+ (p_{12}^+ (l^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes f^+ \mathcal{O}_T^{\beta}) \otimes (k \times id)^+ \mathcal{L})[-2d-2] \\
&\simeq q_{2,+} (k \times id)_+ ((id \times F)^+ p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes g^+ \mathcal{O}_T^{\beta} \otimes (k \times id)^+ \mathcal{L})[-n-2d-3] \\
&\simeq p_+ p_{13,+} ((id \times F)^+ p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes g^+ \mathcal{O}_T^{\beta} \otimes (k \times id)^+ \mathcal{L})[-n-2d-3] \\
&\simeq p_+ p_{13,+} ((id \times F)^+ p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes g^+ \mathcal{O}_T^{\beta} \otimes (id \times F)^+ \mathcal{L}_1)[-3n-2d-3] \\
&\simeq p_+ p_{13,+} (g^+ \mathcal{O}_T^{\beta} \otimes (id \times F)^+ (p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes \mathcal{L}_1))[-2n-d-3] \\
&\simeq p_+ p_{13,+} (p_{13}^+ q^+ \mathcal{O}_T^{\beta} \otimes (id \times F)^+ (p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes \mathcal{L}_1))[-2n-d-3] \\
&\simeq p_+ (q^+ \mathcal{O}_T^{\beta} \otimes p_{13,+} (id \times F)^+ (p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes \mathcal{L}_1))[-2n-d-2] \\
&\simeq p_+ (q^+ \mathcal{O}_T^{\beta} \otimes F^+ p_{2,+} (p_1^+ j_+ \mathcal{O}_{\mathbb{C}^*}^{\beta_0} \otimes \mathcal{L}_1))[-2n-d-2] \\
&\simeq p_+ (q^+ \mathcal{O}_T^{\beta} \otimes F^+ j_{\dagger} \mathcal{O}_{\mathbb{C}^*}^{-\beta_0-1})[-2n-d-1] \\
&\simeq p_+ (q^{\dagger} \mathcal{O}_T^{\beta} \otimes F^{\dagger} j_{\dagger} \mathcal{O}_{\mathbb{C}^*}^{-\beta_0-1})[2n+d+1]
\end{aligned}$$

□

The isomorphism $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \simeq \mathcal{H}^{2n+d+1}(p_+(q^{\dagger} \mathcal{O}_T^{\beta} \otimes F^{\dagger} j_{\dagger} \mathcal{O}_{\mathbb{C}^*}^{-\beta_0-1}))$ which holds for $\tilde{\beta} \in \mathbb{R}^{d+1} \setminus sRes(\tilde{A})$ endows the GKZ-system with the structure of a complex mixed Hodge module. We define the mixed Hodge module structure by

$${}^H \mathcal{M}_{\tilde{A}}^{\tilde{\beta}} := \mathcal{H}^{2n+d+1}(p_*(q^* p \mathcal{C}_T^{\beta} \otimes F^* j_{\dagger} p \mathcal{C}_{\mathbb{C}^*}^{-\beta_0-1})) \quad (31)$$

Now we have to check if the definition (31) coincides with the one of Proposition 4.3 in the case $\beta_0 \in \mathbb{Z}$.

Proposition 4.5. *If $\beta_0 \in \mathbb{Z}$ and $(\beta_0, \beta) \notin sRes(\tilde{A})$ then there is an isomorphism*

$${}^* \mathcal{R}_{\mathbb{C}}^{\circ}(g_* p \mathcal{C}_T^{H,\beta})[n+1] \simeq p_*(q^* p \mathcal{C}_T^{\beta} \otimes F^* j_{\dagger} p \mathcal{C}_{\mathbb{C}^*}^{H,-\beta_0-1})[2n+d+1]$$

Proof. Consider the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc}
\mathbb{C}^* & \xleftarrow{\hat{F}} & U_0 & \xrightarrow{\quad} & U \\
\downarrow j & & \downarrow k_U & & \downarrow j_U \\
\mathbb{C} & \xleftarrow{\tilde{F}} & W_0 \times V & \xrightarrow{j_0 \times id} & \mathbb{P}(W) \times V \xrightarrow{\pi_2} V \\
\uparrow F & \nearrow g_0 \times id & \downarrow \tilde{q} & & \downarrow \pi_1 \\
T \times V & & W_0 & \xrightarrow{j_0} & \mathbb{P}(W) \\
\downarrow q & \nearrow g_0 & & & \\
T & & & &
\end{array}$$

where $W_0 := \mathbb{P}(W) \setminus \{w_0 \neq 0\} = \mathbb{C}^n$ with coordinates w_1, \dots, w_n and $U_0 = U \cap W_0$. We denote by p, p_0, π_1 the projection to the first factor and by \tilde{F} the map $(w_1, \dots, w_n, \lambda_0, \lambda_n) \mapsto \lambda_0 + \sum_{i=1}^n \lambda_i w_i$. We consider the coordinate change ϕ_0

$$\tilde{t}_k = t_k \text{ for } k = 1, \dots, d, \quad \tilde{\lambda}_0 = \lambda_0 + \sum_{i=1}^n \lambda_i t_i = F \quad \text{and} \quad \tilde{\lambda}_i = \lambda_i \text{ for } i = 1, \dots, n$$

on $T \times V$ and the coordinate change ψ_0

$$\tilde{w}_j = w_j \text{ for } j = 1, \dots, n, \quad \tilde{\lambda}_0 = \lambda_0 + \sum_{i=1}^n \lambda_i w_i = \tilde{F} \quad \text{and} \quad \tilde{\lambda}_i = \lambda_i \text{ for } i = 1, \dots, n$$

on $W_0 \times V$. Notice that with respect to these coordinates the maps F and \tilde{F} are given by the coordinate function $\tilde{\lambda}_0$. Let $pr : V \rightarrow \mathbb{C}$ be the projection $(\tilde{\lambda}_0, \dots, \tilde{\lambda}_n) \mapsto \tilde{\lambda}_0$. We also have $\psi_0 \circ (g_0 \times id) \circ \varphi_0^{-1} = g_0 \times id$ and the map q factors as $\pi_2 \circ j_0 \circ (g \times id)$. Hence we get

$$\begin{aligned}
& p_*(q^* p \mathbb{C}_T^{H,\beta} \otimes F^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1})[2n+d+1] \\
& \simeq p_*(q^* p \mathbb{C}_T^{H,\beta} \otimes F^* j_!^p \mathbb{C}_{\mathbb{C}^*}^H)[2n+d+1] \\
& \simeq p_*(p \mathbb{C}_T^{H,\beta} \boxtimes pr^* j_!^p \mathbb{C}_{\mathbb{C}^*}^H)[n+1] \\
& \simeq (\pi_2 \circ (j_0 \times id) \circ (g_0 \times id))_*(p \mathbb{C}_T^{H,\beta} \boxtimes pr^* j_!^p \mathbb{C}_{\mathbb{C}^*}^H)[n+1] \\
& \simeq (\pi_2 \circ (j_0 \times id))_*(g_{0*} p \mathbb{C}_T^{H,\beta} \boxtimes pr^* j_!^p \mathbb{C}_{\mathbb{C}^*}^H)[n+1] \\
& \simeq (\pi_2 \circ (j_0 \times id))_*(\tilde{q}^* g_{0*} p \mathbb{C}_T^\beta \otimes \tilde{F}^* j_!^p \mathbb{C}_{\mathbb{C}^*}^H)[n+1] \\
& \simeq (\pi_2 \circ (j_0 \times id))_*(\tilde{q}^* g_{0*} p \mathbb{C}_T^{H,\beta} \otimes k_{U!} \hat{F}^* p \mathbb{C}_{\mathbb{C}^*}^H)[n+1] \\
& \simeq (\pi_2 \circ (j_0 \times id))_*(\tilde{q}^* g_{0*} p \mathbb{C}_T^{H,\beta} \otimes k_{U!}^p \mathbb{C}_{U_0}^H)[n+1] \\
& \simeq (\pi_2 \circ (j_0 \times id))_*((j_0 \times id)^! \pi_1^* g_*^p \mathbb{C}_T^{H,\beta} \otimes k_{U!}^p \mathbb{C}_{U_0}^H)[n+1] \\
& \simeq \pi_{2*}(\pi_1^* g_*^p \mathbb{C}_T^{H,\beta} \otimes (j_0 \times id)_* k_{U!}^p \mathbb{C}_{U_0}^H)[n+1] \\
& \stackrel{(*)}{\simeq} \pi_{2*}(\pi_1^* g_*^p \mathbb{C}_T^{H,\beta} \otimes j_{U!}^p \mathbb{C}_U^H)[n+1] \\
& \simeq \pi_{2*} j_{U!} j_U^* \pi_1^* g_*^p \mathbb{C}_T^{H,\beta} [n+1] \\
& \simeq {}^* \mathcal{R}_c^\circ(g_*^p \mathbb{C}_T^{H,\beta})[n+1]
\end{aligned}$$

where the isomorphism $(*)$ follows from the fact that $\pi_1^* g_*^p \mathbb{C}_T^\beta$ is localized along the divisor $\mathbb{P}(W) \setminus W_0$. \square

4.3 Calculation in charts

The projection $q : T \times V \rightarrow V$ factors as

$$T \times V \xrightarrow{g \times id} \mathbb{P}(W) \times V \xrightarrow{\pi_2} V$$

Our goal in this section is to compute the underlying \mathcal{D} -module \mathcal{N} of

$${}^H\mathcal{N} := \mathcal{H}^{2n+d+1}(g \times id)_*(q^{*p}\mathbb{C}_T^{H,\beta} \otimes F^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1}) \quad (32)$$

(cf. Lemma 4.7 and notice the fact that $g \times id$ is affine) together with its Hodge filtration on affine charts of $\mathbb{P}(W) \times V$.

We define the map

$$F_u : T \times V \longrightarrow \mathbb{C}$$

$$(t_1, \dots, t_d, \lambda_0, \dots, \lambda_n) \mapsto \lambda_u + \sum_{\substack{i=0 \\ i \neq u}} \lambda_i \underline{t}^{\underline{a}_i - \underline{a}_u} = \left(\lambda_0 + \sum_{i=1} \lambda_i \underline{t}^{\underline{a}_i} \right) \cdot \underline{t}^{-\underline{a}_u}$$

(notice that $F_0 = F$). We need the following result

Lemma 4.6. *There is an isomorphism*

$$F^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1} \simeq F_u^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1}$$

for $u = 0, \dots, n$.

Proof. For $u \in \{0, \dots, n\}$ and $G := (\mathbb{C}^*)^d$ consider the action

$$\mu_u : G \times T \times V \longrightarrow T \times V$$

$$(g_1, \dots, g_d, t_1, \dots, t_d, \lambda_0, \dots, \lambda_n) \mapsto (t_1, \dots, t_d, \lambda_0 \underline{g}^{-\underline{a}_u}, \dots, \lambda_n \underline{g}^{-\underline{a}_u})$$

and the action $G \times \mathbb{C} \rightarrow \mathbb{C}$ given by $(g_1, \dots, g_d, t) \mapsto \underline{g}^{-\underline{a}_u} \cdot t$. It is easy to see that the map $F : T \times V \rightarrow \mathbb{C}$ is equivariant with respect to this action. Let $i : T \rightarrow G \times T$ the embedding $(t_1, \dots, t_d) \mapsto (t_1, \dots, t_d, t_1, \dots, t_d)$. Since $j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1}$ is a G -equivariant mixed Hodge module, the module $F^!j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1}$ is also G -equivariant. Let $p : G \times T \times V \rightarrow T \times V$ the projection. We have isomorphisms

$$F^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1} \simeq i^*p_2^*F^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1} \simeq i^*\mu_u^*F^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1} \simeq F_u^*j_!^p\mathbb{C}_{\mathbb{C}^*}^{H,-\beta_0-1}$$

where the second isomorphism follows from the G -equivariance of $F^*j_!^p\mathbb{C}_T^{H,-\beta_0-1}$. \square

We define a coordinate change ϕ_u by

$$\tilde{t}_k = t_k, \quad (\tilde{\lambda}_i)_{i \neq u} = (\lambda_i)_{i \neq u} \quad \text{and} \quad \tilde{\lambda}_u = \lambda_u + \sum_{i \neq u} \lambda_i \underline{t}^{\underline{a}_i - \underline{a}_u}$$

Denote by $C_u \in GL(d+1, \mathbb{Z})$ the matrix

$$C_u := \begin{pmatrix} 1 & & & & \\ -a_{1u} & 1 & & & \\ \vdots & & \ddots & & \\ -a_{du} & & & & 1 \end{pmatrix}.$$

and define for $\tilde{\beta} = (\beta_0, \beta) \in \mathbb{Z}^{d+1}$:

$$\tilde{\beta}^u := (\beta_0, \beta^u) := C_u \cdot \tilde{\beta}$$

Notice that $\tilde{\beta}^0 = \tilde{\beta}$ since $\underline{a}_0 = 0$.

Lemma 4.7.

1. With respect to the coordinates defined by ϕ_u the complex $q^* p \mathbb{C}_T^{H, \beta^u} \otimes F_u^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} [2n + d + 1]$ is isomorphic to

$$p \mathbb{C}_T^{H, \beta^u} \boxtimes pr_u^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} [n]$$

where $pr_u : V \rightarrow \mathbb{C}$ is the projection $(\tilde{\lambda}_0, \dots, \tilde{\lambda}_n) \mapsto \tilde{\lambda}_u$. In particular we have

$$\mathcal{H}^k \left(q^* p \mathbb{C}_T^{H, \beta^u} \otimes F_u^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} \right) = 0 \quad \text{for } k \neq 2n + d + 1$$

and the underlying \mathcal{D} -module of $\mathcal{H}^{2n+d+1} \left(q^* p \mathbb{C}_T^{H, \beta^u} \otimes F_u^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} \right)$ is given by the exterior product

$$\mathcal{D}_T / \left((\partial_{\tilde{t}_k} \tilde{t}_k + \beta_k^u)_{k=1, \dots, d} \right) \boxtimes \mathcal{D}_V / \left((\partial_{\tilde{\lambda}_i})_{i \neq u}, \tilde{\lambda}_u \partial_{\tilde{\lambda}_u} - \beta_0 \right).$$

2. For $\alpha \in \mathbb{Z}^d$ and $u_1, u_2 \in \{0, \dots, n\}$ the map

$$\mathcal{H}^{2n+d+1} \left(q^* p \mathbb{C}_T^{H, \beta^{u_1}} \otimes F_{u_1}^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} \right) \longrightarrow \mathcal{H}^{2n+d+1} \left(q^* p \mathbb{C}_T^{H, \beta^{u_2} + \alpha} \otimes F_{u_2}^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} \right),$$

given by right multiplication with \underline{t}^α on the level of $\mathcal{D}_{V \times T}$ -modules, is an isomorphism.

Proof. Notice that the map F_u is just the projection $((\tilde{t}_k)_{k=1, \dots, s}, (\tilde{\lambda}_i)_{i \neq u}, \tilde{\lambda}_u) \mapsto \tilde{\lambda}_u$ with respect to the new coordinates. This gives

$$q^* p \mathbb{C}_T^{H, \beta^u} \otimes F_u^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} [2n + d + 1] \simeq p \mathbb{C}_T^{H, \beta^u} \boxtimes pr_u^* j_!^p \mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} [n]$$

(the shifts can be seen by noticing that $q^* [n + 1]$, $F_u^* [n + d]$ and $ri_u^* [n]$ are exact. The rest is clear.

For the second point we define coordinates Let $((\tilde{t}_k)_{k=1, \dots, d}, (\tilde{\lambda}_i)_{i=0, \dots, n})$ resp. $((\bar{t}_k)_{k=1, \dots, d}, (\bar{\lambda}_i)_{i=0, \dots, n})$ corresponding to the maps ϕ_{u_1} resp. ϕ_{u_2} . The coordinate change $\phi_{u_2} \circ \phi_{u_1}^{-1}$ is given by

$$\bar{t}_k = \tilde{t}_k, \quad \bar{\lambda}_{u_1} = \tilde{\lambda}_{u_1} - \sum_{i \neq u_1} \tilde{\lambda}_i \tilde{t}_i^{a_i - a_{u_1}}, \quad \bar{\lambda}_{u_2} = \tilde{\lambda}_{u_1} \tilde{t}_i^{a_{u_1} - a_{u_2}}, \quad \bar{\lambda}_i = \tilde{\lambda}_i$$

for $k = 1, \dots, d$ and $i \neq u_1, u_2$. We get the following transformations:

$$\begin{aligned} \partial_{\tilde{\lambda}_i} &\mapsto \partial_{\bar{\lambda}_i} - \tilde{t}_i^{a_i - a_{u_1}} \partial_{\bar{\lambda}_{u_1}} \equiv \partial_{\bar{\lambda}_i} \\ \tilde{\lambda}_{u_1} \partial_{\tilde{\lambda}_{u_1}} - \beta_0 &\mapsto \bar{\lambda}_{u_2} \tilde{t}_i^{a_{u_2} - a_{u_1}} \partial_{\bar{\lambda}_{u_1}} + \bar{\lambda}_{u_2} \partial_{\bar{\lambda}_{u_2}} - \beta_0 \equiv \bar{\lambda}_{u_2} \partial_{\bar{\lambda}_{u_2}} - \beta_0 \\ \partial_{\tilde{\lambda}_{u_2}} &\mapsto -\tilde{t}_i^{a_{u_2} - a_{u_1}} \partial_{\bar{\lambda}_{u_1}} \\ \partial_{\tilde{t}_k} \tilde{t}_k + \beta_k^{u_1} &\mapsto \partial_{\bar{t}_k} \bar{t}_k - \sum_{i \neq u_1} (a_{ki} - a_{ku_1}) \tilde{\lambda}_i \tilde{t}_i^{a_i - a_{u_1}} \partial_{\bar{\lambda}_{u_1}} + (a_{ku_1} - a_{ku_2}) \bar{\lambda}_{u_2} \partial_{\bar{\lambda}_{u_2}} + \beta_k^{u_1} \\ &\equiv \partial_{\bar{t}_k} \bar{t}_k + (a_{ku_1} - a_{ku_2}) \bar{\lambda}_{u_2} \partial_{\bar{\lambda}_{u_2}} + \beta_k^{u_1} \\ &\equiv \partial_{\bar{t}_k} \bar{t}_k + (a_{ku_1} - a_{ku_2}) \beta_0 + \beta_k^{u_1} \\ &= \partial_{\bar{t}_k} \bar{t}_k + \beta_k^{u_2} \end{aligned}$$

where \equiv means equality modulo the ideal generated by the operators on the left hand side. This shows that

$$\mathcal{D}_V / \left((\partial_{\tilde{\lambda}_i})_{i \neq u_1}, \tilde{\lambda}_{u_1} \partial_{\tilde{\lambda}_{u_1}} - \beta_0 \right) \boxtimes \mathcal{D}_T / \left((\partial_{\tilde{t}_k} \tilde{t}_k + \beta_k^{u_1})_{k=1, \dots, d} \right)$$

is actually equal to

$$\mathcal{D}_V / \left((\partial_{\bar{\lambda}_i})_{i \neq u_2}, \bar{\lambda}_{u_2} \partial_{\bar{\lambda}_{u_2}} - \beta_0 \right) \boxtimes \mathcal{D}_T / \left((\partial_{\bar{t}_k} \bar{t}_k + \beta_k^{u_2})_{k=1, \dots, d} \right)$$

after the change of coordinates $\phi_{u_2} \circ \phi_{u_1}^{-1}$. It is then easy to see that the map

$$\mathcal{D}_{V \times T} / \left((\partial_{\tilde{\lambda}_i})_{i \neq u_2}, \tilde{\lambda}_{u_2} \partial_{\tilde{\lambda}_{u_2}} - \beta_0, (\partial_{\tilde{t}_k} \tilde{t}_k + \beta_k^{u_2})_{k=1, \dots, d} \right) \rightarrow \mathcal{D}_V / \left((\partial_{\bar{\lambda}_i})_{i \neq u_2}, \bar{\lambda}_{u_2} \partial_{\bar{\lambda}_{u_2}} - \beta_0, (\partial_{\bar{t}_k} \bar{t}_k + \beta_k^{u_2} + \alpha_k)_{k=1, \dots, d} \right)$$

is given by right multiplication with \underline{t}^α . This shows the second claim. \square

Let $(w_0 : \dots : w_n)$ be the homogeneous coordinates on $\mathbb{P}(W)$ and denote by $j_u : W_u \hookrightarrow \mathbb{P}(W)$ the chart $w_u \neq 0$ with coordinates $w_{iu} := \frac{w_i}{w_u}$ for $i \neq u$. The map g factors over the chart W_u and gives rise to the map

$$\begin{aligned} g_u : T &\longrightarrow W_u \\ (t_1, \dots, t_n) &\mapsto (\underline{t}^{\underline{a}_0 - \underline{a}_u}, \dots, \underline{t}^{\underline{a}_n - \underline{a}_u}). \end{aligned}$$

We define the maps

$$\begin{aligned} \tilde{F}_u : W_u \times V &\longrightarrow \mathbb{C} \\ (w_{iu})_{i \neq u} &\mapsto \lambda_u + \sum_{\substack{i=0 \\ i \neq u}}^n \lambda_i w_{iu} \end{aligned}$$

As mentioned above we would like to compute the restriction of \mathcal{N} to the affine chart $W_u \times V$. For $u = 0, \dots, n$ we set

$$\begin{aligned} {}^H\mathcal{N}_u &:= {}^H\mathcal{N}|_{W_u \times V} \simeq \mathcal{H}^{2n+d+1}(g_u \times id)_*(q^* {}^p\mathbb{C}_T^{H, \beta^u} \otimes F_u^* j_u^! {}^p\mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1}) \\ &\simeq \mathcal{H}^n(g_u \times id)_* \left({}^p\mathbb{C}_T^{H, \beta^u} \boxtimes pr_u^* j_u^! {}^p\mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} \right) \\ &\simeq \mathcal{H}^n \left({}^p g_{u*} \mathbb{C}_T^{H, \beta^u} \boxtimes pr_u^* j_u^! {}^p\mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1} \right) \\ &\simeq \mathcal{H}^0({}^p g_{u*} \mathbb{C}_T^{H, \beta^u}) \boxtimes \mathcal{H}^n(pr_u^* j_u^! {}^p\mathbb{C}_{\mathbb{C}^*}^{H, -\beta_0 - 1}) \end{aligned}$$

We now apply the main result of section 3 in order to compute the module $g_{u+} \mathcal{O}_T^{\beta^u}$ together with its corresponding Hodge filtration.

