Hypergeometric Hodge modules

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Abstract

We consider mixed Hodge module structures on GKZ-hypergeometric differential systems. We show that the Hodge filtration on these $\mathcal{D}$-modules is given by the order filtration, up to suitable shift. As an application, we prove a conjecture on the existence of non-commutative Hodge structures on the reduced quantum $\mathcal{D}$-module of a nef complete intersection inside a toric variety.

Contents

1 Introduction

This paper deals with Hodge structures on certain hypergeometric differential systems, also known as GKZ-systems (see, e.g., [GKZ90, Ado94]). They occur in various places in mathematics, notably in questions related to mirror symmetry for toric varieties. We have shown in our previous papers [RS15, RS12] how to express variants of the mirror correspondence as an equivalence of differential systems of “GKZ-type”.

However, an important point was left open in these articles: The mirror statements given there actually involve differential systems (i.e., holonomic $\mathcal{D}$-modules) with some additional data, sometimes called *lattices*. These are constructed by a variant of the Fourier-Laplace transformation from regular holonomic *filtered* $\mathcal{D}$-modules. In [Rei14], the first named author has shown that certain GKZ-systems actually carries a much richer structure, namely, they underlie *mixed Hodge modules* in the sense of M. Saito (see [Sai90]). The filtration in question is the Hodge filtration on these modules, but a concrete description of it is missing in [RS15, RS12]. As a consequence, the most important Hodge theoretic property

of the differential system entering in the mirror correspondence was formulated only as a conjecture in [RS12] (conjecture 6.13): the so-called *reduced quantum $\mathcal{D}$-module*, which governs certain Gromov-Witten invariants of nef complete intersections in toric varieties conjecturally underlies a *variation of non-commutative Hodge structures*. We prove this conjecture here (see Theorem 4.6), it appears as a consequence of the main result of the present paper, which determines the Hodge filtration on the GKZ-systems. More precisely, as GKZ-systems are defined as cyclic quotients of the ring of (algebraic) differential operators on an affine space, we obtain (Theorem 3.30) that this Hodge filtration is given by the filtration induced from the order of differential operators up to a suitable shift.
Let us give a short overview on the content of this article. The main result is obtained in two major steps, which occupy the sections two and three. First we study embeddings of tori into affine spaces given by a monomial map $h_B : (\mathbb{C}^*)^r \hookrightarrow \mathbb{C}^s; (t_1, \ldots, t_r) \mapsto (\underline{t}^{\underline{b}_1}, \ldots, \underline{t}^{\underline{b}_s})$, where $\underline{t}^{\underline{b}_i} = \prod_{k=1}^r t_j^{b_{ki}}$ and where the matrix of columns $B = (\underline{b}_i)_{i=1,\ldots,s}$ satisfies certain combinatorial properties related to the geometry of the semi-group ring $\mathbb{C}[\mathbb{N}B]$. We consider the direct image $\mathcal{H}^0(h_{B*}{}^p\mathbb{Q}^H_{(\mathbb{C}^*)^r})$ in the category of mixed Hodge modules, and calculate its Hodge filtration (Theorem 2.21). If the matrix B we started with satisfy an homogeneity property, then the underlying $\mathcal{D}$-module of this mixed Hodge module is a (monodromic) Fourier-Laplace transformation of the GKZ-system we are interested in. It should be noticed that Theorem 2.21 is of independent interest, its statement is related to the description of the Hodge filtration on various cohomology groups associated to singular toric varieties. We plan to discuss this question in a subsequent work. The main point in Theorem 2.21 is to determine the canonical V-filtration on the direct image module along the boundary divisor $\overline{im(h_B)}\backslash im(h_B)$, i.e., the calculation of some Bernstein polynomials.
The second step, carried out in section three consists in studying the behavior of a projectivized version of the above mentioned direct image module under the so-called *Radon*-transformation. It is well-known (see [Bry86] and [DE03]) that there is a close relation between Fourier-Laplace transformation and Radon transformation of holonomic $\mathcal{D}$-modules, however, the former one does not a priori preserve the category of mixed Hodge modules whereas the latter does. This fact is one of the main points in the prove of the existence of a mixed Hodge module structure on GKZ-systems in [Rei14]. We calculate the behaviour of the Hodge filtration under the various functors entering into the Radon transformation functor, an essential tool for these calculations is the so called *Euler-Koszul-complex* (or some variants of it) as introduced in [MMW05]. The last part of section three deals with the Hodge module structure on the holonomic dual GKZ-system (which is, under the assumptions on the initial data, also a GKZ-system).
In section four we explain the above mentioned conjecture from [RS12] and show how its proof can be deduced from our main result.
While working on this paper, a recent preprint of T. Mochizuki ([Moc15]) appeared where [RS12, Conjecture 6.13] is shown with apparently rather different methods.