Notice that the embedding g_u is given by the $d \times n$ -matrix $A_u = (a_{ki}^u)$ with columns $(\underline{a}_i - \underline{a}_u)$ for $i \in \{0, \dots, n\} \setminus \{u\}$. We need to check if the matrices A_u satisfy the conditions in Theorem 3.16. Recall that $\tilde{\beta}^u := (\beta_0^u, \beta^u) := C_u \cdot \tilde{\beta}$.

Lemma 4.8. *Assume that $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ and $\mathbb{N}\tilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\tilde{A}$, then the matrices A_u satisfy conditions,*

1. $\mathbb{Z}A_u = \mathbb{Z}^d$
2. $\mathbb{N}A_u = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A_u$
3. if $\tilde{\beta} \in \mathfrak{A}_{\tilde{A}}$ then $\beta^u \in \mathfrak{A}_{A_u}$

Proof. Denote by \tilde{A}_u the $(d+1) \times (n+1)$ -matrix with columns $(1, \underline{a}_i - \underline{a}_u)$ for $i \in \{0, \dots, n\}$. We will first show the two properties for the matrix \tilde{A}_u . Notice that we have $C_u \cdot \tilde{A} = C_u \cdot \tilde{A}_0 = \tilde{A}_u$. Since C_u is a linear, invertible map we get $C_u(\mathbb{Z}\tilde{A}) = \tilde{A}_u$, $C_u(\mathbb{N}\tilde{A}) = \tilde{A}_u$ and $C_u(\mathbb{R}_{\geq 0}\tilde{A}) = \mathbb{R}_{\geq 0}\tilde{A}_u$. Therefore the two properties hold for \tilde{A}_u iff they hold for \tilde{A} .

Denote by $p : \mathbb{Z}^{d+1} \rightarrow \mathbb{Z}^d$ the projection to the last d -coordinates. Since p maps $(1, \underline{a}_i - \underline{a}_u)$ to $\underline{a}_i - \underline{a}_u$ it is easy to see that the first two properties also hold for A_u .

It follows easily from the definition that $\tilde{\beta} \in \mathfrak{A}_{\tilde{A}} \Leftrightarrow \tilde{\beta}^u \in \mathfrak{A}_{\tilde{A}_u}$. Hence it is enough to show that $\tilde{\beta} = (\beta_0, \beta) \in \mathfrak{A}_{\tilde{A}}$ implies $\beta \in \mathfrak{A}_A$. We notice first that there is a 1-1 correspondence between facets of $\mathbb{R}_{\geq 0}A$ and facets of $\mathbb{R}_{\geq 0}\tilde{A}$ containing $\tilde{\underline{a}}_0 = (1, 0, \dots, 0)$ given by

$$F \leftrightarrow \tilde{F} = F + \mathbb{R}_{\geq 0} \cdot (1, 0, \dots, 0)$$

If n_F is a primitive, inward, pointing normal vector of a facet F of $\mathbb{R}_{\geq 0}A$, the vector $n_{\tilde{F}} := (0, n_F)$ is a primitive, inward, pointing normal vector of the corresponding facet \tilde{F} of $\mathbb{R}_{\geq 0}\tilde{A}$. Since $\tilde{\underline{c}} := \sum_{i=0}^n \tilde{\underline{a}}_i = (n+1, \underline{c})$, we have $e_{\tilde{F}} = \langle n_{\tilde{F}}, \tilde{\underline{c}} \rangle = \langle (0, n_F), (n+1, \underline{c}) \rangle = \langle n_F, \underline{c} \rangle = e_F$. We get by definition 10

$$\tilde{\beta} \in \mathfrak{A}_{\tilde{A}} = \bigcap_{\tilde{F} \text{ facet}} \{\mathbb{R} \cdot \tilde{F} - [0, e_{\tilde{F}}] \cdot \tilde{\underline{c}}\} \subset \bigcap_{\substack{\tilde{F} \text{ facet} \\ \tilde{\underline{a}}_0 \in \tilde{F}}} \{\mathbb{R} \cdot \tilde{F} - [0, e_{\tilde{F}}] \cdot \tilde{\underline{c}}\} \Rightarrow \beta \in \bigcap_{F \text{ facet}} \{\mathbb{R} \cdot F - [0, e_F] \cdot \underline{c}\} = \mathfrak{A}_A.$$

□

Denote by \mathbb{L}_{A_u} the \mathbb{Z} -module of relations among the columns of A_u . In order to calculate the direct image of \mathcal{O}_T^β under the map g_u , we use Theorem 3.16 where A_u takes the role of the matrix B in loc.cit.

Proposition 4.9. *Consider the \mathcal{D}_{W_u} -module $\check{\mathcal{M}}_{A_u}^{\beta^u}$ as defined in Definition 2.5, that is, $\check{\mathcal{M}}_{A_u}^{\beta^u} = \mathcal{D}_{W_u}/\check{\mathcal{I}}_{A_u}^{\beta^u}$ where the left ideal $\check{\mathcal{I}}_{A_u}^{\beta^u}$ is generated by*

$$\check{\square}_{\underline{m} \in \mathbb{L}_{A_u}} = \prod_{i \neq u: m_i > 0} w_{iu}^{m_i} - \prod_{i \neq u: m_i < 0} w_{iu}^{-m_i}$$

and the Euler vector fields:

$$\begin{aligned} \check{E}_k^u + \beta_k^u &= \sum_{i \neq u} a_{ki}^u \partial_{w_{iu}} w_{iu} + \beta_k^u \\ &= \sum_{i \neq u} (a_{ki} - a_{ku}) \partial_{w_{iu}} w_{iu} + \beta_k^u. \end{aligned}$$

Then the direct image $g_{u+} \mathcal{O}_T^{\beta^u}$ is isomorphic to $\check{\mathcal{M}}_{A_u}^{\beta^u}$. Moreover, the Hodge filtration on $\check{\mathcal{M}}_{A_u}^{\beta^u}$ is the order filtration shifted by $(n-d)$, i.e.

$$F_{p+(n-d)}^H \check{\mathcal{M}}_{A_u}^{\beta^u} = F_p^{ord} \check{\mathcal{M}}_{A_u}^{\beta^u}.$$

Proof. The statement follows from Theorem 3.16 and Lemma 4.8. □

We now want to compute how the \mathcal{D} -modules $g_{u+} \mathcal{O}_T^{\beta^u}$ glue on their common domain of definition. Let $u_1, u_2 \in \{0, \dots, n\}$ and denote by $W_{u_1 u_2}$ the intersection $W_{u_1} \cap W_{u_2}$. We fix $u_1, u_2 \in \{0, \dots, n\}$ with $u_1 < u_2$. We have the following change of coordinates between the charts W_{u_1} and W_{u_2}

$$w_{iu_1} = w_{iu_2} w_{u_1 u_2}^{-1} \quad \text{for } i \neq u_2 \quad \text{and} \quad w_{u_2 u_1} = w_{u_1 u_2}^{-1}$$

which gives the following transformation rules for vector field:

$$w_{iu_1} \partial_{w_{iu_1}} = w_{iu_2} \partial_{w_{iu_2}} \quad \text{for } i \neq u_2 \quad w_{u_2 u_1} \partial_{w_{u_2 u_1}} = - \sum_{i \neq u_2} w_{iu_2} \partial_{w_{iu_2}}. \quad (33)$$

The above mentioned transformation rules define an algebra isomorphism

$$\iota_{u_1 u_2} : D_{W_{u_1}}[w_{u_2 u_1}^{-1}] \longrightarrow D_{W_{u_2}}[w_{u_1 u_2}^{-1}].$$

The module of global sections $\Gamma(W_{u_1 u_2}, g_{u_1+} \mathcal{O}_T^{\beta^{u_1}})$ can be expressed as the quotient $D_{W_{u_1}}[w_{u_2 u_1}^{-1}] / \check{I}_{A_{u_1}}^{\beta^{u_1}}$, where $\check{I}_{A_{u_1}}^{\beta^{u_1}} \subset D_{W_{u_1}}[w_{u_2 u_1}^{-1}] := \mathbb{C}[(w_{iu_1})_{i \neq u_1}][w_{u_2 u_1}^{-1}] \otimes_{\mathbb{C}[(w_{iu_1})_{i \neq u_1}]} D_{W_{u_1}}$ is the left ideal generated by

1. $\check{E}_k^{u_1} + \beta_k^{u_1} = \sum_{\substack{i=0 \\ i \neq u_1}}^n a_{ki}^{u_1} \partial_{w_{iu_1}} w_{iu_1} + \beta_k^{u_1} = \sum_{\substack{i=0 \\ i \neq u_1}}^n a_{ki}^{u_1} w_{iu_1} \partial_{w_{iu_1}} + \sum_{\substack{i=0 \\ i \neq u_1}}^n a_{ki}^{u_1} + \beta_k^{u_1} \quad k = 1, \dots, d$
2. $\check{\square}_{\underline{m}} = \prod_{\substack{m_i > 0 \\ i \neq u_1}} w_{iu_1}^{m_i} - \prod_{\substack{m_i < 0 \\ i \neq u_1}} w_{iu_1}^{-m_i} \quad \underline{m} \in \mathbb{L}_{A_{u_1}}.$

Let $\gamma^{u_1} := \sum_{i \neq u_1} a_{ki}^{u_1}$, then $\check{I}_{A_{u_1}}^{\beta^{u_1} - \gamma^{u_1}} \subset D_{W_{u_1}}[w_{u_2 u_1}^{-1}]$ is the left ideal generated by

1. $\sum_{\substack{i=0 \\ i \neq u_1}}^n a_{ki}^{u_1} w_{iu_1} \partial_{w_{iu_1}} + \beta_k^{u_1} \quad k = 1, \dots, d$
2. $\square_l = \prod_{\substack{l_i > 0 \\ i \neq u_1}} w_{iu_1}^{l_i} - \prod_{\substack{l_i < 0 \\ i \neq u_1}} w_{iu_1}^{-l_i} \quad l \in \mathbb{L}_{A_{u_1}}$

We get the following isomorphism of $D_{W_{u_1}}[w_{u_2 u_1}^{-1}]$ -modules

$$\begin{aligned} D_{W_{u_1}}[w_{u_2 u_1}^{-1}]/\dot{I}_{A_{u_1}}^{\beta^{u_1}} &\longrightarrow D_{W_{u_1}}[w_{u_2 u_1}^{-1}]/\dot{I}_{A_{u_1}}^{\beta^{u_1}-\gamma^{u_1}} \\ 1 &\mapsto \prod_{i \neq u_1} w_{i u_1}^{-1} \end{aligned} \quad (34)$$

which is the image of the isomorphism

$$\begin{aligned} \mathcal{O}_T^{\gamma^{u_1}+\beta_k^{u_1}-1} &\longrightarrow \mathcal{O}_T^{\beta_k^{u_1}-1} \\ 1 &\mapsto \underline{t}^{-\gamma^{u_1}} = \underline{t}^{-\sum_{i \neq u_1} (a_i - a_{u_1})} \end{aligned}$$

under the functor g_{u_1+} (cf. equation (21)). One obtains the same results for the chart W_{u_2} by exchanging u_1 and u_2 above. Using the transformation rules from above, we can identify $D_{W_{u_1}}[w_{u_2 u_1}^{-1}]$ with $D_{W_{u_2}}[w_{u_1 u_2}^{-1}]$ by the transformation rules (33) above which gives a well-defined map

$$\iota_{u_1 u_2} : D_{W_{u_1}}[w_{u_2 u_1}^{-1}]/\dot{I}_{A_{u_1}}^{\beta^{u_1}-\gamma^{u_1}} \longrightarrow D_{W_{u_2}}[w_{u_1 u_2}^{-1}]/\dot{I}_{A_{u_2}}^{\beta^{u_2}-\gamma^{u_2}}$$

We can now give an explicit expression for the gluing map between the various charts of the module $g_+ \mathcal{O}_T^\beta$.

Lemma 4.10. *The isomorphism between $g_{u_1+} \mathcal{O}_T^{\beta^{u_1}}$ and $g_{u_2+} \mathcal{O}_T^{\beta^{u_2}}$ on their common domain of definition $W_{u_1 u_2} = W_{u_1} \cap W_{u_2}$ is given by*

$$\begin{aligned} \Gamma(W_{u_1 u_2}, g_{u_1+} \mathcal{O}_T) &\simeq D_{W_{u_1}}[w_{u_2 u_1}^{-1}]/\dot{I}_{A_{u_1}}^{\beta^{u_1}} \longrightarrow D_{W_{u_2}}[w_{u_1 u_2}^{-1}]/\dot{I}_{A_{u_2}}^{\beta^{u_2}} \simeq \Gamma(W_{u_1 u_2}, g_{u_2+} \mathcal{O}_T) \\ P &\longmapsto \iota_{u_1 u_2}(P) w_{u_1 u_2}^{n+1}. \end{aligned}$$

Proof. This follows easily from the discussion above by concatenating the three maps

$$D_{W_{u_1}}[w_{u_2 u_1}^{-1}]/\dot{I}_{A_{u_1}}^{\beta^{u_1}} \longrightarrow D_{W_{u_1}}[w_{u_2 u_1}^{-1}]/\dot{I}_{A_{u_1}}^{\beta^{u_1}-\gamma^{u_1}} \xrightarrow{\iota} D_{W_{u_2}}[w_{u_1 u_2}^{-1}]/\dot{I}_{A_{u_2}}^{\beta^{u_2}-\gamma^{u_2}} \longrightarrow D_{W_{u_2}}[w_{u_1 u_2}^{-1}]/\dot{I}_{A_{u_2}}^{\beta^{u_2}}$$

and the simple computation

$$\left(\prod_{i \neq u_2} w_{i u_2} \right) \cdot \iota \left(\prod_{i \neq u_1} w_{i u_1}^{-1} \right) = w_{u_1 u_2}^{n+1}$$

□

Consider the following change of coordinates θ_u on $W_u \times V$.

$$\tilde{\lambda}_u = \lambda_u + \sum_{\substack{j=0 \\ j \neq u}}^n \lambda_j w_{j u} \quad , \quad \tilde{\lambda}_i = \lambda_i \quad \text{and} \quad \tilde{w}_{i u} = w_{i u} \quad (35)$$

for $i = 0, \dots, n$ and $i \neq u$. Notice that $\theta_u^{-1} \circ (g_u \times id) \circ \phi_u^- g_u \times id$ and \tilde{F}_u is just the projection $((\tilde{w}_{i u})_{i \neq u}, \tilde{\lambda}_0, \dots, \tilde{\lambda}_n) \mapsto \tilde{\lambda}_u$.

Proposition 4.11. *Consider the original coordinates $((w_{i u})_{i \neq u}, (\lambda_0, \dots, \lambda_n))$ of $W_u \times V$. Then there is an isomorphism of $\mathcal{D}_{W_u \times V}$ -modules $\mathcal{N}_u \simeq \mathcal{D}_{W_u \times V} / \mathcal{K}_{A_u}^{\beta^u}$, where $\mathcal{K}_{A_u}^{\beta^u}$ is the left $\mathcal{D}_{W_u \times V}$ -ideal generated by the following classes of operators*

1. $\sum_{\substack{i=0 \\ i \neq u}}^n a_{k i}^u \partial_{w_{i u}} w_{i u} - \sum_{i=1}^n a_{k i} \lambda_i \partial_{\lambda_i} + \beta_k$
2. $\check{\square}_{\underline{m}} = \prod_{\substack{m_i > 0 \\ i \neq u}} w_{i u}^{m_i} - \prod_{\substack{m_i < 0 \\ i \neq u}} w_{i u}^{-m_i} \quad \underline{m} \in \mathbb{L}_{A_u}$
3. $\partial_{\lambda_i} - w_{i u} \partial_{\lambda_u} \quad \text{for } i = 0, \dots, n \text{ and } i \neq u$
4. $\sum_{j=0}^n \lambda_j \partial_{\lambda_j} - \beta_0$.

Moreover, we have

$$F_{p+(n-d)}^H \mathcal{N}_u \simeq F_p^{ord} \mathcal{D}_{W_u \times V} / \mathcal{K}_u.$$

Proof. Recall that $\mathcal{N}_u = \check{\mathcal{M}}_{A_u}^{\beta^u} \boxtimes \mathcal{D}_V / \left((\partial_{\check{\lambda}_i})_{i \neq u}, \check{\lambda}_u \partial_{\check{\lambda}_u} - \beta_0 \right) = \mathcal{D}_{W_u \times V} / \check{\mathcal{K}}_{A_u}^{\beta^u}$, where

$$\check{\mathcal{K}}_{A_u}^{\beta^u} = \left((\check{E}_k^u + \beta_k^u)_{k=1, \dots, d}, (\check{\square}_{\underline{m}})_{\underline{m} \in \mathbb{L}_{A_u}}, (\partial_{\check{\lambda}_j})_{j \neq u}, (\check{\lambda}_u \partial_{\check{\lambda}_u} - \beta_0) \right).$$

Using the coordinate transformation (35) we see that $\check{\mathcal{K}}_{A_u}^{\beta^u}$ is transformed into the ideal $\mathcal{K}_{A_u}^{\tilde{\beta}}$ generated by the operators

$$\begin{aligned} \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u (\partial_{w_{iu}} - \lambda_i \partial_{\lambda_u}) w_{iu} + \beta_k^u & \quad k = 1, \dots, d \\ \check{\square}_{\underline{m}} = \prod_{\substack{m_i > 0 \\ i \neq u}} w_{iu}^{m_i} - \prod_{\substack{m_i < 0 \\ i \neq u}} w_{iu}^{-m_i} & \quad \underline{m} \in \mathbb{L}_{A_u} \\ \partial_{\lambda_i} - w_{iu} \partial_{\lambda_u} & \quad \text{for } i = 0, \dots, n \text{ and } i \neq u \\ (\lambda_u + \sum_{\substack{j=0 \\ j \neq u}}^n \lambda_j w_{ju}) \partial_{\lambda_u} - \beta_0. & \end{aligned}$$

The last operator can be rewritten (using the relations $\partial_{\lambda_i} - w_{iu} \partial_{\lambda_u}$, i.e. the third class of operators)

$$\sum_{j=0}^n \lambda_j \partial_{\lambda_j} - \beta_0 \equiv (\lambda_u + \sum_{\substack{j=0 \\ j \neq u}}^n \lambda_j w_{ju}) \partial_{\lambda_u} - \beta_0.$$

The Euler-type operators $\sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u (\partial_{w_{iu}} - \lambda_i \partial_{\lambda_u}) w_{iu}$ can be further simplified by writing

$$\begin{aligned} \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u (\partial_{w_{iu}} - \lambda_i \partial_{\lambda_u}) w_{iu} + \beta_k^u & \equiv \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u (\partial_{w_{iu}} w_{iu} - \lambda_i \partial_{\lambda_i}) + \beta_k^u \\ & = \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u \partial_{w_{iu}} w_{iu} - \sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i} + a_{ku} \sum_{i=0}^n \lambda_i \partial_{\lambda_i} + \beta_k^u \\ & \equiv \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u \partial_{w_{iu}} w_{iu} - \sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i} + a_{ku} \beta_0 + \beta_k^u, \\ & = \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u \partial_{w_{iu}} w_{iu} - \sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i} + \beta_k, \end{aligned}$$

where the first equivalence follows by using the relation $\sum_{j=0}^n \lambda_j \partial_{\lambda_j} \equiv (\lambda_u + \sum_{\substack{j=0 \\ j \neq u}}^n \lambda_j w_{ju}) \partial_{\lambda_u}$ from above.

Hence we obtain the presentation $\mathcal{N}_u \simeq \mathcal{D}_{W_u \times V} / \mathcal{K}_{A_u}^{\tilde{\beta}}$, and the statement on the Hodge filtration follows directly from Proposition 4.9. \square

4.4 A Koszul complex

In this section, we will construct a strict resolution of the filtered module (\mathcal{N}_u, F^H) . For this purpose, we first describe an alternative presentation of the ideal $\mathcal{K}_{A_u}^{\tilde{\beta}} \subset \mathcal{D}_{W_u \times V}$. Let A_u^s be the $(d+1) \times (2n+1)$ -matrix with columns $(0, \underline{a}_0 - \underline{a}_u), \dots, (0, \widehat{\underline{a}_u - \underline{a}_u}, \dots, (0, \underline{a}_n - \underline{a}_u), (1, \underline{a}_0), \dots, (1, \underline{a}_n)$ (here the symbol

$\widehat{}$ means that the zero column $(0, \underline{a}_u - \underline{a}_u)$ is omitted). In other words, we have

$$A_u^s = \left(\begin{array}{ccc|c|ccc} 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ \hline & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ \hline & & & & & & \end{array} \right).$$

Lemma 4.12. *If, as before, $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ and $\mathbb{N}\tilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\tilde{A}$ holds, then we have $\mathbb{Z}A_u^s = \mathbb{Z}^{d+1}$ and $\mathbb{N}A_u^s = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}A_u^s$.*

Proof. From $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ we conclude $\mathbb{Z}A_u^s = \mathbb{Z}^{d+1}$ since evidently $\mathbb{Z}\tilde{A} \subset \mathbb{Z}A_u^s$. Hence it remains to show that the semi-group $\mathbb{N}A_u^s$ is normal. We have

$$C_u \cdot A_u^s = \left(\begin{array}{ccc|c|ccc} 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ \hline & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ \hline & & & & & & \end{array} \right) =: (\underline{\bar{a}}_1, \dots, \underline{\bar{a}}_n, \underline{\tilde{a}}_0, \underline{\tilde{a}}_1, \dots, \underline{\tilde{a}}_n) \in M((d+1) \times (2n+1), \mathbb{Z}),$$

where $C_u \in GL(d+1, \mathbb{Z})$ is the matrix already used in Lemma 4.8. It suffices to show the normality property for the semi-group $\mathbb{N}(C_u \cdot A_u^s)$ since C_u is an invertible linear mapping, hence a homeomorphism. Suppose that we are given a linear combination

$$\underline{v} = \sum_{i=1}^n \lambda_i \underline{\bar{a}}_i^u + \sum_{j=0}^n \mu_j \underline{\tilde{a}}_j^u \in \mathbb{Z}^{d+1},$$

where $\lambda_i, \mu_j \in \mathbb{R}_{\geq 0}$. Then $v = \sum_{i=1}^n (\lambda_i + \mu_i) \underline{\bar{a}}_i^u + \underline{\tilde{a}}_0^u \cdot \left(\sum_{j=0}^n \mu_j \right)$. Clearly, $\sum_{j=0}^n \mu_j \in \mathbb{N}$, and moreover, the vector $\sum_{i=1}^n (\lambda_i + \mu_i) \underline{\bar{a}}_i^u$ lies in $\mathbb{R}_{\geq 0}A_u$, but the latter semi-group is normal according to Lemma 4.8. Hence we have $\sum_{i=1}^n (\lambda_i + \mu_i) \underline{\bar{a}}_i^u \in \mathbb{N}A_u$, and therefore $v \in \mathbb{N}\tilde{A}_u \subset \mathbb{N}(C_u \cdot A_u^s)$, as required. \square

We now show that \mathcal{N}_u can be interpreted as a partial Fourier-Laplace transformed GKZ-system. For this we consider the GKZ system $\mathcal{M}_{A_u^s}$ on $\hat{W}_u \times V$ with coordinates $(\hat{w}_{iu})_{i \neq u}, \lambda_0, \dots, \lambda_n$. Let $\text{FL}_{\hat{W}_u}$ be the partial Fourier-Laplace transformation which interchanges $\partial_{\hat{w}_{iu}}$ with $(w_{iu})_{i \neq u}$ and \hat{w}_{iu} with $-\partial_{w_{iu}}$.