To finish this introduction, we will introduce some notation and conventions used throughout the paper. Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension d_X. We denote by $M(\mathcal{D}_X)$ the abelian category of algebraic left $\mathcal{D}_X$-modules on X and the abelian subcategory of (regular) holonomic $\mathcal{D}_X$-modules by $M_h(\mathcal{D}_X)$ (resp. $(M_{rh}(\mathcal{D}_X))$. The full triangulated subcategory in $D^b(\mathcal{D}_X)$, consisting of objects with (regular) holonomic cohomology, is denoted by $D^b_h(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$).
Let $f : X \to Y$ be a map between smooth algebraic varieties. Let $M \in D^b(\mathcal{D}_X)$ and $N \in D^b(\mathcal{D}_Y)$, then we denote by

$$f_+M := Rf_*(\mathcal{D}_{Y\leftarrow X} \overset{L}{\otimes} M) \quad \text{resp.} \quad f^+M := \mathcal{D}_{X\to Y} \overset{L}{\otimes} f^{-1}M[d_X - d_Y]$$

the direct resp. inverse image for $\mathcal{D}$-modules. Recall that the functors f_+, f^+ preserve (regular) holonomicity (see e.g., [HTT08, Theorem 3.2.3]). We denote by $\mathbb{D} : D^b_h(\mathcal{D}_X) \to (D^b_h(\mathcal{D}_X))^{opp}$ the holonomic duality functor. Recall that for a single holonomic $\mathcal{D}_X$-module M, the holonomic dual is also a single holonomic $\mathcal{D}_X$-module ([HTT08, Proposition 3.2.1]) and that holonomic duality preserves regular holonomicity ([HTT08, Theorem 6.1.10]).
For a morphism $f : X \to Y$ between smooth algebraic varieties we additionally define the functors $f_\dagger := \mathbb{D} \circ f_+ \circ \mathbb{D}$ and $f^\dagger := \mathbb{D} \circ f^+ \circ \mathbb{D}$.

Let $MF(\mathcal{D}_X)$ be the category of filtered $\mathcal{D}_X$-modules (M,F) where the ascending filtration $F_\bullet$ satisfies

1. $F_pM = 0$ for $p \ll 0$
2. $\bigcup_p F_pM = M$
3. $(F_p\mathcal{D}_X)F_qM \subset F_{p+q}M$ for $p \in \mathbb{Z}_{\geq 0},\ q \in \mathbb{Z}$

where $F_\bullet\mathcal{D}_X$ is the filtration by the order of the differential operator.

We denote by $MHM(X)$ the abelian category of algebraic mixed Hodge modules and by $D^bMHM(X)$ the corresponding bounded derived category. The forgetful functor to the bounded derived category of regular holonomic $\mathcal{D}$-modules is denoted by

$$Dmod : D^bMHM(X) \longrightarrow D^b_{rh}(\mathcal{D}_X)\,.$$

For each morphism $f : X \to Y$ between complex algebraic varieties, there are induced functors

$$f_*, f_! : D^bMHM(X) \longrightarrow D^bMHM(Y)$$

and

$$f^*, f^! : D^bMHM(Y) \to D^bMHM(X)\,,$$

which are interchanged by $\mathbb{D}$ and which lift the analogous functors $f_+, f_\dagger, f^\dagger, f^+$ on $D^b_{rh}(\mathcal{D}_X)$.
Let $\mathbb{Q}^H_{pt}$ be the unique mixed Hodge structure of weight 0 with $Gr^W_i = Gr^F_i = 0$ and for $i \neq 0$ and underlying vector space $\mathbb{Q}$. Denote by $a_X : X \longrightarrow \{pt\}$ the map to the point and set