Lemma 4.13. *Let $\mathcal{I}_{A_u^s}^{(\vee)\tilde{\beta}}$ be the left $\mathcal{D}_{W_u \times V}$ ideal generated by the operators*

$$\square_{(m,l)}^{(\vee)} := \prod_{\substack{m_i > 0 \\ i \neq u}} w_{iu}^{m_i} \prod_{\substack{l_i > 0 \\ 0 \leq i \leq n}} \partial_{\lambda_i}^{l_i} - \prod_{\substack{m_i < 0 \\ i \neq u}} w_{iu}^{-m_i} \prod_{\substack{l_i > 0 \\ 0 \leq i \leq n}} \partial_{\lambda_i}^{-l_i},$$

where $(\underline{m}, \underline{l}) = ((m_i)_{i \neq u}, l_0, \dots, l_n) \in \mathbb{L}_{A_u^s}$,

$$E_k^{(\vee)} - \beta_k := - \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u \partial_{w_{iu}} w_{iu} + \sum_{i=1}^n a_{ki} \lambda_i \partial_{\lambda_i} - \beta_k \quad \text{for } k = 1, \dots, d$$

(notice that the operators $E_k^{(\vee)}$ are the same operators as in Proposition 4.11 1. above, but multiplied with -1 , which is useful for a Fourier-Laplace transformation that will be performed below) and

$$E_0^{(\vee)} - \beta_0 := \sum_{i=0}^n \lambda_i \partial_{\lambda_i} - \beta_0.$$

Then we have $\mathcal{I}_{A_u^s}^{(\vee)\tilde{\beta}} = \mathcal{K}_u$, and hence the $\mathcal{D}_{W_u \times V}$ -module \mathcal{N}_u is isomorphic to $\mathcal{D}_{W_u \times V} / \mathcal{I}_{A_u^s}^{(\vee)\tilde{\beta}}$. In other words, we have an isomorphism

$$\mathcal{N}_u \simeq \text{FL}_{\hat{W}_u} \mathcal{M}_{A_u^s}^{\tilde{\beta}}.$$

Proof. For the first statement, notice that $\check{\square}_{(m,0)}^{(\vee)}$ equals the operator $\check{\square}_{\underline{m}}$ from the definition of the ideal \mathcal{K}_u . On the other hand, one can obtain all operators $\check{\square}_{(m,l)}^{(\vee)}$ from the operators $\check{\square}_{(m',0)}$ using the relations $\partial_{\lambda_i} - w_{iu}\partial_{\lambda_u}$. The last statement follows by exchanging $\partial_{w_{iu}}$ with $-\hat{w}_{iu}$ and w_{iu} with $\partial_{\hat{w}_{iu}}$ in the classes of operators of type 1., 2., 3. and 4. in the definition of the ideal \mathcal{K}_u . \square

In order to construct a strictly filtered resolution of \mathcal{N}_u , we use the theory of Euler-Koszul complexes, which we explained in section 2.1. It will be applied to the $\mathcal{D}_{\hat{W}_u \times V}$ -module $\mathcal{M}_{A_u^s}$. As before, we work on the level of global sections.

Let $F_{\bullet}^{\hat{\omega}} D_{\hat{W}_u \times V}$ the filtration on $D_{\hat{W}_u \times V}$ corresponding to the weight vector

$$\hat{\omega} = ((\text{weight}(\hat{w}_{iu}))_{i \neq u}, (\text{weight}(\partial_{\hat{w}_{iu}}))_{i \neq u}, \text{weight}(\lambda_0), \dots, \text{weight}(\lambda_n), \text{weight}(\partial_{\lambda_0}), \dots, \text{weight}(\partial_{\lambda_n}))$$

$$:= \left(\underbrace{(1, \dots, 1)}_{n\text{-times}}, \underbrace{(0, \dots, 0)}_{n\text{-times}}, \underbrace{(0, \dots, 0)}_{(n+1)\text{-times}}, \underbrace{(1, \dots, 1)}_{(n+1)\text{-times}} \right).$$

Notice that this filtration corresponds to the order filtration $F_{\bullet}^{\text{ord}} D_{W_u \times V}$ under the Fourier-Laplace transformation functor $\text{FL}_{\hat{W}_u}$. We obtain a filtered resolution $((K_u^{\bullet}, d), F^{\hat{\omega}})$ of $M_{A_u^s}^{\tilde{\beta}}$. Using Remark 2.3 we show that resolution is strict.

Lemma 4.14. *The Euler-Koszul complex $(K_u^{\bullet}, F_{\bullet}^{\hat{\omega}})$ is a resolution of $(M_{A_u^s}^{\tilde{\beta}}, F_{\bullet}^{\hat{\omega}})$ in the category of filtered $D_{\hat{W}_u \times V}$ -modules (with respect to the filtration $F_{\bullet}^{\hat{\omega}} D_{\hat{W}_u \times V}$), i.e., we have a quasi-isomorphism $K_u^{\bullet} \rightarrow M_{A_u^s}$ and the complex K_u^{\bullet} is strictly filtered.*

Proof. By Lemma 2.3 above it is enough to show that $H^{-i}(Gr_{\bullet}^{F_{\bullet}^{\hat{\omega}}} K_u^{\bullet}) = 0$ for $i \geq 1$ and $H^0(Gr_{\bullet}^{F_{\bullet}^{\hat{\omega}}} K_u^{\bullet}) \simeq Gr_{\bullet}^{F_{\bullet}^{\hat{\omega}}} M_{A_u^s}$. Denote by $GD_{\hat{W}_u \times V} = Gr_{\bullet}^{\hat{\omega}} D_{\hat{W}_u \times V}$ the associated graded object of $D_{\hat{W}_u \times V}$, by $(\hat{v}_{iu})_{i \neq u}$ the symbol of $(\partial_{\hat{w}_{iu}})_{i \neq u}$ and by μ_j the symbol of ∂_{λ_j} in $GD_{\hat{W}_u \times V}$. Since $\check{\square}_{(m,l)}$ is homogeneous in (∂_{λ_j}) and $\text{ord}_{\hat{\omega}}(\partial_{\hat{w}_{iu}}) = 0$ for all $i \neq u$ we have

$$Gr^{\hat{\omega}}(D_{\hat{W}_u \times V}/D_{\hat{W}_u \times V} J_{A_u^s}^g) = GD_{\hat{W}_u \times V}/J_{A_u^s}^g$$

where $J_{A_u^s}^g$ is generated by

$${}^g\check{\square}_{(m,l)} := \prod_{\substack{m_i > 0 \\ i \neq u}} \hat{v}_{iu}^{m_i} \prod_{\substack{l_i > 0 \\ 0 \leq i \leq n}} \mu_i^{l_i} - \prod_{\substack{m_i < 0 \\ i \neq u}} \hat{v}_{iu}^{-k_i} \prod_{\substack{l_i > 0 \\ 0 \leq i \leq n}} \mu_i^{-l_i}$$

Notice that

$$GD_{\hat{W}_u \times V}/J_{A_u^s}^g \simeq \mathbb{C}[(\hat{w}_{iu})_{i \neq u}, \lambda_0, \dots, \lambda_n] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A_u^s]$$

The associated graded complex $Gr^{\hat{\omega}} K_u^{\bullet}$ is isomorphic to a Koszul complex

$$Gr^{\hat{\omega}} K_u^{\bullet} \simeq \text{Kos}(GD_{\hat{W}_u \times V}/\hat{J}_{A_u^s}^g, ({}^gE_k^u)_{k=0, \dots, d})$$

where ${}^gE_k^u$ is defined by

$${}^g\hat{E}_k^u := \sum_{i \neq u} a_{ki}^u \hat{w}_{iu} \hat{v}_{iu} + \sum_{i=1}^n a_{ki} \lambda_i \mu_i \quad \text{for } k = 1, \dots, d$$

and

$${}^gE_0^u := \sum_{i=0}^n \lambda_i \mu_i$$

It is shown in [BZGM15, Theorem 1.2] that the ${}^gE_k^u$ are part of a system of parameters. Since $\mathbb{N}A_u^s$ is a normal semi-group (see Lemma 4.12 above), the ring $GD_{\hat{W}_u \times V}/J_{A_u^s}^g$ is Cohen-Macaulay. Hence $({}^gE_k^u)_{k=0, \dots, d}$ is a regular sequence in $GD_{\hat{W}_u \times V}/J_{A_u^s}^g$. This shows $H^{-i}(Gr_{\bullet}^{\hat{\omega}} K_u^{\bullet}) = 0$ for $i \geq 1$. On the other hand, it follows from [SST00, Theorem 4.3.5] that $H^0(Gr_{\bullet}^{\hat{\omega}} K_u^{\bullet}) = GD_{\hat{W}_u \times V}/(J_{A_u^s}^g + ({}^gE_k^u)_{k=0, \dots, d}) \simeq Gr_{\bullet}^{\hat{\omega}} M_{A_u^s}^{\tilde{\beta}}$, as required. \square

As a consequence, we obtain the filtered resolution of \mathcal{N}_u we are looking for. Let $J_{A_u^s}^{(\vee)}$ be the ideal in $D_{W_u \times V}$ generated by the box operators $\square_{(\underline{m}, l)}^{(\vee)}$ for $(\underline{m}, l) \in \mathbb{L}_{A_u^s}$. Put

$$K_u^{(\vee)-l} = \bigoplus_{0 \leq i_1 < \dots < i_l \leq d} D_{W_u \times V} / J_{A_u^s}^{(\vee)} e_{i_1, \dots, i_l}$$

and define

$$K_u^{(\vee)\bullet} := \text{Kos}(D_{W_u \times V} / J_{A_u^s}^{(\vee)}, (E_k^u - \beta_k)_{k=0, \dots, d}),$$

where E_k^u denote the (pairwise commuting) endomorphisms of $D_{W_u \times V} / J_{A_u^s}^{(\vee)}$ induced from right multiplication by E_k^u on $D_{W_u \times V}$. Define a filtration $\{F_\bullet K_u^{(\vee)\bullet}\}$ on $K_u^{(\vee)\bullet}$ by

$$F_p K_u^{(\vee)l} := \bigoplus_{0 \leq i_1 < \dots < i_p \leq d} F_{p-l+(n-d)}^{\text{ord}} D_{W_u \times V} / J_{A_u^s}^{(\vee)}.$$

Then we have

Proposition 4.15. *We have a filtered quasi-isomorphism $(K_u^{(\vee)\bullet}, F_\bullet) \simeq (N_u, F_\bullet^{\text{ord}}) \simeq (N_u, F_{\bullet+(n-d)}^H)$, i.e. the complex $(K_u^{(\vee)\bullet}, F_\bullet)$ is a resolution of $(N_u, F_{\bullet+(n-d)}^H)$ in the category of filtered $D_{W_u \times V}$ -modules.*

Proof. The filtered quasi-isomorphism $(K_u^{(\vee)\bullet}, F_\bullet) \simeq (N_u, F_\bullet^{\text{ord}})$ is obtained by applying the Fourier-Laplace functor $\text{FL}_{\hat{W}_u}$ to the (filtered) Euler-Koszul complex $(K_u^{(\vee)\bullet}, F_\bullet^{\hat{\omega}})$ from above (using Lemma 4.14). The second filtered (quasi-)isomorphism $(N_u, F_\bullet^{\text{ord}}) \simeq (N_u, F_{\bullet+(n-d)}^H)$ is just the content of Proposition 4.11. \square

4.5 \mathcal{R} -modules

In the following the Rees construction of a filtered \mathcal{D} -module will be helpful, we are following [Sab05]. Let X be a smooth variety of dimension n . The order filtration of \mathcal{D}_X gives rise to the Rees ring $R_F \mathcal{D}_X$. Given a filtered \mathcal{D}_X -module $(\mathcal{M}, F_\bullet \mathcal{M})$ we construct the corresponding graded $R_F \mathcal{D}_X$ -module $R_F \mathcal{M} := \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M} z^k$. In local coordinates the sheaf of rings $R_F \mathcal{D}_X$ is given by

$$R_F \mathcal{D}_X = \mathcal{O}_X[z] \langle z \partial_{x_1}, \dots, z \partial_{x_n} \rangle$$

Denote by \mathcal{X} the product $X \times \mathbb{C}$. We will consider the sheaf

$$\mathcal{R}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_X[z]} R_F \mathcal{D}_X$$

and its ring of global sections

$$R_{\mathcal{X}} := \Gamma(\mathcal{X}, \mathcal{R}_{\mathcal{X}}) = \mathcal{O}_X(X)[z] \langle z \partial_{x_1}, \dots, z \partial_{x_n} \rangle$$

Given a $R_F \mathcal{D}_X$ -module $R_F \mathcal{M}$ the corresponding $\mathcal{R}_{\mathcal{X}}$ -module

$$\mathcal{M} := \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_X[z]} R_F \mathcal{M}$$

This gives an exact functor \mathcal{T} from the category of filtered \mathcal{D}_X -modules $MF(\mathcal{D}_X)$ to the category of $\mathcal{R}_{\mathcal{X}}$ -modules $\text{Mod}(\mathcal{R}_{\mathcal{X}})$

$$\begin{aligned} \mathcal{T} : MF(\mathcal{D}_X) &\longrightarrow \text{Mod}(\mathcal{R}_{\mathcal{X}}) \\ (\mathcal{M}, F_\bullet \mathcal{M}) &\mapsto \mathcal{M} \end{aligned}$$

We denote by $\text{Mod}_{qc}(\mathcal{R}_{\mathcal{X}})$ the category of $\mathcal{R}_{\mathcal{X}}$ -modules which are quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -modules. We denote by $\Omega_{\mathcal{X}}^1 = z^{-1} \Omega_{X \times \mathbb{C}/\mathbb{C}}^1$ the sheaf of algebraic 1-forms on \mathcal{X} relative to the projection $\mathcal{X} \rightarrow \mathbb{C}$ having at most a pole of order one along $z = 0$. If we put $\Omega_{\mathcal{X}}^k = \wedge^k \Omega_{\mathcal{X}}^1$, we get a deRham complex

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}} \xrightarrow{d} \Omega_{\mathcal{X}}^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_{\mathcal{X}}^n \longrightarrow 0$$

where the differential d is induced by the relative differential $d_{X \times \mathbb{C}/\mathbb{C}}$. If X is a smooth affine variety we get the following equivalence of categories.

Lemma 4.16. *Let X be a smooth affine variety. The functor*

$$\Gamma(\mathcal{X}, \bullet) : \text{Mod}_{qc}(\mathcal{R}_{\mathcal{X}}) \longrightarrow \text{Mod}(R_{\mathcal{X}})$$

is exact and gives an equivalence of categories.

Proof. The proof is completely parallel to the \mathcal{D} -module case (see e.g. [HTT08, Proposition 1.4.4]). \square

One can also define a notion of direct image in the category of \mathcal{R} -modules. Since we only need the case of a projection, we will restrict to this special situation. Let X, Y smooth algebraic varieties and $f : X \times Y \rightarrow Y$ be the projection to the second factor. Similarly as above we have a relative de Rham complex $\Omega_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{\bullet} = z^{-1} \Omega_{X \times Y \times \mathbb{C} / Y \times \mathbb{C}}^{\bullet}$. If \mathcal{M} is an $\mathcal{R}_{\mathcal{X} \times \mathcal{Y}}$ -module the relative de Rham complex $DR_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}(\mathcal{M})$ is locally given by

$$d(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^n \left(\frac{dx_i}{z} \wedge \omega \right) \otimes z \partial_{x_i} m$$

where $(x_i)_{1 \leq i \leq n}$ is a local coordinate of X . The direct image with respect to f is then defined as

$$f_+ \mathcal{M} := Rf_* DR_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}(\mathcal{M})[n]$$

Recall that for a filtered \mathcal{D} -module $(\mathcal{M}, F_{\bullet} \mathcal{M})$ the direct image under f is given by

$$f_+ \mathcal{M} = Rf_* \left(0 \rightarrow \mathcal{M} \rightarrow \Omega_{X \times Y / Y}^1 \otimes \mathcal{M} \rightarrow \dots \rightarrow \Omega_{X \times Y / Y}^n \otimes \mathcal{M} \rightarrow 0 \right) [n]$$

together with its filtration

$$F_p f_+ \mathcal{M} = Rf_* \left(0 \rightarrow F_p \mathcal{M} \rightarrow \Omega_{X \times Y / Y}^1 \otimes F_{p+1} \mathcal{M} \rightarrow \dots \rightarrow \Omega_{X \times Y / Y}^n \otimes F_{p+n} \mathcal{M} \rightarrow 0 \right) [n]$$

It is a straightforward exercise to check that the functor \mathcal{T} commutes with the direct image functor f_+ .

We will apply this to the filtered \mathcal{D} -module $(\mathcal{N}, F_{\bullet}^H \mathcal{N})$ as defined in equation (32) in order to compute $\mathcal{H}^0 \pi_{2+} \mathcal{N} \simeq \mathcal{H}^{2n+d+1}(p_+(q^\dagger \mathcal{O}_T^\beta \otimes F^\dagger j_+ \mathcal{O}_{\mathbb{C}^*}^{-\beta-1}))$ together with its corresponding Hodge filtration. We will denote by $\mathcal{P} \times \mathcal{V}$ the space $\mathbb{P}(W) \times V \times \mathbb{C}$. The corresponding \mathcal{R} -module is

$$\mathcal{N} := \mathcal{T}(\mathcal{N}) = \mathcal{O}_{\mathcal{P} \times \mathcal{V}} \otimes_{\mathcal{O}_{\mathbb{P}(W) \times V}[z]} R_{FH} \mathcal{N}$$

The direct image with respect to π_2 is then given by

$$\pi_{2+} \mathcal{N} \simeq R\pi_{2*} \left(0 \rightarrow \mathcal{N} \rightarrow \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^1 \otimes \mathcal{N} \rightarrow \dots \rightarrow \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^n \otimes \mathcal{N} \rightarrow 0 \right) [n] \quad (36)$$

Since this is rather hard to compute, we will replace the complex

$$0 \rightarrow \mathcal{N} \rightarrow \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^1 \otimes \mathcal{N} \rightarrow \dots \rightarrow \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^n \otimes \mathcal{N} \rightarrow 0$$

by a quasi-isomorphic one. For this we will construct a resolution of \mathcal{N} . Let $\mathcal{W}_u \times \mathcal{V} := W_u \times V \times \mathbb{C}$ and denote by \mathcal{N}_u the restriction of \mathcal{N} to $\mathcal{W}_u \times \mathcal{V}$. We write $R_{\mathcal{W}_u \times \mathcal{V}} = \Gamma(\mathcal{W}_u \times \mathcal{V}, \mathcal{R}_{\mathcal{W}_u \times \mathcal{V}})$, then the module of global sections of \mathcal{N}_u is the $R_{\mathcal{W}_u \times \mathcal{V}}$ -module

$$N_u := \Gamma(\mathcal{W}_u \times \mathcal{V}, \mathcal{N}_u).$$

Proposition 4.17. *The $R_{\mathcal{W}_u \times \mathcal{V}}$ -module N_u is isomorphic to*

$$z^{n-d} \cdot R_{\mathcal{W}_u \times \mathcal{V}} / I_u$$

where I_u is generated by

$$\bar{\square}_{(m,l)} := \prod_{\substack{m_i > 0 \\ i \neq u}} w_{iu}^{m_i} \prod_{\substack{l_i > 0 \\ 0 \leq i \leq n}} (z \partial_{\lambda_i})^{l_i} - \prod_{\substack{m_i < 0 \\ i \neq u}} w_{iu}^{-m_i} \prod_{\substack{l_i < 0 \\ 0 \leq i \leq n}} (z \partial_{\lambda_i})^{-l_i},$$

where $(m, l) = ((m_i)_{i \neq u}, l_0, \dots, l_n) \in \mathbb{L}_{A_u^s}$,

$$\bar{E}_k^u - \beta_k := - \sum_{\substack{i=0 \\ i \neq u}}^n a_{ki}^u z \partial_{w_{iu}} w_{iu} + \sum_{i=1}^n a_{ki} \lambda_i z \partial_{\lambda_i} - \beta_k \quad \text{for } k = 1, \dots, d$$

and

$$\bar{E}_0^u - \beta_0 := \sum_{i=0}^n \lambda_i z \partial_{\lambda_i} - \beta_0$$

Proof. This follows easily from Lemma 4.13 and Lemma 4.16. \square

We will now define a Koszul complex K_u^\bullet in the category of R_u -modules which corresponds to the Koszul complex $\overset{(\vee)}{K}_u^\bullet$ alluded to above. Write J_u for the left ideal in $R_{\mathcal{W}_u \times \mathcal{V}}$ generated by all operators $\bar{\square}_{(k,l)}$ for $(k, l) \in \mathbb{L}_{A_u^s}$, then a computation similar to formula (2) shows that the maps

$$\begin{aligned} R_{\mathcal{W}_u \times \mathcal{V}} / J_u &\longrightarrow R_{\mathcal{W}_u \times \mathcal{V}} / J_u \\ P &\mapsto P \cdot (\bar{E}_k^u - \beta_k) \quad \text{for } k = 0, \dots, d \end{aligned} \quad (37)$$

are well-defined. Since $[\bar{E}_{k_1}^u - \beta_{k_1}, \bar{E}_{k_2}^u - \beta_{k_2}] = 0$ for $k_1, k_2 \in \{0, \dots, d\}$ we can build a Koszul complex

$$K_u^\bullet := \text{Kos} \left(z^{n-d} \cdot R_{\mathcal{W}_u \times \mathcal{V}} / J_u, (\bar{E}_k^u - \beta_k)_{k=0, \dots, d} \right)$$

whose terms are given by

$$z^{n-2d-1} \cdot R_{\mathcal{W}_u \times \mathcal{V}} / J_u \rightarrow \dots \rightarrow \bigoplus_{i=1}^n z^{n-d-1} \cdot R_{\mathcal{W}_u \times \mathcal{V}} / J_u e_1 \wedge \hat{e}_i \wedge \dots \wedge e_d \rightarrow z^{n-d} \cdot R_{\mathcal{W}_u \times \mathcal{V}} / J_u e_1 \wedge \dots \wedge e_d$$

Lemma 4.18. *The Koszul complex K_u^\bullet is a resolution of N_u .*

Proof. In order to prove the Lemma it is enough to apply the exact Rees functor \mathcal{S} to the Koszul complex $\overset{(\vee)}{K}_u^\bullet$ which is a strict resolution of N_u in the category of filtered $D_{W_u \times V}$ -modules by Proposition 4.15. \square

We denote by \mathcal{K}_u^\bullet the corresponding resolution of $\mathcal{N}_u = \mathcal{N}_{|W_u \times V}$. We are now able to construct a resolution of \mathcal{N} .