$$\mathbb{Q}^H_X := a^*_X\mathbb{Q}^H_{pt}\,.$$

The shifted object ${}^p\mathbb{Q}^H_X := \mathbb{Q}^H_X[d_X]$ lies in $MHM(X)$ and is equal to $(\mathcal{O}_X, F, \mathbb{Q}_X[d_X], W)$ with $Gr^F_p = 0$ for $p \neq 0$ and $Gr^W_i = 0$ for $i \neq d_X$. We have $\mathbb{D}\mathbb{Q}^H_X \simeq a^!_X\mathbb{Q}^H_{pt}$ and, since X is smooth, the isomorphism

$$\mathbb{D}\mathbb{Q}^H_X \simeq \mathbb{Q}^H_X(d_X)[2d_X]\,. \tag{1}$$

Here (d_X) denotes the Tate twist (see e.g., [Sai90, page 257])

2 Hodge filtration on torus embeddings

Let B be a $(r \times s)$-integer matrix with columns $(\underline{b}_1, \ldots, \underline{b}_s)$, satisfying

$$\mathbb{Z}B := \sum_{i=1}^{s} \mathbb{Z}\underline{b}_i = \mathbb{Z}^r$$

and set $\underline{b}_0 := 0$. We will sometimes associate to the matrix B the homogenized matrix $\widetilde{B}$ with columns $\underline{\widetilde{b}}_i := (1, \underline{b}_i)$ for $i = 0, \ldots, s$. Notice that $\mathbb{Z}\widetilde{B} = \mathbb{Z}^{r+1}$ holds and that the matrix $\widetilde{B}$ is pointed, by which we mean that 0 is the only unit in the semigroup $\mathbb{N}\widetilde{B}$.

The matrix B gives rise to a map from a torus $T = (\mathbb{C}^*)^r$ with coordinates $(t_1, \ldots, t_r)$ into the affine space $W = \mathbb{C}^s$ with coordinates $w_1, \ldots, w_s$:

$$\begin{aligned} h_B : T &\longrightarrow W \\ (t_1, \ldots, t_r) &\mapsto (\underline{t}^{\underline{b}_1}, \ldots, \underline{t}^{\underline{b}_s})\,, \end{aligned}$$

where $\underline{t}^{\underline{b}_i} := \prod_{k=1}^{r} t_k^{b_{ki}}$. Notice that the map h_B is affine and a locally closed embedding, hence the direct image functor for $\mathcal{D}_T$-modules $(h_B)_+$ is exact.

The aim of this section is to give a presentation of $(h_B)_+\mathcal{O}_T$ as a cyclic D_W-module and to compute explicitly its Hodge filtration as a mixed Hodge module.

2.1 Torus embeddings

Definition 2.1. *Let $\beta \in \mathbb{C}^r$. Write $\mathbb{L}_B$ for the $\mathbb{Z}$-module of relations among the columns of B and write $\mathcal{D}_W$ for the sheaf of rings of algebraic differential operators on W. Define*

$$\check{\mathcal{M}}_B^\beta := \mathcal{D}_W / \left((\check{\Box}_{\underline{l}})_{\underline{l} \in \mathbb{L}_B}, (\check{E}_k + \beta_k)_{k=1,\dots,r} \right) ,$$

where

$$\begin{aligned} \check{E}_k &:= \sum_{i=1}^{s} b_{ki} \partial_{w_i} w_i \quad \textit{for } k = 1, \dots, r \\ \check{\Box}_{\underline{m} \in \mathbb{L}_B} &:= \prod_{m_i > 0} w_i^{m_i} - \prod_{m_i < 0} w_i^{-m_i} . \end{aligned} \tag{2}$$

We will often work with the D_W-module of global sections

$$\check{M}_B^\beta := \Gamma(W, \check{\mathcal{M}}_B^\beta)$$

of the $\mathcal{D}_W$-module $\check{\mathcal{M}}_B^\beta$.