Proposition 4.19. *There exists a resolution \mathcal{K}^\bullet of \mathcal{N} in the category of $\mathcal{R}_{\mathcal{P} \times \mathcal{V}}$ -modules which is locally given by*

$$\Gamma(\mathcal{W}_u \times \mathcal{V}, \mathcal{K}^\bullet) = K_u^\bullet$$

Proof. The resolution \mathcal{K}^\bullet is constructed by providing glueing maps between the $R_{\mathcal{W}_{u_1 u_2} \times \mathcal{V}}$ -modules

$$\Gamma(\mathcal{W}_{u_1 u_2} \times \mathcal{V}, \mathcal{K}_{u_1}^\bullet) \simeq K_{u_1}^\bullet[w_{u_2 u_1}^{-1}] \longrightarrow \Gamma(\mathcal{W}_{u_1 u_2} \times \mathcal{V}, \mathcal{K}_{u_2}^\bullet) \simeq K_{u_2}^\bullet[w_{u_1 u_2}^{-1}]$$

which are compatible with the glueing maps on

$$\Gamma(\mathcal{W}_{u_1 u_2} \times \mathcal{V}, \mathcal{N}_{u_1}) \simeq N_{u_1}[w_{u_2 u_1}^{-1}] \longrightarrow \Gamma(\mathcal{W}_{u_1 u_2} \times \mathcal{V}, \mathcal{N}_{u_2}) \simeq N_{u_2}[w_{u_1 u_2}^{-1}]$$

Notice that the latter maps are given by

$$\begin{aligned} N_{u_1}[w_{u_2 u_1}^{-1}] &\longrightarrow N_{u_2}[w_{u_1 u_2}^{-1}] \\ P &\mapsto \iota_{u_1 u_2}(P) w_{u_1 u_2}^{n+1} \end{aligned}$$

which follows from Lemma 4.10 and by tracing back the functors applied to $g_{u+} \mathcal{O}_T$. Using the same argument as in Lemma 4.10 shows that the maps

$$\begin{aligned} K_{u_1}^\bullet[w_{u_2 u_1}^{-1}] &\longrightarrow K_{u_2}^\bullet[w_{u_1 u_2}^{-1}] \\ P &\mapsto \iota_{u_1 u_2}(P) w_{u_1 u_2}^{n+1} \end{aligned}$$

are well defined. We have to check that they give rise to a morphism of complexes. But this follows from the commutativity of the diagram

$$\begin{array}{ccc}
P & \longrightarrow & \iota_{u_1 u_2}(P)w_{u_1 u_2}^{n+1} \\
\\
\begin{array}{ccc}
R_{\mathcal{W}_{u_1} \times \mathcal{V}}/J_{u_1} & \longrightarrow & R_{\mathcal{W}_{u_2} \times \mathcal{V}}/J_{u_2} \\
\uparrow \cdot \bar{E}_k^{u_1} - \beta_k & & \uparrow \cdot \bar{E}_k^{u_2} - \beta_k \\
R_{\mathcal{W}_{u_1} \times \mathcal{V}}/J_{u_1} & \longrightarrow & R_{\mathcal{W}_{u_2} \times \mathcal{V}}/J_{u_2}
\end{array} \\
\\
P & \longrightarrow & \iota_{u_1 u_2}(P)w_{u_1 u_2}^{n+1}
\end{array}$$

□

4.6 A quasi-isomorphism

We now apply the relative de Rham functor $DR_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}$ to the resolution \mathcal{K}^\bullet and get a double complex $\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{K}^\bullet$:

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{n-1} \otimes \mathcal{K}^0 & \xrightarrow{II d^{n,0}} & \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^n \otimes \mathcal{K}^0 \\
& & \uparrow Id^{n-1,0} & & \uparrow Id^{n,0} \\
\cdots & \longrightarrow & \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{n-1} \otimes \mathcal{K}^{-1} & \xrightarrow{II d^{n,-1}} & \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^n \otimes \mathcal{K}^{-1} \\
& & \uparrow & & \uparrow \\
& & \vdots & & \vdots
\end{array}$$

The corresponding total complex is denoted by $Tot\left(\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{K}^\bullet\right)$.

Proposition 4.20. *The following natural morphisms of complexes*

$$\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{N} \longleftarrow Tot\left(\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{K}^\bullet\right) \longrightarrow \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^n \otimes \mathcal{K}^\bullet / II d\left(\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{n-1} \otimes \mathcal{K}^\bullet\right) =: \mathcal{L}^\bullet$$

are quasi-isomorphisms.

Proof. Since the double complex $\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{K}^\bullet$ is bounded we can associate to it two spectral sequences which both converge. The first one is given by taking cohomology in the vertical direction which gives the IE_1 -page of the spectral sequence. Since \mathcal{K}^\bullet is a resolution of \mathcal{N} and $\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^l$ is a locally free (i.e. flat) $\mathcal{O}_{\mathcal{P} \times \mathcal{V}}$ -module for every $l = 1, \dots, n$, the only terms which are non-zero are the $IE_1^{0,q}$ -terms which are isomorphic to $\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^l \otimes \mathcal{N}$. Hence the first spectral sequence degenerates at the second page which shows that $\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{N} \leftarrow Tot\left(\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{K}^\bullet\right)$ is a quasi-isomorphism.

We now look at the second spectral sequence which is given by taking cohomology in the horizontal direction. We claim that $II E_1^{p,q} = 0$ for $q \neq n$. It is enough to check this locally on the charts $\mathcal{W}_u \times \mathcal{V}$ and moreover using Lemma 4.16 on the level of global sections. Notice that the complex

$$\Gamma(\mathcal{W}_u \times \mathcal{V}, \Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{K}^l)$$

is isomorphic to a direct sum of Koszul complexes $Kos^\bullet(z^{-d-l}R_{\mathcal{W}_u \times \mathcal{V}}/J_{A_u^s}, \frac{1}{z}(z\partial_{w_{iu}} \cdot)_{i \neq u})$ where each summand is given by

$$z^{n-d-l}R_{\mathcal{W}_u \times \mathcal{V}}/J_{A_u^s} \longrightarrow \cdots \longrightarrow z^{-d-l}R_{\mathcal{W}_u \times \mathcal{V}}/J_{A_u^s} e_1 \wedge \cdots \wedge e_n.$$

The quotient $R_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s}$ can be written as

$$\mathbb{C}[z, (z\partial_{w_{iu}})_{i \neq u}] \otimes_{\mathbb{C}[z]} \left(\mathbb{C}[z, \lambda_0, \dots, \lambda_n, (w_{iu})_{i \neq u}] \langle z\partial_{\lambda_0}, \dots, z\partial_{\lambda_n} \rangle / \left((\overline{\square}_{(m,l)})_{(m,l) \in \mathbb{L}_{A_u^s}} \right) \right)$$

Since the operators $z\partial_{w_{iu}} \cdot$ act only on the first term in the tensor product, we immediately see that ${}_{II}E_1^{p,q} = 0$ for $q \neq n$.

The fact that $\text{Tot} \left(\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{H}^\bullet \right) \rightarrow \mathcal{L}^\bullet$ is a quasi-isomorphism follows from the fact that ${}_{II}E^{p,q} = 0$ for $q \neq n$, i.e. the second spectral sequence degenerates at the second page. \square

The next result is an explicit local description of the complex \mathcal{L}^\bullet .

Proposition 4.21. *For any $u \in \{0, \dots, n\}$ define the ring*

$$S_{\mathcal{W}_u \times \mathcal{V}} := \mathbb{C}[z, \lambda_0, \dots, \lambda_n, (w_{iu})_{i \neq u}] \langle z\partial_{\lambda_1}, \dots, z\partial_{\lambda_n} \rangle$$

and denote by \mathcal{S} the sheaf of rings on $\mathcal{W} \times \mathcal{V}$ which is locally given by

$$\Gamma(\mathcal{W}_u \times \mathcal{V}, \mathcal{S}) = S_{\mathcal{W}_u \times \mathcal{V}}$$

with glueing maps

$$\begin{aligned} S_{\mathcal{W}_{u_1} \times \mathcal{V}}[w_{u_2 u_1}^{-1}] &\longrightarrow S_{\mathcal{W}_{u_2} \times \mathcal{V}}[w_{u_1 u_2}^{-1}] \\ P &\longmapsto \iota_{u_1 u_2}(P) \end{aligned}$$

Denote by $J_{A_u^s}$ the left $S_{\mathcal{W}_u \times \mathcal{V}}$ -ideal generated by the Box operators $\overline{\square}_{(k,l)}$ for $(k,l) \in \mathbb{L}_{A_u^s}$. Note that this is a slight abuse of notation, as the ideal generated by the same set of operators in the ring $R_{\mathcal{W}_u \times \mathcal{V}}$ was also denoted by $J_{A_u^s}$, but which is justified by the fact that these generators do not contain the variables $z\partial_{w_{iu}}$. Then the complex \mathcal{L}^\bullet is given locally by

$$\Gamma(\mathcal{W}_u \times \mathcal{V}, \mathcal{L}^\bullet) \simeq \text{Kos}^\bullet(z^{-d} S_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s}, (\tilde{E}_k - \beta_k)_{k=0, \dots, d}) \quad (38)$$

whose terms are given by

$$z^{-2d-1} S_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s} \longrightarrow \dots \longrightarrow z^{-d} S_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s} e_1 \wedge \dots \wedge e_d$$

where

$$\begin{aligned} \tilde{E}_k - \beta_k &:= \sum_{i=1}^n a_{ki} \lambda_i z \partial_{\lambda_i} - \beta_k \quad \text{for } k = 1, \dots, d \\ \tilde{E}_0 - \beta_0 &:= \sum_{i=0}^n \lambda_i z \partial_{\lambda_i} - \beta_0 \end{aligned}$$

Proof. It follows from Proposition 4.20 that the 0-th cohomology of the complex $\left(\Omega_{\mathcal{P} \times \mathcal{V} / \mathcal{V}}^{\bullet+n} \otimes \mathcal{H}^p, {}_{II}d^{\bullet,p} \right)$ is a direct sum of terms of the form $H^0(z^{-d} \text{Kos}^\bullet(R_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s}, \frac{1}{z}(z\partial_{w_{iu}} \cdot)_{i \neq u}))$. Taking the cokernel of left multiplication on $R_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s}$ by $z\partial_{w_{iu}}$ shows that we have an isomorphism of $S_{\mathcal{W}_u \times \mathcal{V}}$ -modules

$$H^0(z^{-d} \text{Kos}^\bullet(R_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s}, (z\partial_{w_{iu}} \cdot)_{i \neq u})) \simeq z^{-d} S_{\mathcal{W}_u \times \mathcal{V}} / J_{A_u^s}.$$

Hence equation (38) follows. \square

The ideals $J_{A_u^s}$ glue to an ideal $\mathcal{J} \subset \mathcal{S}$. Notice that the Euler vector fields $(\tilde{E}_k - \beta_k)_{k=0, \dots, d}$ are global sections of \mathcal{S} . We recall from Proposition 4.19 that the glueing maps for $\Gamma(\mathcal{W}_u \times \mathcal{V}, \Omega_{\mathcal{P} \times \mathcal{V}}^n \otimes \mathcal{H}^p)$ are given by:

$$\bigwedge_{\substack{i=0 \\ i \neq u_1}}^n dw_{iu_1} \otimes P \longmapsto \bigwedge_{\substack{i=0 \\ i \neq u_2}}^n dw_{iu_2} \cdot (w_{u_1 u_2})^{-n-1} \otimes \iota_{u_1 u_2}(P) w_{u_1 u_2}^{n+1}$$

Since both powers of $w_{u_1 u_2}$ on the right hand side cancel when considering the quotient \mathcal{L}^p , we see that

$$\mathcal{L}^\bullet \simeq \text{Kos}^\bullet(z^{-d} \mathcal{S} / \mathcal{J}, (\tilde{E}_k - \beta_k)_{k=0, \dots, d})$$

Summarizing, Proposition 4.20 and Proposition 4.21 show that instead of computing the direct image (36) we can compute

$$R\pi_{2*}(\mathcal{L}^\bullet) \simeq R\pi_{2*}(z^{-d} \text{Kos}^\bullet(\mathcal{S} / \mathcal{J}, (\tilde{E}_k - \beta_k)_{k=0, \dots, d}))$$

4.7 Computation of the direct image

Because of Lemma 4.16 it is enough to work on the level of global sections:

$$\Gamma R\pi_{2*}(\mathcal{L}^\bullet) \simeq R\Gamma R\pi_{2*}(\mathcal{L}^\bullet) \simeq R\Gamma(\mathcal{L}^\bullet) \simeq R\Gamma(Kos^\bullet(z^{-d}\mathcal{S}/\mathcal{I}, (\tilde{E} - \beta_k)_{k=0,\dots,d})) \quad (39)$$

where the first isomorphism follows from the exactness of $\Gamma(\mathcal{V}, \bullet)$.

We will show that each term of the complex \mathcal{L}^\bullet is Γ -acyclic. For this it is enough to show that \mathcal{S}/\mathcal{I} is Γ -acyclic. Recall that $\mathcal{P} \times \mathcal{V} = \mathbb{C}_z \times \mathbb{P}(W) \times V$. We denote by $\mathcal{W} \times \mathcal{V}$ the space $\mathbb{C}_z \times W \times V$. Let

$$S := \mathbb{C}[z, w_0, \dots, w_n, \lambda_0, \dots, \lambda_n] \langle z\partial_{\lambda_0}, \dots, z\partial_{\lambda_n} \rangle$$

and consider the S -module

$$S/J_{A^s}$$

where the left ideal J_{A^s} is generated by

$$\square_{(k,l)} = \prod_{k_i > 0} w_i^{k_i} \prod_{l_i > 0} (z\partial_{\lambda_i})^{l_i} - \prod_{k_i < 0} w_i^{-k_i} \prod_{l_i < 0} (z\partial_{\lambda_i})^{-l_i} \quad \text{for } (k, l) \in \mathbb{L}_{A^s}$$

and the matrix A^s is given by

$$A^s := (\underline{a}_0^s, \dots, \underline{a}_n^s, \underline{b}_0^s, \dots, \underline{b}_n^s) := \begin{pmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & a_{11} & \dots & a_{1n} & 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{d1} & \dots & a_{dn} & 0 & a_{d1} & \dots & a_{dn} \end{pmatrix}$$

and \mathbb{L}_{A^s} is the \mathbb{Z} -module of relations among the columns of A^s . Notice that S/J_{A^s} is \mathbb{Z} -graded by the degree of the w_i . Denote by S_{w_u} the localization of S with respect to w_u , then one easily sees that the degree zero part $[S_{w_u}/J_{A^s}]_0$ of S_{w_u}/J_{A^s} is equal to $\Gamma(\mathcal{W}_u \times \mathcal{V}, \mathcal{S}/\mathcal{I}) \simeq S_{\mathcal{W}_u \times \mathcal{V}}/J_{A^s}$ if we identify w_i/w_u with w_{iu} . Let $\widetilde{S/J_{A^s}}$ be the associated sheaf on $\mathcal{P} \times \mathcal{V}$ having global sections $[S/J_{A^s}]_0$, then we obviously have

$$\widetilde{S/J_{A^s}} \simeq \mathcal{S}/\mathcal{I}$$

Define

$$\Gamma_*(\mathcal{S}/\mathcal{I}) := \bigoplus_{a \in \mathbb{Z}} \Gamma(\mathcal{P} \times \mathcal{V}, (\mathcal{S}/\mathcal{I})(a))$$

We want to use the following result applied to the graded module S/J_{A^s}

Proposition 4.22. [Gro61, Proposition 2.1.3] *There is the following exact sequence of \mathbb{Z} -graded S -modules*

$$0 \longrightarrow H_{(\underline{w})}^0(S/J_{A^s}) \longrightarrow S/J_{A^s} \longrightarrow \Gamma_*(\mathcal{S}/\mathcal{I}) \longrightarrow H_{(\underline{w})}^1(S/J_{A^s}) \longrightarrow 0$$

and for each $i \geq 1$ the following isomorphisms

$$\bigoplus_{a \in \mathbb{Z}} H^i(\mathcal{P} \times \mathcal{V}, (\mathcal{S}/\mathcal{I})(a)) \simeq H_{(\underline{w})}^{i+1}(S/J_{A^s}) \quad (40)$$

where (\underline{w}) is the ideal in $\mathbb{C}[z, \lambda_0, \dots, \lambda_n, w_0, \dots, w_n]$ generated by w_0, \dots, w_n .

Proof. In the category of $\mathbb{C}[z, \lambda_0, \dots, \lambda_n, w_0, \dots, w_n]$ -modules, the statement follows from [Gro61, Proposition 2.1.3]. The statement in the category of S -modules follows from the proof given there. \square

In order to compute the local cohomology of S/J_{A^s} we introduce a variant of the Ishida complex (see e.g. [BH93, Theorem 6.2.5]). Let $T := \mathbb{C}[w_0, \dots, w_n, z\partial_{\lambda_0}, \dots, z\partial_{\lambda_n}] \subset S$ be a commutative subring and let $\mathbb{C}[\mathbb{N}A^s]$ be the affine semigroup algebra of A^s , i.e.

$$\mathbb{C}[\mathbb{N}A^s] = \{y^{\underline{c}} \in \mathbb{C}[y_0^\pm, \dots, y_{n+1}^\pm] \mid \underline{c} \in \mathbb{N}A^s \subset \mathbb{Z}^{d+2}\}$$

We have a map

$$\begin{aligned}\Phi_{A^s} : T &\longrightarrow \mathbb{C}[\mathbb{N}A^s] \\ w_i &\mapsto y_i^{a_i^s} \\ z\partial_{\lambda_i} &\mapsto y_i^{b_i^s}\end{aligned}$$

Notice that the kernel K_{A^s} of Φ_{A^s} is equal to the ideal in T generated by the elements $\square_{(k,l)}$, hence

$$T/K_{A^s} \simeq \mathbb{C}[\mathbb{N}A^s]$$

Remark 4.23. *The \mathbb{Z} -grading of T by the degree of the w_i induces a \mathbb{Z} -grading on $\mathbb{C}[\mathbb{N}A^s]$ since the operators $\square_{(k,l)}$ are homogeneous. The semi-group ring $\mathbb{C}[\mathbb{N}A^s] \subset \mathbb{C}[\mathbb{Z}^{d+2}]$ carries also a natural \mathbb{Z}^{d+2} -grading. Looking at the matrix A^s one sees that the \mathbb{Z} -grading coming from T is the first component of this \mathbb{Z}^{d+2} -grading.*

We regard $\mathbb{C}[\mathbb{N}A^s]$ as a T -module using the map Φ_{A^s} , which gives the isomorphisms

$$S/J_{A^s} \simeq S \otimes_T T/K_{A^s} \simeq S \otimes_T \mathbb{C}[\mathbb{N}A^s]$$

We want to express the local cohomology of S/J_{A^s} by the local cohomology of the commutative ring $\mathbb{C}[\mathbb{N}A^s]$. For this, let I be the ideal in $\mathbb{C}[\mathbb{N}A^s]$ generated by $y_0^{a_0^s}, \dots, y_n^{a_n^s}$, then we have the following change of rings formula:

Lemma 4.24. *There is the following isomorphism of \mathbb{Z} -graded S -modules:*

$$H_{(w)}^k(S/J_{A^s}) \simeq S \otimes_T H_I^k(\mathbb{C}[\mathbb{N}A^s])$$

Proof. Notice that if S was commutative this would be a standard property of the local cohomology groups. Here we have to adapt the proof slightly. First notice that it is enough to compute $H_{(w)}^k(S/J_{A^s})$ with an injective resolution of T -modules. To see why, let I^\bullet be an injective resolution (in the category of S -modules) of S/J_{A^s} . Since S is a free, hence flat, T -module, it follows from $\text{Hom}_S(S \otimes_T M, I) \simeq S \otimes_T \text{Hom}_T(M, I)$, that an injective S -module is also an injective T -module. Therefore we have

$$H_{(w)}^k(S/J_{A^s}) \simeq H^k \Gamma_{(w)}(I^\bullet) = H^k \Gamma_{I'}(I^\bullet) \simeq H_{I'}^k(S/J_{A^s})$$

where I' is the ideal in T generated by w_0, \dots, w_n and the second isomorphism follows from the equality

$$\Gamma_{(w)}(I^k) = \{x \in I^k \mid \forall i \exists k_i \text{ such that } w_i^{k_i} x = 0\} = \Gamma_{I'}(I^k)$$

Let J^\bullet be an injective resolution of T/K_{A^s} . In order to show the claim consider the following isomorphisms

$$\begin{aligned}S \otimes_T H_I^k(\mathbb{C}[\mathbb{N}A^s]) &\simeq S \otimes_T H_{I'}^k(T/K_{A^s}) \\ &\simeq S \otimes_T H^k \Gamma_{I'}(J^\bullet) \\ &\simeq H^k(S \otimes_T \Gamma_{I'}(J^\bullet)) \\ &\simeq H^k \Gamma_{I'}(S \otimes_T J^\bullet) \\ &\simeq H_{I'}^k(S/J_{A^s})\end{aligned}$$

where the third isomorphism follows from the fact that S is a flat T -module and the fifth isomorphism follows from the fact that $S \otimes_T J^\bullet$ is a T -injective resolution of $S/J_{A^s} \simeq S \otimes_T T/K_{A^s}$. \square

4.8 Local cohomology of semi-group rings

Let \mathcal{F} be the face lattice of $\mathbb{R}_{\geq 0}A^s$ and denote by \mathcal{F}_σ the sub-lattice of faces which lie in the face σ spanned by a_0^s, \dots, a_n^s . For a face σ of $\mathbb{R}_{\geq 0}A^s$ consider the multiplicatively closed set

$$U_\sigma := \{y^{\underline{c}} \mid \underline{c} \in \mathbb{N}(A^s \cap \sigma)\}$$

and denote by $\mathbb{C}[\mathbb{N}A^s]_\sigma = \mathbb{C}[\mathbb{N}A^s + \mathbb{Z}(A^s \cap \sigma)]$ the localization. We put

$$L_\sigma^k = \bigoplus_{\substack{\tau \in \mathcal{F}_\sigma \\ \dim \tau = k}} \mathbb{C}[\mathbb{N}A^s]_\tau$$

and define maps $f^k : L_\sigma^k \rightarrow L_\sigma^{k+1}$ by specifying its components

$$f_{\tau', \tau}^k : \mathbb{C}[\mathbb{N}A^s]_{\tau'} \rightarrow \mathbb{C}[\mathbb{N}A^s]_\tau \quad \text{to be} \quad \begin{cases} 0 & \text{if } \tau' \not\subset \tau \\ \epsilon(\tau', \tau) \text{nat} & \text{if } \tau' \subset \tau \end{cases}$$

where ϵ is a suitable incidence function on \mathcal{F}_σ and nat the natural localization morphism. The Ishida complex with respect to the face σ is

$$L_\sigma^\bullet : 0 \rightarrow L_\sigma^0 \rightarrow L_\sigma^1 \rightarrow \dots \rightarrow L_\sigma^{d+1} \rightarrow 0$$

The Ishida complex with respect to the face σ can be used to calculate local cohomology groups of $\mathbb{C}[\mathbb{N}A^s]$.

Proposition 4.25. *As above, denote by $I \subset \mathbb{C}[\mathbb{N}A^s]$ the ideal generated by the elements $\Phi_{A^s}(w_i) = y^{\underline{a}_i^s}$. Then for all k we have the isomorphism*

$$H_I^k(\mathbb{C}[\mathbb{N}A^s]) \simeq H^k(L_\sigma^\bullet)$$

Proof. The proof can be easily adapted from [BH93, Theorem 6.2.5]. For the convenience of the reader we sketch it here together with the necessary modifications. In order to show the claim we have to prove that the functors $N \mapsto H^k(L_\sigma^\bullet \otimes N)$ form a universal δ -functor (see e.g. [Har77]). If we can additionally show that

$$H_I^0(\mathbb{C}[\mathbb{N}A^s]) \simeq H^0(L_\sigma^\bullet) \tag{41}$$

the claim follows by [Har77, Corollary III.1.4]. Let $\mathcal{F}_\sigma(1)$ be the set of one-dimensional faces in \mathcal{F}_σ and notice that

$$H_{I'}^0(\mathbb{C}[\mathbb{N}A^s]) \simeq \ker \left(\mathbb{C}[\mathbb{N}A^s] \longrightarrow \bigoplus_{\tau \in \mathcal{F}_\sigma(1)} \mathbb{C}[\mathbb{N}A^s]_\tau \right) \simeq H^0(L_\sigma^\bullet \otimes_T M)$$

where $I' \subset \mathbb{C}[\mathbb{N}A^s]$ is the ideal generated by $\{y^{\underline{a}_i^s} \mid \mathbb{R}_{\geq 0} \underline{a}_i^s \in \mathcal{F}_\sigma(1)\}$. In order to show (41) we have to show that $\text{rad } I' = I$ since obviously $H_{I'}^0(\mathbb{C}[\mathbb{N}A^s]) = \overline{H}_{\text{rad } I'}^0(\mathbb{C}[\mathbb{N}A^s])$. Since $I' \subset I$ and $I = \text{rad } I$ (I is a prime ideal corresponding to the face spanned by $\underline{a}_0^s, \dots, \underline{a}_n^s$), it is enough to check that a multiple of every $y^\underline{c} \in I$ lies in I' . But this follows easily from the fact that the elements $\{\underline{a}_i^s \mid \mathbb{R}_{\geq 0} \underline{a}_i^s \in \mathcal{F}_\sigma(1)\}$ span the same cone over \mathbb{Q} as the elements $\{\underline{a}_0^s, \dots, \underline{a}_n^s\}$.

The proof that $N \mapsto H^k(L_\sigma^\bullet \otimes_T N)$ is a δ -functor is completely parallel to the proof in [BH93]. \square

Notice that the complex L_σ^\bullet is \mathbb{Z}^{d+2} -graded since $\mathbb{C}[\mathbb{N}A^s]$ is \mathbb{Z}^{d+2} -graded. In order to analyze the cohomology of L_σ^\bullet we look at its \mathbb{Z}^{d+2} -graded parts. For this we have to determine when $(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \neq 0$ (and therefore $(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \simeq \mathbb{C}$) for $x \in \mathbb{Z}^{d+2}$.

We are following [BH93, Chapter 6.3]. Denote by C_{A^s} the cone $\mathbb{R}_{\geq 0} A^s \subset \mathbb{R}^{d+2}$. Let $x, y \in \mathbb{R}^{d+2}$. We say that y is visible from x if $y \neq x$ and the line segment $[x, y]$ does not contain a point $y' \in C_{A^s}$ with $y' \neq y$. A subset S is visible from X if each $v \in S$ is visible from x .

Recall that the cone C_{A^s} is given by the intersection of finitely many half-spaces

$$H_\tau^+ := \{x \in \mathbb{R}^{d+2} \mid \langle n_\tau, x \rangle \geq 0\} \quad \tau \in \mathcal{F}(d+1)$$

where $\mathcal{F}(d+1)$ is the set of $d+1$ -dimensional faces (facets) of C_{A^s} . We set

$$x^0 = \{\tau \mid \langle n_\tau, x \rangle = 0\}, \quad x^+ = \{\tau \mid \langle n_\tau, x \rangle > 0\}, \quad x^- = \{\tau \mid \langle n_\tau, x \rangle < 0\}$$

Lemma 4.26. [BH93, Lemma 6.3.2, 6.3.3]

1. A point $y \in C_{A^s}$ is visible from $x \in \mathbb{R}^{d+2} \setminus C_{A^s}$ if and only if $y^0 \cap x^- \neq \emptyset$.
2. Let $x \in \mathbb{Z}^{d+2}$ and τ be a face of C_{A^s} . The \mathbb{C} -vector space $(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \neq 0$ if and only if τ is not visible from x .

Recall that the facet $\sigma \in \mathcal{F}(d+1)$ is spanned by $\underline{a}_0^s, \dots, \underline{a}_n^s$. It is the unique maximal element in the face lattice $\mathcal{F}_\sigma \subset \mathcal{F}$. Denote by H_σ its supporting hyperplane (i.e. $\sigma = C_{A^s} \cap H_\sigma$) which is given by

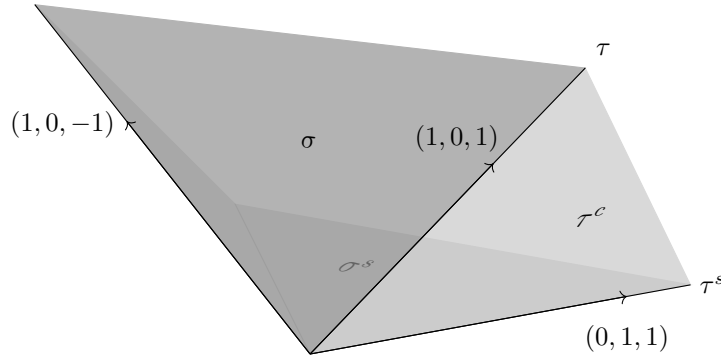
$$H_\sigma = \{x \in \mathbb{R}^{d+2} \mid \langle n_\sigma, x \rangle = 0\}$$

where $n_\sigma = (0, 1, 0, \dots, 0)$. Let $\tau \in \mathcal{F}_\sigma$ be a k -dimensional face contained in σ and set $I_\tau := \{i \mid \underline{a}_i^s \in \tau\}$. Notice that the vectors $\{\underline{a}_i^s \mid i \in I_\tau\}$ span the face τ . This face τ gives rise to two other faces, namely its "shadow" τ^s which is spanned by the vectors $\{\underline{b}_i^s \mid \underline{a}_i^s \in \tau\}$ and the unique $k+1$ -dimensional face τ^c which contains both τ and τ^s . Let $\{\tau_1, \dots, \tau_m\} = \mathcal{F}_\sigma(d)$ be the faces of dimension d contained in σ , which give rise to the facets $\tau_1^c, \dots, \tau_m^c$.

Example 4.27. Consider the matrix

$$A^s = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}$$

where the face σ is generated by $(1, 0, 0)$, $(1, 0, 1)$, $(1, 0, -1)$ and its shadow σ^s is generated by $(0, 1, 0)$, $(0, 1, 1)$, $(0, 1, -1)$. The facet τ^c is generated by τ and its shadow τ^s .



First notice that by Lemma 4.26.1 the facet σ is visible from a point $x \in \mathbb{R}^{d+2}$ if and only if $\langle n_\sigma, x \rangle < 0$. If $\langle n_\sigma, x \rangle \geq 0$ holds, it follows from Lemma 4.26.1. that a face $\tau_i \subset \sigma$ is visible from x if and only if the facet τ_i^c is visible from x , i.e. $\langle n_{\tau_i^c}, x \rangle < 0$.

We define

$$S := \mathbb{Z}^{d+2} \cap (\mathbb{R}(\underline{a}_0^s, \dots, \underline{a}_n^s) + \mathbb{R}_{\geq 0}(\underline{b}_0^s, \dots, \underline{b}_n^s)),$$

this is the set of \mathbb{Z}^{d+2} -degrees occurring in $\mathbb{C}[\mathbb{N}A^s]_\sigma$. Notice also that we have

$$H_\sigma = \mathbb{R}(\underline{a}_0^s, \dots, \underline{a}_n^s) \quad \text{and} \quad H_\sigma^+ = \mathbb{R}(\underline{a}_0^s, \dots, \underline{a}_n^s) + \mathbb{R}_{\geq 0}(\underline{b}_0^s, \dots, \underline{b}_n^s).$$

Given a point $x \in S$ with $\langle n_\sigma, x \rangle \geq 0$ we will construct a point $y_x \in \mathbb{Z}^{d+2}$ which lies in H_σ such that τ_i is visible from x if and only if it is visible from y_x for all $i = 1, \dots, m$. Denote by z_x the projection of x to the sub-vector space generated by $\underline{b}_0^s, \dots, \underline{b}_n^s$. Since the semi-group generated by these vectors is saturated, we can express z_x by a linear combination with positive integers

$$z_x = \sum_{i=0}^n r_i^x \underline{b}_i^s \quad \text{with} \quad r_i^x \in \mathbb{N}$$

Since we have $0 = \langle n_{\tau_i^c}, \underline{a}_j^s - \underline{b}_j^s \rangle = \langle n_{\tau_i^c}, (1, -1, 0, \dots, 0) \rangle$ for any $\underline{a}_j^s, \underline{b}_j^s \in \tau_i^c$ the first two components of the vector $n_{\tau_i^c}$ are equal. Hence, if we set

$$y_x := x + \sum_{j=0}^n r_j^x \underline{a}_j^s - \sum_{i=0}^n r_i^x \underline{b}_i^s$$

we easily see that

$$\langle n_{\tau_i^c}, x \rangle = \langle n_{\tau_i^c}, y_x \rangle \quad \text{for } i = 1, \dots, m. \quad (42)$$

It follows that τ_i is visible from any point $x \in S$ iff it is visible from y_x , as required. Let us remark that the vectors x and y_x only differ in the first two components, because the same is true for the pair of vectors $(\underline{a}_i^s, \underline{b}_i^s)$ for all $i \in \{0, \dots, n\}$.

Lemma 4.28. *In the above situation, let $x \in S$. Then $y_x \in S \cap H_\sigma$ and we have*

$$(L_\sigma^\bullet)_x = (L_\sigma^\bullet)_{y_x}$$

Proof. For the first point, notice that the vector $x - z_x$ is precisely the projection of x to H_σ . On the other hand, we have $y_x = x - z_x + \sum_{i=0}^n n_i^x \underline{a}_i^s$, and $\sum_{i=0}^n n_i^x \underline{a}_i^s$ is an element of H_σ anyhow.

The second statement is an easy consequence of Lemma 4.26.2. More precisely, Equation (42) shows that the visibility of some facet τ_i^c is the same from x and from y_x . Moreover, σ is not visible from both x and y_x (i.e., $\langle n_\sigma, x \rangle \geq 0$, $\langle n_\sigma, y_x \rangle \geq 0$), hence, also the visibility of τ_i is the same from x and from y_x . We conclude that any localization $\mathbb{C}[\mathbb{N}A^s]_\tau$ (for any face $\tau \subset \sigma$) vanishes in degree x iff it vanishes in degree y_x . This yields the desired equality $(L_\sigma^\bullet)_x = (L_\sigma^\bullet)_{y_x}$. \square

We are now able to compute the cohomology of the Ishida complex with respect to the face σ . Set

$$H_{\tau_i^c}^- := \{x \in \mathbb{R}^{d+2} \mid \langle n_{\tau_i^c}, x \rangle < 0\} \quad \text{for } i = 1, \dots, m$$

and define

$$S^- := \mathbb{Z}^{d+2} \cap H_\sigma^+ \cap \bigcap_{i=1}^m H_{\tau_i^c}^-.$$

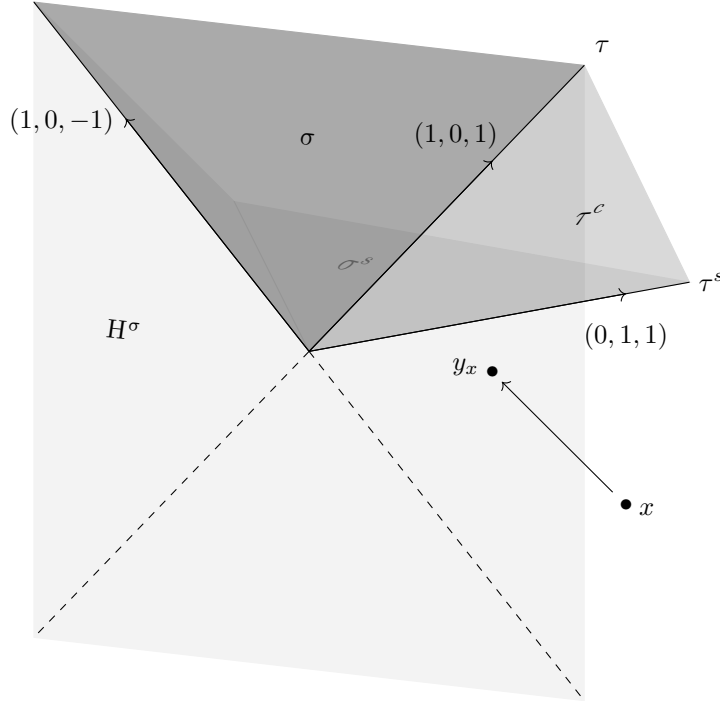
Notice that $S = \mathbb{Z}^{d+2} \cap H_\sigma^+$, hence we have a natural inclusion $S^- \subset S$.

Example 4.29. *We consider again the matrix*

$$A^s = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}$$

and take the point $x = (-1, 1, 1)$. Its projection to $\mathbb{R}(\underline{b}_0^s, \dots, \underline{b}_n^s)$ is $(0, 1, 1)$, hence we get

$$y_x = x + (1, 0, 1) - (0, 1, 1) = (0, 0, 1) \in H^\sigma.$$



Proposition 4.30. *Let A^s as above. Take any $x \in \mathbb{Z}^{d+2}$ and denote by L_σ^\bullet the Ishida complex with respect to the face σ generated by $\underline{a}_0^s, \underline{a}_1^s, \dots, \underline{a}_n^s$.*

1. *If $x \notin S$, then $(L_\sigma^\bullet)_x = 0$.*
2. *If $x \in S \setminus S^-$, then $H^i(L_\sigma^\bullet)_x = 0$ for all i*
3. *If $x \in S^-$, then $H^i(L_\sigma^\bullet)_x = 0$ for $i \neq d+1$ and $H^{d+1}(L_\sigma^\bullet)_x \simeq \mathbb{C}$.*

Proof. The first point follows from the fact that we have $(\mathbb{C}[\mathbb{N}A^s]_\sigma)_x = 0$ for $x \notin S$, hence $(L_\sigma^i)_x = 0$ for all i . For the proof of the second and third point, it is sufficient to consider the case where $x \in H_\sigma$: Namely, in both cases we have $x \in S$ so that Lemma 4.28 apply. We can thus replace x by y_x , i.e., $(L_\sigma^\bullet)_x = (L_\sigma^\bullet)_{y_x}$. Moreover, $x \in S^-$ if and only if $y_x \in S^- \cap H_\sigma$ by formula (42). Hence we will suppose in the remainder of this proof that $x \in S \cap H_\sigma$.

We will reduce statements 2. and 3. for $x \in S \cap H_\sigma$ to the computation of the local cohomology of a semi-group ring with respect to a maximal ideal via the Ishida complex as done in [BH93, Theorem 6.3.4]. For this, we will use the matrix $\tilde{A} = (\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n)$, which can be seen as the matrix of the first $n+1$ columns of A^s , with the second row deleted. The semigroup $\mathbb{N}\tilde{A}$ (resp. the cone $C_{\tilde{A}}$) embeds into $\mathbb{N}A^s$ (resp. into C_{A^s}) via the map $\tilde{a}_i \mapsto \underline{a}_i^s$, and these embeddings are compatible with the embeddings $\mathbb{R}^{r+1} \hookrightarrow \mathbb{R}^{r+2}$ (resp. $\mathbb{Z}^{r+1} \hookrightarrow \mathbb{Z}^{r+2}$) given by

$$(x_1, x_3, x_4, \dots, x_{d+2}) \mapsto (x_1, 0, x_3, x_4, \dots, x_{d+2}).$$

The following equality of semi-groups holds true:

$$S^- \cap H_\sigma = \mathbb{Z}^{d+1} \cap \text{Int}(-C_{\tilde{A}}), \quad (43)$$

where both intersections are taken in \mathbb{Z}^{d+2} . To show this, notice that $C_{\tilde{A}} = \mathbb{R}_{\geq 0}(\underline{a}_0^s, \dots, \underline{a}_n^s) \cap H_\sigma$, that $\tau_i^c \cap H_\sigma = \tau_i$ and hence

$$\bigcap_{i=1}^m (H_{\tau_i^c}^- \cap H_\sigma) = \text{Int}(-C_{\tilde{A}})$$

Consider the projection map

$$p: \mathbb{R}^{d+2} \longrightarrow \mathbb{R}^{d+1} \\ (x_1, x_2, x_3, \dots, x_{d+2}) \mapsto (x_1, x_3, \dots, x_{d+2})$$

which forgets the second component, then for all $\tau \subset \sigma$ and all elements $x \in S \cap H_\sigma$ we have that

$$(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \simeq \left(\mathbb{C}[\mathbb{N}\tilde{A}]_{p(\tau)} \right)_{p(x)}$$

Under this isomorphism the \mathbb{Z}^{d+2} -graded part $(L_\sigma^\bullet)_x$ of the Ishida complex with respect to the face σ goes over to the \mathbb{Z}^{d+1} -graded part $(L_\sigma^\bullet)_{p(x)}$ of the Ishida complex considered in [BH93] (i.e., the Ishida complex of the semi-group $\mathbb{C}[\mathbb{N}\tilde{A}]$ with respect to the maximal ideal generated by (w_0, \dots, w_n)). Using formula (43), the proposition follows now from Theorem 6.3.4 in loc.cit. \square

We finish this section by the following easy consequence, which will be crucial in the proof of the main result (Theorem 4.35 below).

Corollary 4.31. *In the above situation, we have $H^i(L_\sigma^\bullet) = 0$ for all $i \neq d+1$, $H^{d+1}(L_\sigma^\bullet)_x = 0$ for all $x \in \mathbb{Z}^{d+2} \setminus S^-$ and $\deg_{\mathbb{Z}}(H^{d+1}(L_\sigma^\bullet)_x) < 0$ for $x \in S^-$, where $\deg_{\mathbb{Z}}(-)$ refers to the \mathbb{Z} -grading of $H^i(L_\sigma^\bullet)$ corresponding to the first row of A^s .*

In other words, the cohomology groups of the Ishida complex (with respect to the face σ) are concentrated in negative degrees.