Remark 2.2. *Notice that $\check{\mathcal{M}}_B^\beta$ is just a Fourier-Laplace transformation (in all variables) of the GKZ-system $\mathcal{M}_B^\beta$ (cf. Definition 3.1).*

The semigroup ring associated to the B is

$$\mathbb{C}[\mathbb{N}B] \simeq \mathbb{C}[\underline{w}] / \left((\check{\Box}_{\underline{m}})_{\underline{m} \in \mathbb{L}_B} \right) ,$$

where $\mathbb{C}[\underline{w}]$ is the commutative ring $\mathbb{C}[w_1, \dots, w_s]$ and the isomorphism follows from [MS05, Theorem 7.3]. The rings $\mathbb{C}[\underline{w}]$ and $\mathbb{C}[\mathbb{N}B]$ are naturally $\mathbb{Z}^r$-graded if we define $\deg(w_j) = \underline{b}_j$ for $j = 1, \dots, s$. This is compatible with the $\mathbb{Z}^d$-grading of the Weyl algebra D given by $\deg(\partial_{w_j}) = -\underline{b}_j$ and $\deg(w_j) = \underline{b}_j$.

Definition 2.3 ([MMW05] Definition 5.2)**.** *Let N be a finitely generated $\mathbb{Z}^r$-graded $\mathbb{C}[\underline{w}]$-module. An element $\alpha \in \mathbb{Z}^r$ is called a true degree of N if N_α is non-zero. A vector $\alpha \in \mathbb{C}^r$ is called a quasi-degree of N, written $\alpha \in qdeg(N)$, if α lies in the complex Zariski closure $qdeg(N)$ of the true degrees of N via the natural embedding $\mathbb{Z}^r \hookrightarrow \mathbb{C}^r$.*

Schulze and Walther now define the following set of parameters:

Definition 2.4 ([SW09])**.** *The set*

$$sRes(B) := \bigcup_{j=1}^{s} sRes_j(B) ,$$

where

$$sRes_j(B) := \{\beta \in \mathbb{C}^r \mid \beta \in -(\mathbb{N}+1)\underline{b}_j - qdeg(\mathbb{C}[\mathbb{N}B]/(w_j))\}$$

is called the set of strongly resonant parameters of B.

Notice that Schulze and Walther [SW09] use the GKZ-system $\mathcal{M}_B^\beta$ and the convention $deg(\partial_{\lambda_j}) = \underline{b}_j$. We will use $\check{\mathcal{M}}_B^\beta$ and $deg(w_j) = \underline{b}_j$ instead.
For a pointed matrix B Schulze and Walther computed the direct image of the twisted structure sheaf

$$\mathcal{O}_T\, \underline{t}^\beta := \mathcal{D}_T / \mathcal{D}_T \cdot (\partial_{t_1} t_1 + \beta_1, \dots \partial_{t_r} t_r + \beta_r)$$

under the morphism h_B.

Theorem 2.5 ([SW09] Theorem 3.6, Corollary 3.7)**.** *Let B a pointed $(r \times s)$ integer matrix satisfying $\mathbb{Z}B = \mathbb{Z}^r$ Then the following are equivalent*

1. $\beta \notin sRes(B)$.
2. $\check{\mathcal{M}}_B^\beta \simeq (h_B)_+ (\mathcal{O}_T\, \underline{t}^\beta)$.
3. *Left multiplication with w_i is invertible on $\check{M}_B^\beta$ for $i = 1, \ldots, s$.*

In this section we want to generalize the implication 1. $\Rightarrow$ 2. to the case of a non-pointed matrix B. Notice that if we start with a (not necessarily pointed) matrix B which satisfies $\mathbb{Z}B = \mathbb{Z}^r$ then its homogenization $\widetilde{B}$ is pointed.

Consider now the augmented map

$$\begin{aligned} h_{\widetilde{B}} : \widetilde{T} &\longrightarrow \widetilde{W} \\ (t_0, \ldots, t_r) &\mapsto (t_0 \underline{t}^{\underline{b}_0}, t_0 \underline{t}^{\underline{b}_1}, \ldots, t_0 \underline{t}^{\underline{b}_s}), \end{aligned} \tag{3}$$

where $\widetilde{T} = (\mathbb{C}^*)^{r+1}$ and $\widetilde{W} = \mathbb{C}^{s+1}$ with coordinates $w_0, \ldots, w_s$. Let $\widetilde{W}_0$ be the subvariety of $\widetilde{W}$ given by $w_0 \neq 0$ and denote by $k_0 : \widetilde{W}_0 \to \widetilde{W}$ the canonical embedding. The map $h_{\widetilde{B}}$ factors through $\widetilde{W}_0$ which gives rise to a map h_0 with $h_{\widetilde{B}} = k_0 \circ h_0$. We get the following commutative diagram