Proof. The first two statements are precisely those from Proposition 4.30, points 1. and 2. In order to show the third one, notice that for any $x \in S$, we have $\deg_{\mathbb{Z}}(x) \leq \deg_{\mathbb{Z}}(y_x)$ (this follows from the very definition of the vector y_x). Now let $x \in S^-$, and suppose that $H^{d+1}(L_\sigma^\bullet)_x \neq 0$. From Lemma 4.28 we deduce that

$$H^{d+1}(L_\sigma^\bullet)_x = H^{d+1}(L_\sigma^\bullet)_{y_x},$$

and as already remarked above, $y_x \in S^- \cap H_\sigma$ because $x \in S^-$. However, we deduce from formula (43) that $\deg_{\mathbb{Z}}(y_x) < 0$ if $y_x \in S^- \cap H_\sigma$, so that we obtain $\deg_{\mathbb{Z}}(H^{d+1}(L_\sigma^\bullet)_x) < 0$, as required. \square

4.9 Proof of the main theorem

Corollary 4.32. *The \mathcal{S} -modules \mathcal{S}/\mathcal{J} are Γ -acyclic.*

Proof. If we consider the degree zero part of formula (40), then it suffices to show that the \mathbb{Z} -graded local cohomology S -modules $H_{(w)}^*(S/J_{A_s})$ are concentrated in negative degrees. By Proposition 4.25 and Lemma 4.24, these local cohomology groups are calculate by the Ishida complex $L_{\mathcal{S}}^{\bullet}$, i.e., we have isomorphisms

$$H_{(w)}^k(S/J_{A_s}) \simeq S \otimes_T H_I^k(\mathbb{C}[\mathbb{N}A^s]) \simeq S \otimes_T H^k(L_{\mathcal{S}}^{\bullet}).$$

The cohomology groups $H^k(L_{\mathcal{S}}^{\bullet})$ are concentrated in negative degrees by Corollary 4.31 (and tensoring with S do not change the \mathbb{Z} -degree which is counted with respect to the degree of the variables w_0, \dots, w_n). Hence the result follows. \square

Proposition 4.33. *There is the following isomorphism in $D^b(\mathcal{R}_V)$:*

$$\Gamma\pi_{2+}\mathcal{N} \simeq \Gamma(Kos^{\bullet}(z^{-d}\mathcal{S}/\mathcal{J}, (\tilde{E} - \beta_k)_{k=0, \dots, d}))$$

Proof. By formula (36), Proposition 4.20 and Proposition 4.21 we have the isomorphisms

$$\begin{aligned} \Gamma\pi_{2+}\mathcal{N} &\simeq \Gamma R\pi_{2*}(\Omega_{\mathcal{S} \times \mathcal{Y} | \mathcal{Y}}^{\bullet+n} \otimes \mathcal{N}) \\ &\simeq R\Gamma R\pi_{2*}(\Omega_{\mathcal{S} \times \mathcal{Y} | \mathcal{Y}}^{\bullet+n} \otimes \mathcal{N}) \\ &\simeq R\Gamma(\Omega_{\mathcal{S} \times \mathcal{Y} | \mathcal{Y}}^{\bullet+n} \otimes \mathcal{N}) \\ &\simeq R\Gamma(\mathcal{L}^{\bullet}) \end{aligned} \tag{44}$$

Using the last isomorphism in (39) and Corollary 4.32 we get

$$R\Gamma(\mathcal{L}^{\bullet}) \simeq R\Gamma(Kos^{\bullet}(z^{-d}\mathcal{S}/\mathcal{J}, (\tilde{E})_{k=0, \dots, d})) \simeq \Gamma(Kos^{\bullet}(z^{-d}\mathcal{S}/\mathcal{J}, (\tilde{E})_{k=0, \dots, d}))$$

\square

Denote by $R_{\mathcal{Y}}$ the ring

$$R_{\mathcal{Y}} = \mathbb{C}[z, \lambda_0, \dots, \lambda_n][z\partial_{\lambda_0}, \dots, z\partial_{\lambda_n}],$$

let $J_{\tilde{A}}^{\lambda} \subset R_{\mathcal{Y}}$ be the left ideal generated by

$$\square_{\underline{l}}^{\lambda} = \prod_{l_i > 0} (z\partial_{\lambda_i})^{l_i} - \prod_{l_i < 0} (z\partial_{\lambda_i})^{-l_i} \quad \text{for } \underline{l} \in \mathbb{L}_{\tilde{A}}$$

and let $I_{\tilde{A}}^{\lambda} \subset R_{\mathcal{Y}}$ be the left ideal generated by $J_{\tilde{A}}^{\lambda}$ and the operators

$$\begin{aligned} \tilde{E}_k - \beta_k &:= \sum_{i=1}^n a_{ki} \lambda_i z \partial_{\lambda_i} - \beta_k \quad \text{for } k = 1, \dots, d \\ \tilde{E}_0 - \beta_0 &:= \sum_{i=0}^n \lambda_i z \partial_{\lambda_i} - \beta_0 \end{aligned}$$

Lemma 4.34. *There is the following isomorphism of $R_{\mathcal{Y}}$ -modules*

$$\Gamma \mathcal{H}^0(\pi_{2+}\mathcal{N}) \simeq \mathcal{H}^0(\Gamma\pi_{2+}\mathcal{N}) \simeq z^{-d}R_{\mathcal{Y}}/I_{\tilde{A}}^{\lambda}$$

Proof. The first isomorphism follows from Lemma 4.16. The second isomorphism follows from Proposition 4.33, the isomorphism

$$\Gamma(\mathcal{S}/\mathcal{J}) \simeq R_{\mathcal{Y}}/J_{\tilde{A}}^{\lambda}$$

and the isomorphism

$$z^{-d}R_{\mathcal{Y}}/I_{\tilde{A}}^{\lambda} \simeq H^0\left(Kos^{\bullet}\left(z^{-d}R_{\mathcal{Y}}/J_{\tilde{A}}^{\lambda}, (\tilde{E}_k - \beta_k)_{k=0, \dots, d}\right)\right)$$

\square

We are now able to prove the main theorem of this paper. Let \tilde{A} be the $(d+1) \times (n+1)$ integer matrix

$$\tilde{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{d1} & \dots & a_{dn} \end{pmatrix}$$

given by a matrix $A = (a_{jk})$ such that $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ and such that $\mathbb{N}\tilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\tilde{A}$.

Theorem 4.35. *Let \tilde{A} be an integer matrix as above and $\tilde{\beta} \in \mathfrak{A}_{\tilde{A}}$. The GKZ-system $\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$ carries the structure of a mixed Hodge module whose Hodge-filtration is given by the shifted order filtration, i.e.*

$$(\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet}^H) \simeq (\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet+d}^{ord}).$$

Proof. Recall from Proposition 4.4 that we have the isomorphism

$${}^H\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \simeq \mathcal{H}^{2n+d+1}(p_*(q^*p^*\mathcal{O}_T^{\beta} \otimes F^*j_!^p\mathcal{O}_{\mathbb{C}^*}^{-\beta_0-1})) \in \text{MHM}(V).$$

The underlying \mathcal{D}_V -module of this mixed Hodge module is

$$\mathcal{H}^{2n+d+1}(p_+(q^!\mathcal{O}_T^{\beta} \otimes_{\mathcal{O}} F^!(j_!\mathcal{O}_{\mathbb{C}^*}^{-\beta_0-1}))) \simeq \mathcal{H}^0(\pi_{2+}\mathcal{N}).$$

We have already computed the Hodge filtration of \mathcal{N} . In order to compute the Hodge filtration under the direct image of π_2 , we will use the results obtained above and read off the Hodge filtration from the corresponding \mathcal{R}_V -module $\mathcal{S}(\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}, F^H)$. We have the following isomorphisms

$$\Gamma\mathcal{S}(\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}, F^H) \simeq \Gamma\mathcal{S}(\mathcal{H}^0(\pi_{2+}\mathcal{N}, F^H)) \simeq \Gamma\mathcal{H}^0(\pi_{2+}\mathcal{N}) \simeq z^{-d}\mathbb{R}_{\mathcal{V}}/I_{\tilde{A}}^{\lambda}$$

Using these isomorphisms the claim follows easily. \square

4.10 Duality

For applications like the one presented in the next section, it will be useful to extend the computation of the Hodge filtration on $\mathcal{M}_{\tilde{A}}^0$ to the dual Hodge-module $\mathbb{D}\mathcal{M}_{\tilde{A}}^0$. This is possible under the assumption made in the above main theorem (Theorem 4.35) plus the extra requirement that the semi-group ring $\mathbb{C}[\mathbb{N}\tilde{A}]$ is Gorenstein. More precisely, it follows from [Wal07], that under these assumptions, the \mathcal{D}_V -module $\mathbb{D}\mathcal{M}_{\tilde{A}}^0$ is still a GKZ-system. Hence it is reasonable to expect that its Hodge filtration will also be the order filtration up to a suitable shift.

The Gorenstein condition for normal semi-group rings has a well-known combinatorial expression (see [BH93, Corollary 6.3.8]), namely, $\mathbb{C}[\mathbb{N}\tilde{A}]$ is Gorenstein iff there is a vector \tilde{c} such that the set of interior points $\text{int}(\mathbb{N}\tilde{A})$ (i.e., the intersection of $\text{int}(\mathbb{R}_{\geq 0}\tilde{A}) \cap \mathbb{Z}^{d+1}$) is given by $\tilde{c} + \mathbb{N}\tilde{A}$.

Theorem 4.36. *Suppose that $\tilde{A} \in M(d+1 \times n+1, \mathbb{Z})$ is such that $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$, $\mathbb{N}\tilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\tilde{A}$ and such that $\text{int}(\mathbb{N}\tilde{A}) = \tilde{c} + \mathbb{N}\tilde{A}$ for some $\tilde{c} = (c_0, c) \in \mathbb{Z}^{d+1}$ and $\tilde{\beta} \in \mathfrak{A}_{\tilde{A}}$. Then we have*

$$\mathbb{D}\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \simeq \mathcal{M}_{\tilde{A}}^{-\tilde{\beta}-\tilde{c}},$$

and the Hodge filtration on $\mathbb{D}\mathcal{M}_{\tilde{A}}^{\tilde{\beta}}$ is the order filtration, shifted by $n + c_0$, i.e., we have

$$F_p^H \mathbb{D}\mathcal{M}_{\tilde{A}}^{\tilde{\beta}} \simeq F_{p-n-c_0}^{ord} \mathcal{M}_{\tilde{A}}^{-\tilde{\beta}-\tilde{c}}.$$

Proof. The proof is very much parallel to [RS15, Proposition 2.19] resp. [RS17, Theorem 5.4], we will give the main ideas here once again for the convenience of the reader. We work again with the modules of global sections, and write $D_V := \mathbb{C}[\lambda_0, \dots, \lambda_n] \langle \partial_{\lambda_0}, \dots, \partial_{\lambda_n} \rangle$ and $S_{\tilde{A}}$ for the commutative ring $\mathbb{C}[\partial_{\lambda_0}, \dots, \partial_{\lambda_n}] / (\square_l)_{l \in \mathbb{L}_{\tilde{A}}}$. These rings are \mathbb{Z}^{d+1} -graded by $\deg(\lambda_i) = -\tilde{a}_i$, $\deg(\partial_{\lambda_i}) = \tilde{a}_i$.

In order to calculate $\mathbb{D}M_{\tilde{A}}^{\tilde{\beta}}$ together with its Hodge filtration, we need to find a strictly filtered free resolution $(L_{\bullet}, F_{\bullet}) \xrightarrow{\sim} (M_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet}^H) = (M_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet+d}^{ord})$. We have already used in the previous sections of this paper resolutions of ‘‘Koszul’’-type for various (filtered) \mathcal{D} -modules. Here we consider the Euler-Koszul complex

$$K^{\bullet} := \text{Kos}(D_V \otimes_{\mathbb{C}[\partial_{\lambda}]} S_{\tilde{A}}, (E_k - \beta_k)_{k=0, \dots, d}),$$

as defined in section 2.1 and a generalization to \mathbb{Z}^{d+1} -graded $\mathbb{C}[\partial_{\lambda}]$ -modules (for details see [MMW05]).

A free resolution of $M_{\tilde{A}}^{\tilde{\beta}}$ is constructed as follows: Take a $\mathbb{C}[\partial]$ -free graded resolution of $T^{\bullet} \rightarrow S_{\tilde{A}}$, and define L^{\bullet} to be the total complex $\text{Tot}(K^{\bullet}(E - \beta, D_V \otimes_{\mathbb{C}[\partial_{\lambda}]} T^{\bullet}))$. Notice that the double complex $K^{\bullet}(E - \beta, D_V \otimes_{\mathbb{C}[\partial_{\lambda}]} T^{\bullet})$ exists since $K^{\bullet}(E - \beta, D_V \otimes_{\mathbb{C}[\partial]} -)$ is a functor from the category of \mathbb{Z}^{d+1} -graded $\mathbb{C}[\partial]$ -modules to the category of (bounded complexes of) \mathbb{Z}^{d+1} -graded D_V -modules. Then we have $L^{-k} = 0$ for all $k > n + 1$ (notice that the length of the Euler-Koszul complexes is $d + 1$, and the length of the resolution $T_{\bullet} \rightarrow P$ is $n - d + 1$, hence the total complex has length $(d + 1) + (n - d + 1) - 1 = n + 1$). Moreover, the last term L^{-n-1} of this complex is simply equal to D_V (and so is the first one L^0).

As we have $\text{int}(\mathbb{N}\tilde{A}) = \tilde{c} + \mathbb{N}\tilde{A}$, the ring $\mathbb{C}[\mathbb{N}\tilde{A}] \simeq S_{\tilde{A}}$ is Gorenstein, more precisely, we have $\omega_{S_{\tilde{A}}} \simeq S_{\tilde{A}}(\tilde{c})$, where $\omega_{S_{\tilde{A}}}$ is the canonical module of $S_{\tilde{A}}$. Then a spectral sequence argument (see also [Wal07, Proposition 4.1]), using

$$\text{Ext}_{\mathbb{C}[\partial]}^i(S_{\tilde{A}}, \omega_{\mathbb{C}[\partial]}) \simeq \begin{cases} 0 & \text{if } i < n - d \\ S_{\tilde{A}}(\tilde{c}) & \text{if } i = n - d \end{cases}$$

shows that

$$\mathbb{D}M_{\tilde{A}}^{\tilde{\beta}} \cong M_{\tilde{A}}^{-\tilde{\beta} - \tilde{c}}.$$

In order to calculate the Hodge filtration on $M_{\tilde{A}}^{-\tilde{c}}$, we remark that the Euler-Koszul complex is naturally filtered by putting

$$F_p K^{-l} := \bigoplus_{0 \leq i_1 < \dots < i_l \leq l} F_{p+d-l}^{ord}(D_V \otimes_{\mathbb{C}[\partial]} S_{\tilde{A}}) e_{i_1 \dots i_l}.$$

Notice that $D_V \otimes_{\mathbb{C}[\partial]} S_{\tilde{A}} \simeq D_V / (\square_l)_{l \in \mathbb{L}_{\tilde{A}}}$, so that this D_V -module has an order filtration induced from $F_{\bullet}^{ord} D_V$. In order to show that $(K^{\bullet}, F_{\bullet}) \rightarrow (M_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet}^H)$ is a filtered quasi-isomorphism, it suffices (by Lemma 2.2) to show that $\text{Gr}_{\bullet}^F K^{\bullet} \rightarrow \text{Gr}_{\bullet}^{F^H} M_{\tilde{A}}^{\tilde{\beta}}$ is a quasi-isomorphism. This follows from [SST00, Formula 4.32, Lemma 4.3.7], as $\mathbb{C}[\mathbb{N}\tilde{A}]$ is Cohen-Macaulay due to the normality assumption on \tilde{A} . The final step is to endow the free resolution $L^{\bullet} = \text{Tot}(K^{\bullet}(E - \beta, D_V \otimes_{\mathbb{C}[\partial_{\lambda}]} T^{\bullet}))$ with a strict filtration F_{\bullet} and to show that $(L_{\bullet}, F_{\bullet}) \xrightarrow{\sim} (M_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet}^H)$. As the resolution $T_{\bullet} \rightarrow S_{\tilde{A}}$ is taken in the category of \mathbb{Z}^{d+1} -graded $\mathbb{C}[\partial]$ -modules, the morphisms of this resolution are homogenous for the (\mathbb{Z}) -grading $\deg(\lambda_i) = -1$ and $\deg(\partial_{\lambda_i}) = 1$ (notice that this is the grading given by the first component of the \mathbb{Z}^{d+1} -grading of the ring $D_V \otimes_{\mathbb{C}[\partial]} S_{\tilde{A}}$). Hence these morphisms are naturally filtered for the order filtration $F_{\bullet}^{ord}(D_V \otimes_{\mathbb{C}[\partial]} S_{\tilde{A}})$ and they are even strict: for a map given by homogenous operators from $\mathbb{C}[\partial]$ taking the symbols has simply no effect, so that $\text{Gr}_{\bullet}^F(D_V \otimes_{\mathbb{C}[\partial]} T_{\bullet}) \rightarrow \text{Gr}_{\bullet}^{F^{ord}}(D_V \otimes_{\mathbb{C}[\partial]} S_{\tilde{A}})$ is a filtered quasi-isomorphism (and similarly for the sums occurring in the terms K^{-l}). However, we have to determine the \mathbb{Z} -degree (for the grading $\deg(\partial_{\lambda_i}) = 1$) of the highest (actually, the only nonzero) cohomology module $\text{Ext}_{\mathbb{C}[\partial]}^{n-d}(S_{\tilde{A}}, \omega_{\mathbb{C}[\partial]})$: it is the first component of the difference of the degree of $\omega_{\mathbb{C}[\partial]}$ (i.e., the first component of the sum of the columns of \tilde{A}), which is $n + 1$, and the first component of the degree of $\omega_{S_{\tilde{A}}}$, which is c_0 . Now the shift of the filtration between $M_{\tilde{A}}^{\tilde{\beta}}$ and the dual module $M_{\tilde{A}}^{-\tilde{\beta} - (c_0, c)}$ is the sum of the length of the complex $K^{\bullet}(E - \beta, D_V \otimes_{\mathbb{C}[\partial]} S_{\tilde{A}})$, i.e., $d + 1$, and the above \mathbb{Z} -degree of $\text{Ext}_{\mathbb{C}[\partial]}^{n-d}(S_{\tilde{A}}, \omega_{\mathbb{C}[\partial]})$, i.e. $n + 1 - c_0$. Hence the filtration $F_{\bullet} L^{-n-1}$ is again the shifted order filtration, more precisely, we have

$$F_p L^{-n-1} = F_{p+d-(d+1)-(n+1-c_0)}^{ord} D_V = F_{p-n-2+c_0}^{ord} D_V.$$

Now it follows from [Sai94, page 55] that

$$\mathbb{D}(M_{\tilde{A}}^{\tilde{\beta}}, F_{\bullet}^H) \simeq \text{Hom}_{D_V}((L^{\bullet}, F_{\bullet}), ((D_V \otimes \Omega_V^{n+1})^{\vee}, F_{\bullet-2(n+1)} D_V \otimes (\Omega_V^{n+1})^{\vee}))$$

so that finally we obtain

$$F_p^H \mathbb{D} M_{\tilde{A}}^{\tilde{\beta}} = F_{p-n-c_0}^{ord} M_{\tilde{A}}^{-\tilde{\beta}-(c_0, c)}.$$

□

From Proposition 4.2, we know that up to multiplication with a non-zero constant, we have the morphism

$$\begin{aligned} \phi : F_{p+d-c_0}^{ord} M_{\tilde{A}}^{-(c_0, c)} &= F_{p+n+d}^H \mathbb{D}(M_{\tilde{A}}^0) = F_p^H \mathbb{D}(M_{\tilde{A}}^0)(-n-d) \longrightarrow F_p^H M_{\tilde{A}}^0 = F_{p+d}^{ord} M_{\tilde{A}}^0 \\ P &\mapsto P \cdot \partial^{(c_0, c)} \end{aligned}$$

where $\partial^{(c_0, c)} := \prod_{i=0}^n \partial_{\lambda_i}^{k_i}$ for any $\underline{k} = (k_0, \dots, k_n)$ with $\tilde{A} \cdot \underline{k} = (c_0, c)$. Since \tilde{A} is homogeneous we have $\sum k_i = c_0$. As a consequence, we obtain the following result.

Corollary 4.37. *Under the above assumptions on \tilde{A} , the morphism*

$$\begin{aligned} \phi : (M_{\tilde{A}}^{-(c_0, c)}, F_{\bullet, -c_0}^{ord}) &\longrightarrow (M_{\tilde{A}}^0, F_{\bullet}^{ord}) \\ P &\longmapsto P \cdot \partial^{(c_0, c)} \end{aligned}$$

(where $\partial^{(c_0, c)}$ is as above) is strictly filtered.

Remark 4.38. *If $\mathbb{C}[\mathbb{N}\tilde{A}]$ is not Gorenstein but normal (and therefore Cohen-Macaulay) then the proof of Theorem 4.36 shows that*

$$\mathbb{D} M_{\tilde{A}}^{\tilde{\beta}} \simeq \mathcal{H}^0(E + \beta, D_V \otimes \text{Ext}^{n-d}(S_{\tilde{A}}, \omega_{\mathbb{C}[\partial_{\lambda}]})) \simeq \mathcal{H}^0(E + \beta, D_V \otimes \omega_{S_{\tilde{A}}})$$

Recall that the canonical module $\omega_{S_{\tilde{A}}}$ of $S_{\tilde{A}}$ is isomorphic to $\mathbb{C}[\text{int}(\mathbb{N}\tilde{A})]$ in the category of \mathbb{Z}^{d+1} -graded $\mathbb{C}[\partial_{\lambda}]$ -modules. The module $\omega_{S_{\tilde{A}}}$ carries a \mathbb{Z} -grading given by the first component of the \mathbb{Z}^{d+1} -grading. Hence $D_V \otimes_{\mathbb{C}[\partial_{\lambda}]} \omega_{S_{\tilde{A}}}$ carries an order filtration which induces a filtration F_{\bullet}^{ord} on $\mathcal{H}^0(E + \beta, D_V \otimes \omega_{S_{\tilde{A}}})$. We therefore get

$$F_p^H \mathbb{D} M_{\tilde{A}}^{\tilde{\beta}} \simeq F_{p-n-c_0}^{ord} \mathcal{H}^0(E + \beta, D_V \otimes \omega_{S_{\tilde{A}}}).$$

where $c_0 := \min\{\deg_{\mathbb{Z}}(P) \mid P \in \mathbb{C}[\text{int}(\mathbb{N}\tilde{A})]\}$. Let $\tilde{c} \in \text{deg}(\mathbb{C}[\text{int}(\mathbb{N}\tilde{A})])$ with $\tilde{c} = (c_0, c)$. Similar to [Wal07, Proposition 4.4] it can be shown that the inclusion $S_{\tilde{A}}[-\tilde{c}] \hookrightarrow \mathbb{C}[\text{int}(\mathbb{N}\tilde{A})]$ induces an isomorphism $M_{\tilde{A}}^{-\tilde{\beta}-\tilde{c}} \xrightarrow{\simeq} \mathcal{H}^0(E + \beta, D_V \otimes \omega_{S_{\tilde{A}}})$ however we do not expect $(M_{\tilde{A}}^{-\tilde{\beta}-\tilde{c}}, F_{\bullet}^{ord}) \longrightarrow (\mathcal{H}^0(E + \beta, D_V \otimes \omega_{S_{\tilde{A}}}), F_{\bullet}^{ord})$ to be a filtered isomorphism.

4.11 Hodge structures on affine hypersurfaces of tori

In this subsection we explain how our main result implies in a rather direct way a classical Theorem of Batyrev concerning the description of the Hodge filtration of the relative cohomology of smooth affine hypersurfaces in algebraic tori.

We first want to recap the sheaf theoretic definition of relative cohomology. Let X be a topological space and K be a closed subset. Denote by $j : X \setminus K \rightarrow X$ the open embedding of the complement. The relative cohomology of the pair (X, K) is defined as the following hypercohomology:

$$H^i(X, K; \mathbb{C}) := \mathbb{H}^i(X, j_! j^{-1} \mathbb{Q}_X)$$

If X and K are quasi-projective varieties the relative cohomology of the pair (X, K) carries a mixed Hodge structure, which is given by $\mathbb{H}^i(X, j_! j^{-1} \mathbb{Q}_X^H)$.

We want to compute this in the following situation: Consider as in section 4 the family of Laurent polynomials $\varphi_A : T \times \Lambda \rightarrow V = \mathbb{C}_{\lambda_0} \times \Lambda$, where $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ and $\mathbb{N}\tilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\tilde{A}$. Let $\Delta := \text{Conv}(\underline{a}_0, \underline{a}_1, \dots, \underline{a}_n)$ be the convex hull of the exponents of φ_A , where $\underline{a}_0 := 0$. Let $\tau \subset \Delta$ be a face of Δ , $x \in V$ and

$$F_{A, x}^{\tau} = \sum_{i: \underline{a}_i \in \tau} x_i t^{\underline{a}_i}$$

Definition 4.39. The fiber $\varphi_A^{-1}(x)$ is non-degenerate if for every face τ of Δ the equations

$$F_{A,x}^\tau = t_1 \frac{\partial F_{A,x}^\tau}{\partial t_1} = \dots = t_d \frac{\partial F_{A,x}^\tau}{\partial t_d} = 0$$

have no common solution in T .

Let $x \in V$ such that the fiber $\varphi_A^{-1}(x)$ is non-degenerate. We give a model of $H^i(T, \varphi_A^{-1}(x); \mathbb{C})$ as the quotient of a graded semi-group ring and compute explicitly its Hodge filtration. This recovers a result of Stienstra [Sti98, Theorem 7] using results of Batyrev [Bat93].

Lemma 4.40. Let $x \in V$ and $i_x : \{x\} \rightarrow V$ be the inclusion. Suppose $\varphi_A^{-1}(x)$ is non-degenerate. Then

1. the fiber $\varphi_A^{-1}(x)$ is smooth.
2. the map i_x is noncharacteristic with respect to $\mathcal{M}_{\tilde{A}}^0$.

Proof. The first statement follows directly from the definition for $\tau = \Delta$. The second statement follows from [Ado94, Lemma 3.3]. \square

Consider the following diagram (cf. diagram (27))

$$\begin{array}{ccccc}
Y & \longrightarrow & U & & \\
\downarrow & & \downarrow j_U & \searrow \pi_2^U & \\
T \times V & \xrightarrow{g \times id} & \mathbb{P}(W) \times V & \xrightarrow{\pi_2} & V \\
\uparrow & & \uparrow i_Z & \nearrow \pi_2^Z & \\
\Gamma & \longrightarrow & Z & &
\end{array}$$

where Y resp. Γ are the pull-backs such that both squares to the left are cartesian. Notice that Γ is given inside $T \times V$ by the equation $\lambda_0 + \sum_{i=1}^n \lambda_i t^{a_i} = 0$, hence Γ is the graph of φ_A and Y is its complement in $T \times V$. Restricting this diagram to some $x \in V$ we therefore get

$$\begin{array}{ccccc}
T \setminus \varphi^{-1}(x) & \xrightarrow{\bar{g}} & U_x & & \\
\downarrow \bar{j} & & \downarrow j & \searrow \pi_x^U & \\
T & \xrightarrow{g} & \mathbb{P}(W) & \xrightarrow{\pi_x} & \{x\} \\
\uparrow \tilde{i} & & \uparrow i & \nearrow \pi_x^Z & \\
\varphi^{-1}(x) & \xrightarrow{\tilde{g}} & Z_x & &
\end{array} \tag{45}$$

We will need the following statement

Lemma 4.41. Let $x \in V$ such that $\varphi_A^{-1}(x)$ is smooth then we have an isomorphism in $D^b(\text{MHM}(\mathbb{P}(W)))$:

$$g_* \bar{j}_! \bar{j}^{-1} \mathbb{Q}_T^H \simeq j_* \bar{g}_* \mathbb{Q}_{T \setminus \varphi^{-1}(x)}^H$$

Proof. The statement follows using the following chain of isomorphisms

$$g_* \bar{j}_! \bar{j}^{-1} \mathbb{Q}_T^H \stackrel{!}{\simeq} j_* j^{-1} g_* \mathbb{Q}_T^H \simeq j_* \bar{g}_* \bar{j}^{-1} \mathbb{Q}_T^H \simeq j_* \bar{g}_* \mathbb{Q}_{T \setminus \varphi^{-1}(x)}^H$$

where the second isomorphism follows from base change. It remains to show the first isomorphism. Notice that we have the following triangles

$$\begin{array}{ccccc}
j_! j^{-1} g_* & \longrightarrow & g_* & \longrightarrow & i_! i^* g_* \xrightarrow{+1} \\
g_* \bar{j}_! \bar{j}^{-1} & \longrightarrow & g_* & \longrightarrow & g_* \tilde{i}_! \tilde{i}^* \xrightarrow{+1}
\end{array}$$

So it is enough to show $i_! i^* g_* \mathbb{Q}_T^H \simeq g_* \tilde{i}_! \tilde{i}^* \mathbb{Q}_T^H$. But this can be seen as follows:

$$i_! i^* g_* \mathbb{Q}_T^H \simeq i_! i^! g_* \mathbb{Q}_T^H[2] \simeq i_! \tilde{g}_* \tilde{i}^! \mathbb{Q}_T^H[2] \simeq i_! \tilde{g}_* \tilde{i}^* \mathbb{Q}_T^H \simeq g_* \tilde{i}_! \tilde{i}^* \mathbb{Q}_T^H$$

where we used the smoothness of $\varphi^{-1}(x)$ in the first and third isomorphism. \square

In order to prove the statement that the restriction of the GKZ-system is isomorphic to a relative cohomology group, we have to rewrite the GKZ-system as a Radon transform. For this consider the following diagram

$$\begin{array}{ccccc}
T \setminus \varphi^{-1}(x) & \xrightarrow{\bar{g}} & U_x & \xrightarrow{\pi_x^U} & \{x\} \\
\bar{i}_x \downarrow & & \downarrow i_{U_x} & & \downarrow i_x \\
Y & \longrightarrow & U & \xrightarrow{\pi_2^U} & V \\
\pi_1^Y \downarrow & & \downarrow \pi_1^U & & \\
T & \xrightarrow{g} & \mathbb{P}(W) & &
\end{array}$$

where all squares are cartesian.

Proposition 4.42. *Let $x \in V$ such that $\varphi_A^{-1}(x)$ is non-degenerate, then there is an isomorphism of mixed Hodge structures:*

$$i_x^* \mathcal{M}_A^0 \simeq H^d(T, \varphi_A^{-1}(x); \mathbb{C})$$

and $H^i(T, \varphi_A^{-1}(x); \mathbb{C}) = 0$ for $i \neq d$.

Proof. Consider the following isomorphisms

$$\begin{aligned}
i_x^* \pi_{2!}^U (\pi_1^U)^* g_*^p \mathbb{Q}_T^H &\simeq \pi_{x!}^U i_{U_x}^* (\pi_1^U)^* g_*^p \mathbb{Q}_T^H && \text{base change} \\
&\simeq \pi_{x!}^U (\pi_1^U \circ i_{U_x})^* g_*^p \mathbb{Q}_T^H \\
&\simeq \pi_{x!}^U (\pi_1^U \circ i_{U_x})^! g_*^p \mathbb{Q}_T^H && (\pi_1^U \circ i_{U_x}) \text{ open} \\
&\simeq \pi_{x!}^U \bar{g}_* (\pi_1^Y \circ \bar{i}_x)^! g_*^p \mathbb{Q}_T^H && \text{base change} \\
&\simeq \pi_{x!}^U \bar{g}_* (\pi_1^Y \circ \bar{i}_x)^* g_*^p \mathbb{Q}_T^H && (\pi_1^Y \circ \bar{i}_x) \text{ open} \\
&\simeq \pi_{x!}^U \bar{g}_*^p \mathbb{Q}_{T \setminus \varphi^{-1}(x)}^H
\end{aligned}$$

We can rewrite this further by looking at a part of diagram (45):

$$\begin{array}{ccccc}
T \setminus \varphi^{-1}(x) & \xrightarrow{\bar{g}} & U_x & & \\
\bar{j} \downarrow & & \downarrow j & \searrow \pi_x^U & \\
T & \xrightarrow{g} & \mathbb{P}(W) & \xrightarrow{\pi_x} & \{x\}
\end{array}$$

We have

$$\pi_{x*} g_* \bar{j}_! \bar{j}^{-1} g_*^p \mathbb{Q}_T^H \simeq \pi_{x*} j_! \bar{g}_*^p \mathbb{Q}_{T \setminus \varphi^{-1}(x)}^H \simeq \pi_{x!}^U \bar{g}_*^p \mathbb{Q}_{T \setminus \varphi^{-1}(x)}^H \simeq i_x^* \pi_{2!}^U (\pi_1^U)^* g_*^p \mathbb{Q}_T^H$$

where the last isomorphism follows from the calculation above. If we take cohomology and keep in mind that i_x is non-characteristic we get

$$\begin{aligned}
H^i(T, \varphi_A^{-1}(x); \mathbb{C}) &\simeq H^{i-d}(\pi_{x*} g_* \bar{j}_! \bar{j}^{-1} g_*^p \mathbb{Q}_T^H) \\
&\simeq \mathcal{H}^{i-d} i_x^* (\pi_{2!}^U (\pi_1^U)^* g_*^p \mathbb{Q}_T^H) \\
&\simeq i_x^* \mathcal{H}^{i-d+n+1} (\pi_{2!}^U (\pi_1^U)^* g_*^p \mathbb{Q}_T^H) \\
&\simeq i_x^* \mathcal{H}^{i-d+n+1} (*\mathcal{R}_c^{\circ}(g_*^p \mathbb{Q}_T^H))
\end{aligned}$$

Since $\mathcal{H}^k(*\mathcal{R}_c^{\circ}(g_*^p \mathbb{Q}_T^H)) = 0$ for $k \neq n+1$ and $\mathcal{H}^k(*\mathcal{R}_c^{\circ}(g_*^p \mathbb{Q}_T^H)) = \mathcal{M}_A^0$ for $k = n+1$ the claim follows. \square

Denote by $S_{\tilde{A}} := \mathbb{C}[\mathbb{N}\tilde{A}] \subset \mathbb{C}[u_0^{\pm}, \dots, u_d^{\pm}]$ the semigroup ring generated by $\underline{u}^{\tilde{a}_0}, \dots, \underline{u}^{\tilde{a}_n}$ where $\tilde{A} = (\tilde{a}_0, \dots, \tilde{a}_n)$ is the matrix from 26.

Define the following differential operators

$$D_k := \sum_{i=0}^n \left(\tilde{a}_{ki} x_i \underline{u}^{\tilde{a}_i} + u_i \partial_{u_i} \right) \quad \text{for } k = 0, \dots, d \text{ and fixed } x = (x_0, \dots, x_n) \in V$$

which act on $\mathbb{C}[u_0^\pm, \dots, u_d^\pm]$ and which preserve $S_{\tilde{A}}$. For $\underline{u}^l = \underline{u}^{l_0 \cdot \tilde{a}_0} \cdot \dots \cdot \underline{u}^{l_n \cdot \tilde{a}_n} \in S_{\tilde{A}}$ we define the degree $\deg(\underline{u}^l) = \sum_{i=0}^n l_i$. Define a descending filtration F^\bullet of \mathbb{C} -vector spaces on $S_{\tilde{A}}$ where $F^{d+1}S_{\tilde{A}} = 0$ and the filtration step $F^{d-k}S_{\tilde{A}}$ is spanned by monomials \underline{u}^l with $\deg(\underline{u}^l) \leq k$.

Theorem 4.43. *Let $x \in V$ such that $\varphi_{\tilde{A}}^{-1}(x)$ is non-degenerate, then the following isomorphism of filtered vector spaces holds*

$$(i_x^* \mathcal{M}_{\tilde{A}}^0, F^\bullet) \simeq (S_{\tilde{A}} / (D_k S_{\tilde{A}})_{k=0, \dots, d}, F^{-\bullet})$$

Proof. Since we assumed that i_x is non-characteristic with respect to $\mathcal{M}_{\tilde{A}}^0$ the only non-zero cohomology group of $i_x^* \mathcal{M}_{\tilde{A}}^0$ is

$$\mathcal{H}^0 i_x^* \mathcal{M}_{\tilde{A}}^0 \simeq (\lambda_i - x_i)_{i=0, \dots, n} \setminus \mathcal{M}_{\tilde{A}}^0$$

We define a \mathbb{C} -linear map

$$\begin{aligned} \Psi' : S_{\tilde{A}} &\longrightarrow (\lambda_i - x_i)_{i=0, \dots, n} \setminus \mathcal{M}_{\tilde{A}}^0 \\ \underline{u}^{l_0 \cdot \tilde{a}_0} \cdot \dots \cdot \underline{u}^{l_n \cdot \tilde{a}_n} &\mapsto \partial_{\lambda_0}^{l_0} \cdot \dots \cdot \partial_{\lambda_n}^{l_n} \end{aligned}$$

We want to show that this map factors over $S_{\tilde{A}} / (D_k S_{\tilde{A}})_{k=0, \dots, d}$ so that Ψ' descends to a map

$$\Psi : S_{\tilde{A}} / (D_k S_{\tilde{A}})_{k=0, \dots, d} \longrightarrow (\lambda_i - x_i)_{i=0, \dots, n} \setminus \mathcal{M}_{\tilde{A}}^0$$

Let $P = \underline{u}^{l_0 \cdot \tilde{a}_0} \cdot \dots \cdot \underline{u}^{l_n \cdot \tilde{a}_n}$, then $\Psi'(P) = \partial_{\lambda_0}^{l_0} \cdot \dots \cdot \partial_{\lambda_n}^{l_n}$ and

$$\begin{aligned} \Psi'(D_k P) &= \Psi \left(\sum_{i=0}^n \tilde{a}_{ki} x_i \underline{u}^{a_i} P + \sum_{i=0}^n \tilde{a}_{ki} l_i P \right) \\ &= \left(\sum_{i=0}^n \tilde{a}_{ki} x_i \partial_{\lambda_i} + \sum_{i=0}^n \tilde{a}_{ki} l_i \right) \Psi(P) \\ &= \left(\sum_{i=0}^n \tilde{a}_{ki} \lambda_i \partial_{\lambda_i} + \sum_{i=0}^n \tilde{a}_{ki} l_i \right) \Psi(P) \\ &= \Psi(P) \cdot \left(\sum_{i=0}^n \tilde{a}_{ki} \lambda_i \partial_{\lambda_i} \right) \\ &= 0 \end{aligned}$$

We will now construct an inverse Θ to Ψ . If $P \in D_V$ is a normally ordered element, we denote by $\bar{P} \in S_{\tilde{A}}$ the element which is obtained from P by replacing λ_i with x_i and ∂_{λ_i} with $\underline{u}^{l_i \cdot \tilde{a}_i}$, i.e. if $P = \lambda_0^{k_0} \dots \lambda_n^{k_n} \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n}$ the element \bar{P} is given by $x_0^{k_0} \dots x_n^{k_n} \underline{u}^{l_0 \cdot \tilde{a}_0} \dots \underline{u}^{l_n \cdot \tilde{a}_n}$. This gives the map

$$\begin{aligned} \Theta' : (\lambda_i - x_i)_{i=0, \dots, n} \setminus D_V &\longrightarrow S_{\tilde{A}} / (D_k S_{\tilde{A}})_{k=0, \dots, d} \\ P &\mapsto \bar{P} \end{aligned}$$

We want to show that Θ' factors over $(\lambda_i - x_i)_{i=0, \dots, n} \setminus \mathcal{M}_{\tilde{A}}^0$ so that Θ' descends to a map

$$\begin{aligned} \Theta : (\lambda_i - x_i)_{i=0, \dots, n} \setminus \mathcal{M}_{\tilde{A}}^0 &\longrightarrow S_{\tilde{A}} / (D_k S_{\tilde{A}})_{k=0, \dots, d} \\ P &\mapsto \bar{P} \end{aligned}$$

We have to show that $\Theta'(P \cdot E_k) = 0$ and $\Theta'(P \cdot \square_l) = 0$ for $k = 0, \dots, d$ and $l \in \mathbb{L}$. We can assume that

P is a monomial $\lambda_0^{j_0} \dots \lambda_n^{j_n} \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n}$:

$$\begin{aligned}
\Theta'(\lambda_0^{j_0} \dots \lambda_n^{j_n} \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n} E_k) &= \Theta'(\lambda_0^{j_0} \dots \lambda_n^{j_n} \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n} \left(\sum_{i=0}^n \tilde{a}_{ki} \lambda_i \partial_{\lambda_i} \right)) \\
&= \Theta'(\lambda_0^{j_0} \dots \lambda_n^{j_n} \left(\sum_{i=0}^n (\tilde{a}_{ki} \lambda_i \partial_{\lambda_i} + \tilde{a}_{ki} l_i) \right) \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n}) \\
&= x_0^{j_0} \dots x_n^{j_n} \left(\sum_{i=0}^n (\tilde{a}_{ki} x_i \underline{u}^{a_i} + \tilde{a}_{ki} l_i) \right) \underline{u}^{l_0 \cdot a_0} \dots \underline{u}^{l_n \cdot a_n} \\
&= \left(\sum_{i=0}^n (\tilde{a}_{ki} x_i \underline{u}^{a_i} + \tilde{a}_{ki} l_i) \right) \cdot x_0^{j_0} \dots x_n^{j_n} \cdot \underline{u}^{l_0 \cdot a_0} \dots \underline{u}^{l_n \cdot a_n} \\
&= \left(\sum_{i=0}^n (\tilde{a}_{ki} x_i \underline{u}^{a_i} + u_i \partial_{u_i}) \right) \cdot x_0^{j_0} \dots x_n^{j_n} \cdot \underline{u}^{l_0 \cdot a_0} \dots \underline{u}^{l_n \cdot a_n} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\Theta'(\lambda_0^{j_0} \dots \lambda_n^{j_n} \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n} \square_{\underline{l}}) &= \Theta'(\lambda_0^{j_0} \dots \lambda_n^{j_n} \partial_{\lambda_0}^{l_0} \dots \partial_{\lambda_n}^{l_n} \left(\prod_{l_i > 0} \partial_{\lambda_i}^{l_i} - \prod_{l_i < 0} \partial_{\lambda_i}^{-l_i} \right)) \\
&= x_0^{j_0} \dots x_n^{j_n} \underline{u}^{l_0 \cdot a_0} \dots \underline{u}^{l_n \cdot a_n} \left(\prod_{l_i > 0} \underline{u}^{l_i \cdot a_i} - \prod_{l_i < 0} \underline{u}^{-l_i \cdot a_i} \right) \\
&= x_0^{j_0} \dots x_n^{j_n} \underline{u}^{l_0 \cdot a_0} \dots \underline{u}^{l_n \cdot a_n} \left(\underline{u}^{\sum_{l_i > 0} l_i \cdot a_i} - \underline{u}^{-\sum_{l_i < 0} l_i \cdot a_i} \right) \\
&= 0
\end{aligned}$$

where the last equality follows from $\sum_{i=0}^n l_i a_i = 0$. This shows that Ψ is an isomorphism. The statement about the Hodge-filtration follows from Theorem 4.35, the fact that x is a smooth point of \mathcal{M}_A^0 and the definition of the inverse image of filtered \mathcal{D} -modules (cf. [Sai88] chapter 3.5, notice that no shift in the Hodge filtration occurs since we are dealing with left \mathcal{D} -modules instead of right \mathcal{D} -modules as in loc.cit.). \square

5 Landau-Ginzburg models and non-commutative Hodge structures

In this final section we will give a first application of our main result. It is concerned with Hodge theoretic properties of differential systems occurring in toric mirror symmetry. More precisely, we will prove [RS17, Conjecture 6.13] showing that the so-called *reduced quantum \mathcal{D} -module* of a nef complete intersection inside a smooth projective toric variety underlies a (variation of) non-commutative Hodge structure(s). We will recall as briefly as possible the necessary notations and results of loc.cit. and then deduce this conjecture from our main Theorem 4.35.

Let X_{Σ} be smooth, projective and toric with $\dim_{\mathbb{C}}(X_{\Sigma}) = k$. Put $m := k + b_2(X_{\Sigma})$. Let $\mathcal{L}_1, \dots, \mathcal{L}_l$ be globally generated line bundles on X_{Σ} (in particular, they are nef according to [Ful93, Section 3.4]) and assume that $-K_{X_{\Sigma}} - \sum_{i=1}^l c_1(\mathcal{L}_i)$ is nef. Put $\mathcal{E} := \oplus_{i=1}^l \mathcal{L}_i$, and let \mathcal{E}^{\vee} the dual vector bundle. Its total space $\mathbb{V}(\mathcal{E}^{\vee}) := \mathbf{Spec}_{\mathcal{O}_{X_{\Sigma}}}(\mathrm{Sym}_{\mathcal{O}_{X_{\Sigma}}}(\mathcal{E}))$ is a quasi-projective toric variety with defining fan Σ' . The matrix $A \in M((k+l) \times (m+l), \mathbb{Z})$ whose columns are the primitive integral generators of the rays of Σ' then satisfies the conditions in Theorem 4.36. More precisely, we have $\mathbb{Z}\tilde{A} = \mathbb{Z}^{d+1}$ and it follows from [RS17, Proposition 5.1] that the semi-group $\mathbb{N}\tilde{A}$ is normal and that we have $\mathrm{int}(\mathbb{N}\tilde{A}) = \tilde{c} + \mathbb{N}\tilde{A}$, where $\tilde{c} = \sum_{i=m+1}^{m+l} e_i = (l+1, \underline{0}, \underline{1})$, e_i being the i 'th standard vector in \mathbb{Z}^{1+m+l} .

The strictly filtered duality morphism ϕ from Corollary 4.37 is more concretely given as

$$\begin{aligned} \phi : (\mathcal{M}_{\tilde{A}}^{-(l+1,0,1)}, F_{\bullet-l-1}^{ord}) &\longrightarrow (\mathcal{M}_{\tilde{A}}^0, F_{\bullet}^{ord}) \\ P &\longmapsto P \cdot \partial_{\lambda_0} \cdot \partial_{\lambda_{m+1}} \cdot \dots \cdot \partial_{\lambda_{m+l}}. \end{aligned}$$

Proposition 5.1. *The image of ϕ underlies a pure Hodge module of weight $m+k+2l$, where the Hodge filtration is given by*

$$F_{\bullet}^H \text{im}(\phi) = \text{im}(\phi) \cap F_{\bullet+k+l}^{ord} \mathcal{M}_{\tilde{A}}^0.$$

Proof. This is a consequence of [RS17, Theorem 2.16] and of Proposition 4.2. \square

A main point in the paper [RS17] is to consider the partial localized Fourier transformations of the GKZ-systems $\mathcal{M}_{\tilde{A}}^{\beta}$. We recall the main construction and refer to [RS17, Section 3.1] for details (in particular concerning the definition and properties of the Fourier-Laplace functor

FL and its “localized” version

FL^{loc}). Let (as done already in section in 4.1) Λ be the affine space \mathbb{C}^{m+l} with coordinates $\lambda_1, \dots, \lambda_{m+l}$ (so that $V = \mathbb{C}_{\lambda_0} \times \Lambda$) and put $\widehat{V} := \mathbb{C}_z \times \Lambda$. Let $\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$ be the $\mathcal{D}_{\widehat{V}}$ -module $\mathcal{D}_{\widehat{V}}[z^{-1}]/\mathcal{I}$, where \mathcal{I} is the left ideal generated by the operators $\widehat{\square}_l$ (for all $l \in \mathbb{L}_A$), $\widehat{E}_j - \beta_j z$ (for $j = 1, \dots, k+l$) and $\widehat{E} - \beta_0 z$, which are defined by

$$\begin{aligned} \widehat{\square}_l &:= \prod_{i: l_i < 0} (z \cdot \partial_{\lambda_i})^{-l_i} - \prod_{i: l_i > 0} (z \cdot \partial_{\lambda_i})^{l_i}, \\ \widehat{E} &:= z^2 \partial_z + \sum_{i=1}^{m+l} z \lambda_i \partial_{\lambda_i}, \\ \widehat{E}_j &:= \sum_{i=1}^{m+l} a_{ji} z \lambda_i \partial_{\lambda_i}. \end{aligned}$$

We denote the corresponding $\mathcal{D}_{\widehat{V}}$ -module by $\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$. Then we have (see [RS17, Lemma 3.2])

$$FL_{\Lambda}^{loc} \left(\mathcal{M}_{\tilde{A}}^{(\beta_0, \beta)} \right) = \widehat{\mathcal{M}}_A^{(\beta_0+1, \beta)}.$$

Consider the filtration on $\mathcal{D}_{\widehat{V}}$ for which z has degree -1 , ∂_z has degree 2 and $\deg(\lambda_i) = 0$, $\deg(\partial_{\lambda_i}) = 1$. Write $\text{MF}^z(\mathcal{D}_{\widehat{V}})$ for the category of well-filtered $\mathcal{D}_{\widehat{V}}$ -modules (that is, $\mathcal{D}_{\widehat{V}}$ -modules equipped with a filtration compatible with the filtration on $\mathcal{D}_{\widehat{V}}$ just described and such that the corresponding Rees module is coherent over the corresponding Rees ring). Denote by G_{\bullet} the induced filtrations on the module $\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$, which are $\mathcal{R}_{\mathbb{C}_z \times \Lambda}$ -modules. We have

$$G_0 \widehat{\mathcal{M}}_A^{(\beta_0, \beta)} = \mathcal{R}_{\mathbb{C}_z \times \Lambda} / \mathcal{R}_{\mathbb{C}_z \times \Lambda} (\widehat{\square}_l)_{l \in \mathbb{L}_A} + \mathcal{R}_{\mathbb{C}_z \times \Lambda} \widehat{E} + \mathcal{R}_{\mathbb{C}_z \times \Lambda} (\widehat{E}_j)_{j=1, \dots, k+l}$$

and $G_k \widehat{\mathcal{M}}_A^{(\beta_0, \beta)} = z^k \cdot G_0 \widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$. In general, the modules $\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$ and their filtration steps may be quite complicated. However, we have considered in [RS17] their restriction to a specific Zariski open subset $\Lambda^{\circ} \subset \left(\Lambda \setminus \bigcup_{i=1}^{m+l} \{w_i = 0\} \right) \subset \Lambda$ (called W° in [RS17, Remark 3.8]), which contains the critical locus of the family of Laurent polynomials associated to the matrix A (but excludes certain singularities at infinity of this family). Denote by ${}^{\circ}\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$ the restriction $(\widehat{\mathcal{M}}_A^{(\beta_0, \beta)})|_{\mathbb{C}_z \times \Lambda^{\circ}}$ together with the induced filtration $G_{\bullet} {}^{\circ}\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$. Then $G_k {}^{\circ}\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$ is $\mathcal{O}_{\mathbb{C}_z \times \Lambda^{\circ}}$ -locally free for all k . Moreover, the multiplication by z is invertible on $\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$, filtered with respect to G_{\bullet} (shifting the filtration by one) and so is its inverse. Hence, we have a strict morphism

$$\cdot z : ({}^{\circ}\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}, G_{\bullet}) \longrightarrow ({}^{\circ}\widehat{\mathcal{M}}_A^{(\beta_0-1, \beta)}, G_{\bullet+1}).$$

We also need a slightly modified version of the Fourier-Laplace transformed GKZ-systems. More precisely, define the modules ${}^{\circ}\widehat{\mathcal{N}}_A^{\beta}$ as the cyclic quotients of $\mathcal{D}_{\mathbb{C}_z \times \Lambda^{\circ}}[z^{-1}]$ by the left ideal generated by $\widetilde{\square}_l$ for $l \in \mathbb{L}_A$ and $\widehat{E}_j - z\beta_j$ for $j = 0, \dots, k+c$, where

$$\begin{aligned} \widetilde{\square}_l &:= \prod_{i \in \{1, \dots, m\}: l_i > 0} \lambda_i^{l_i} (z \cdot \partial_i)^{l_i} \prod_{i \in \{m+1, \dots, m+l\}: l_i > 0} \prod_{\nu=1}^{l_i} (\lambda_i (z \cdot \partial_i) - z \cdot \nu) \\ &\quad - \prod_{i=1}^{m+l} \lambda_i^{l_i} \cdot \prod_{i \in \{1, \dots, m\}: l_i < 0} \lambda_i^{-l_i} (z \cdot \partial_i)^{-l_i} \prod_{i \in \{m+1, \dots, m+l\}: l_i < 0} \prod_{\nu=1}^{-l_i} (\lambda_i (z \cdot \partial_i) - z \cdot \nu). \end{aligned}$$

Consider the *invertible* morphism

$$\Psi : \circ\widehat{\mathcal{N}}_A^{(0,0,0)} \longrightarrow \circ\widehat{\mathcal{M}}_A^{-(2l,0,1)} \quad (46)$$

given by right multiplication with $z^l \cdot \prod_{i=m+1}^{m+l} \lambda_i$ (recall that $\lambda_i \neq 0$ on Λ°). We define $\widetilde{\phi}$ to be the composition $\widetilde{\phi} := \widehat{\phi} \circ \Psi$, where $\widehat{\phi}$ is the morphism

$$\widehat{\phi} : \circ\widehat{\mathcal{M}}_A^{-(2l,0,1)} \longrightarrow \circ\widehat{\mathcal{M}}_A^{(-l,0,0)},$$

given by right multiplication with $\partial_{\lambda_{m+1}} \cdots \partial_{\lambda_{m+l}}$. In concrete terms, we have:

$$\begin{aligned} \widetilde{\phi} : \circ\widehat{\mathcal{N}}_A^{(0,0,0)} &\longrightarrow \circ\widehat{\mathcal{M}}_A^{(-l,0,0)}, \\ x &\longmapsto \widehat{\phi}(x \cdot z^l \cdot \lambda_{m+1} \cdots \lambda_{m+l}) = x \cdot (z\lambda_{m+1}\partial_{m+1}) \cdots (z\lambda_{m+l}\partial_{m+l}). \end{aligned}$$

We have an induced filtration $G_\bullet \circ\widehat{\mathcal{N}}_A^{(0,0,0)}$ which satisfies

$$G_0 \circ\widehat{\mathcal{N}}_A^{(0,0,0)} = \mathcal{R}_{\mathbb{C}_z \times \Lambda^\circ} / \mathcal{R}_{\mathbb{C}_z \times \Lambda^\circ}(\widetilde{\square}_l)_{l \in \mathbb{L}_A} + \mathcal{R}_{\mathbb{C}_z \times \Lambda^\circ}(\widehat{E}_j - z\beta_j)_{j=0, \dots, m+l}$$

and $G_k \circ\widehat{\mathcal{N}}_A^{(0,0,0)} = z^k \cdot G_0 \circ\widehat{\mathcal{N}}_A^{(0,0,0)}$

In order to obtain the lattices G_\bullet we need to extend the functor FL_Λ^{loc} to the category of filtered \mathcal{D} -modules.

Definition 5.2. Let $(\mathcal{M}, F_\bullet) \in \text{MF}(\mathcal{D}_V) = \text{MF}(\mathcal{D}_{\mathbb{C}_{\lambda_0} \times \Lambda})$. Define $\mathcal{M}[\partial_{\lambda_0}^{-1}] := \mathcal{D}_V[\partial_{\lambda_0}^{-1}] \otimes_{\mathcal{D}_V} \mathcal{M}$ and consider the natural localization morphism $\widehat{\text{loc}} : \mathcal{M} \rightarrow \mathcal{M}[\partial_{\lambda_0}^{-1}]$. We define the saturation of F_\bullet to be

$$F_k \mathcal{M}[\partial_{\lambda_0}^{-1}] := \sum_{j \geq 0} \partial_{\lambda_0}^{-j} \widehat{\text{loc}}(F_{k+j} \mathcal{M}). \quad (47)$$

and we denote by $G_\bullet \widehat{\mathcal{M}}$ the filtration induced from $F_k \mathcal{M}[\partial_{\lambda_0}^{-1}]$ on $\widehat{\mathcal{M}} :=$

$FL_\Lambda^{loc}(M) \in M_h(\mathcal{D}_{\widehat{V}}) = M_h(\mathcal{D}_{\mathbb{C}_z \times \Lambda})$. Notice that for $(\mathcal{M}, F_\bullet) = (\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}, F_\bullet^{ord})$, the two definitions of G_\bullet coincide: As we have

$$F_k^{ord} \mathcal{M}_A^{(\beta_0-1, \beta)}[\partial_{\lambda_0}^{-1}] = \text{im}(\partial_{\lambda_0}^k \mathbb{C}[\lambda_0, \lambda_1, \dots, \lambda_{m+l}] \langle \partial_{\lambda_0}^{-1}, \partial_{\lambda_0}^{-1} \partial_{\lambda_1}, \dots, \partial_{\lambda_0}^{-1} \partial_{\lambda_{m+l}} \rangle) \text{ in } \mathcal{M}_A^{(\beta_0-1, \beta)}[\partial_{\lambda_0}^{-1}],$$

the filtration induced by $F_k^{ord} \mathcal{M}_A^{(\beta_0-1, \beta)}[\partial_{\lambda_0}^{-1}]$ on $\widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$ is precisely $G_k \widehat{\mathcal{M}}_A^{(\beta_0, \beta)}$.

We denote by $(FL_\Lambda^{loc}, \text{Sat})$ the induced functor from the category $\text{MF}(\mathcal{D}_V)$ to the category $\text{MF}^z(\mathcal{D}_{\widehat{\Lambda}})$ which sends (\mathcal{M}, F_\bullet) to $(\widehat{\mathcal{M}}, G_\bullet)$.

From the above duality considerations, we deduce the following result.

Proposition 5.3. *The morphism*

$$\widetilde{\phi} : \circ\widehat{\mathcal{N}}_A^{(0,0,0)} \longrightarrow \circ\widehat{\mathcal{M}}_A^{(-l,0,0)}$$

is strict with respect to the filtration G_\bullet , in particular, we have

$$\widetilde{\phi} \left(G_0 \circ\widehat{\mathcal{N}}_A^{(0,0,0)} \right) = G_0 \circ\widehat{\mathcal{M}}_A^{(-l,0,0)} \cap \text{im}(\widetilde{\phi})$$

Moreover, the object $(\text{im}(\widetilde{\phi}), G_\bullet)$ is obtained via the functor $(FL_\Lambda^{loc}, \text{Sat})$ from $(\text{im}(\phi), F_\bullet^H = F_{\bullet+k+l}^{ord})$, which underlies a pure Hodge module of weight $m+k+2l$ by Proposition 5.1.

Proof. The morphism Ψ is invertible, filtered (shifting the filtration by $-l$) and its inverse is also filtered. Hence it is strict. Therefore the strictness of $\widetilde{\phi}$ follows from the strictness of $z\widehat{\phi}$. We will deduce it from the strictness property of the morphism ϕ in Corollary 4.37.

Notice that the morphism $\widehat{\phi}$ is obtained from ϕ by linear extension in $\partial_{\lambda_0}^{-1}$. Recall that the morphism

$$\phi : (\mathcal{M}_{\widehat{A}}^{-(l+1,0,1)}, F_{\bullet}^{ord}) \longrightarrow (\mathcal{M}_{\widehat{A}}^0, F_{\bullet+l+1}^{ord})$$

was strict, hence equation (47) yields the strictness of

$$\widehat{\phi} : (\widehat{\mathcal{M}}_A^{-(2l,0,1)}, G_{\bullet}) \longrightarrow (\widehat{\mathcal{M}}_A^{(-l,0,0)}, G_{\bullet+l})$$

Finally, as already noticed above, this yields the strictness of

$$\widetilde{\phi} = \widehat{\phi} \circ \Psi : (\widehat{\mathcal{N}}_A^{(0,0,0)}, G_{\bullet}) \longrightarrow (\widehat{\mathcal{M}}_A^{(-l,0,0)}, G_{\bullet}).$$

□

The next corollary is now a direct consequence of [Sab08, Corollary 3.15].

Corollary 5.4. *The free $\mathcal{O}_{\mathbb{C}_z \times \Lambda^\circ}$ -module $G_0 \widehat{\mathcal{M}}_A^{(-l,0,0)} \cap \text{im}(\widetilde{\phi})$ underlies a variation of pure polarized non-commutative Hodge structures on Λ° (see [Sab11] for a detailed discussion of this notion).*

The main result in [RS17] concerns a mirror statement for several quantum \mathcal{D} -modules which are associated to the toric variety X_Σ and the split vector bundle \mathcal{E} . In particular, one can consider the reduced quantum \mathcal{D} -module $\overline{\text{QDM}}(X_\Sigma, \mathcal{E})$ which is a vector bundle on $\mathbb{C}_z \times H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*$, where $B_\varepsilon^* := \{q \in (\mathbb{C}^*)^{b_2(X_\Sigma)}, |0 < |q| < \varepsilon\}$ together with a flat connection

$$\nabla : \overline{\text{QDM}}(X_\Sigma, \mathcal{E}) \rightarrow \overline{\text{QDM}}(X_\Sigma, \mathcal{E}) \otimes_{\mathcal{O}_{\mathbb{C}_z \times H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*}} z^{-1} \Omega_{\mathbb{C}_z \times H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*}^1 (\log(\{0\} \times H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*)).$$

We refer to [MM17] for a detailed discussion of the definition of $\overline{\text{QDM}}(X_\Sigma, \mathcal{E})$, a short version can be found in [RS17, Section 4.1]. Notice that in loc.cit., $\overline{\text{QDM}}(X_\Sigma, \mathcal{E})$ is defined on some larger set, but in mirror type statements only its restriction to $H^0(X_\Sigma, \mathbb{C}) \times \mathbb{C}_z \times B_\varepsilon^*$ is considered. In the sequel, we will need to consider a Zariski open subset of $\mathcal{KM}^\circ \subset (\mathbb{C}^*)^{b_2(X_\Sigma)}$ which contains B_ε^* . We recall the main result from [MM17], which gives a GKZ-type description of $\overline{\text{QDM}}(X_\Sigma, \mathcal{E})$. We present it in a slightly different form, taking into account [RS17, Proposition 6.9]. Let $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ be the sheaf of Rees rings on $\mathbb{C}_z \times \mathcal{KM}^\circ$, and $R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ its module of global sections. If we write q_1, \dots, q_r for the coordinates on $(\mathbb{C}^*)^r$ (with $r := b_2(X_\Sigma)$), then $R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ is generated by $zq_i \partial_{q_i}$ and $z^2 \partial_z$ over $\mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$.

Theorem 5.5. *For any $\mathcal{L} \in \text{Pic}(X_\Sigma)$, write $\widehat{\mathcal{L}} \in R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ for the associated “quantized operator” as defined in [MM17, Notation 4.2.] or [RS17, Theorem 6.7]. Define the left ideal J of $R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ by*

$$J := R_{\mathbb{C}_z \times \mathcal{KM}^\circ} (Q_l)_{l \in \mathbb{L}_{A'}} + R_{\mathbb{C}_z \times \mathcal{KM}^\circ} \cdot \widehat{E},$$

where

$$\begin{aligned} Q_l &:= \prod_{i \in \{1, \dots, m\} : l_i > 0} \prod_{\nu=0}^{l_i-1} (\widehat{\mathcal{D}}_i - \nu z) \prod_{j \in \{1, \dots, c\} : l_{m+j} > 0} \prod_{\nu=1}^{l_{m+j}} (\widehat{\mathcal{L}}_j + \nu z) \\ &- \underline{q}^l \cdot \prod_{i \in \{1, \dots, m\} : l_i < 0} \prod_{\nu=0}^{-l_i-1} (\widehat{\mathcal{D}}_i - \nu z) \prod_{j \in \{1, \dots, c\} : l_{m+j} < 0} \prod_{\nu=1}^{-l_{m+j}} (\widehat{\mathcal{L}}_j + \nu z), \end{aligned}$$

$$\widehat{E} := z^2 \partial_z - \widehat{K}_{\mathbb{V}(\mathcal{E}^\vee)}.$$

Here we write $\mathcal{D}_i \in \text{Pic}(X_\Sigma)$ for a line bundle associated to the torus invariant divisor D_i , where $i = 1, \dots, m$. Let $K \subset R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ be the ideal

$$K := \left\{ P \in R_{\mathbb{C}_z \times \mathcal{KM}^\circ} \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} : \prod_{i=0}^k \prod_{j=1}^c (\widehat{\mathcal{L}} + p + i) P \in J \right\}$$

and \mathcal{K} the associated sheaf of ideals in $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$.

Suppose as above that the bundle $-K_{X_\Sigma} - \sum_{j=1}^l \mathcal{L}_j$ is nef, and moreover that each individual bundle \mathcal{L}_j is ample. Then there is a map $\text{Mir} : B_\varepsilon^* \rightarrow H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*$ such that we have an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules

$$(\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ} / \mathcal{K})|_{\mathbb{C}_z \times B_\varepsilon^*} \xrightarrow{\cong} (\text{id}_{\mathbb{C}_z} \times \text{Mir})^* \overline{\text{QDM}}(X_\Sigma, \mathcal{E}).$$

In order to relate the quantum \mathcal{D} -module $\overline{\text{QDM}}(X_\Sigma, \mathcal{E})$ with our results on GKZ-systems, we will use the restriction map $\bar{\rho} : \mathcal{KM}^\circ \hookrightarrow \Lambda$ as constructed in [RS17] (discussion before Definition 6.3. in loc.cit.). Then it follows from the results of loc.cit., Proposition 6.10, that we have an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules

$$\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ} / \mathcal{K} \cong (\text{id}_{\mathbb{C}_z} \times \bar{\rho})^* \left(\tilde{\phi} \left(G_0 \circ \widehat{\mathcal{N}}_A^{(0,0,0)} \right) \right)$$

Now we can deduce from Corollary 5.4 the main result of this section.

Theorem 5.6. *Consider the above situation of a k -dimensional toric variety X_Σ , globally generated line bundles $\mathcal{L}_1, \dots, \mathcal{L}_l$ such that $-K_{X_\Sigma} - \mathcal{E}$ is nef, where $\mathcal{E} = \bigoplus_{j=1}^l \mathcal{L}_j$, \mathcal{L}_j being ample for $j = 1, \dots, l$. Then the smooth $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -module $(\text{id}_{\mathbb{C}_z} \times \text{Mir})^* \overline{\text{QDM}}(X_\Sigma, \mathcal{E})$ (i.e., the vector bundle over $\mathbb{C}_z \times \mathcal{KM}^\circ$ together with its connection operator ∇) underlies a variation of pure polarized non-commutative Hodge structures.*

Proof. The strictness of $\tilde{\phi}$ as shown in Proposition 5.3 shows that $G_0 \widehat{\mathcal{M}}_A^{(-l,0,0)} \cap \text{im}(\tilde{\phi}) = \tilde{\phi}(G_0 \mathcal{N}^{(0,0,0)})$, hence, by Corollary 5.4, $\tilde{\phi}(G_0 \mathcal{N}^{(0,0,0)})$ underlies a variation of pure polarized non-commutative Hodge structures on Λ^0 . Hence the assertion follows from the mirror statement of Theorem 5.5. \square

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