$$\begin{tikzcd} \widetilde{T} \arrow[r, "h_0"] \arrow[rr, bend left, "h_{\widetilde{B}}"] \arrow[d, "\pi"'] & \widetilde{W}_0 \arrow[r, "k_0"] \arrow[d, "\pi_0"] & \widetilde{W} \\ T \arrow[r, "h_B"] & W & \end{tikzcd} \tag{4}$$

where π is the projection which forgets the first coordinate and π_0 is given by

$$\begin{aligned} \pi_0 : \widetilde{W}_0 &\longrightarrow W \\ (w_0, w_1, \ldots, w_s) &\mapsto (w_1/w_0, \ldots, w_s/w_0). \end{aligned}$$

Lemma 2.6. *For each $\beta_0 \in \mathbb{Z}$ we have an isomorphism:*

$$\mathcal{H}^0 \left((h_B)_+ \mathcal{O}_T\, \underline{t}^\beta\right) \simeq \mathcal{H}^0 \left((\pi_0)_+ k_0^+ \left((h_{\widetilde{B}})_+\, \mathcal{O}_{\widetilde{T}}\, \underline{t}^{(\beta_0,\beta)}\right)\right).$$

Proof. We show the claim by using the following isomorphisms

$$\begin{aligned} \mathcal{H}^0 h_{B+} \mathcal{O}_T\, \underline{t}^\beta &\simeq \mathcal{H}^0 h_{B+} \mathcal{H}^0 \pi_+ \mathcal{O}_{\widetilde{T}}\, \underline{t}^{(\beta_0,\beta)} \simeq \mathcal{H}^0 (h_B)_+ \pi_+ \mathcal{O}_{\widetilde{T}}\, \underline{t}^{(\beta_0,\beta)} \simeq \mathcal{H}^0 (\pi_0)_+ (h_0)_+ \mathcal{O}_{\widetilde{T}}\, \underline{t}^{(\beta_0,\beta)} \\ &\simeq \mathcal{H}^0 (\pi_0)_+ k_0^+ (k_0)_+ (h_0)_+ \mathcal{O}_{\widetilde{T}}\, \underline{t}^{(\beta_0,\beta)} \simeq \mathcal{H}^0 (\pi_0)_+ k_0^+ (h_{\widetilde{B}})_+ \mathcal{O}_{\widetilde{T}}\, \underline{t}^{(\beta_0,\beta)}. \end{aligned}$$

The first isomorphism follows from the fact that π is a projection with fiber $\mathbb{C}^*$, the second isomorphism follows from the exactness of $(h_B)_+$ and the fourth by the fact that $k_0^+ (k_0)_+ \simeq id_{\widetilde{W}_0}$.

□

Proposition 2.7. *Let B be a $r \times s$ integer matrix satisfying $\mathbb{Z}B = \mathbb{Z}^r$ and let $\beta \in \mathbb{Z}^r$ with $\beta \notin sRes(B)$, then $\mathcal{H}^0 \left((h_B)_+ \mathcal{O}_T\, \underline{t}^\beta\right)$ is isomorphic to $\check{\mathcal{M}}_B^\beta$*

Proof. The proof relies on Lemma 2.6 and the theorem of Schulze and Walther in the pointed case. Notice that we can find a $\beta_0 \in \mathbb{Z}$ with $\beta_0 >> 0$ such that $(\beta_0, \beta) \notin sRes(\widetilde{B})$ by [Rei14, Lemma 1.16]. Consider the following map on $\widetilde{W}_0$:

$$\begin{aligned} f : \widetilde{W}_0 &\longrightarrow W \times \mathbb{C}^*_{w_0} \\ (w_0, \ldots w_n) &\mapsto ((w_0, w_1/w_0, \ldots, w_s/w_0) \end{aligned}$$

together with the canonical projection $p : W \times \mathbb{C}^*_{w_0} \to W$ which forgets the last coordinate. This factorizes $\pi_0 = p \circ f$, which gives

$$\begin{aligned}\mathcal{H}^0\left((h_B)_+\mathcal{O}_T\,\underline{t}^\beta\right) \simeq &\mathcal{H}^0\left((\pi_0)_+\left((h_{\widetilde{B}})_+\mathcal{O}_{\widetilde{T}}\,\underline{t}^{(\beta_0,\beta)}\right)_{|\widetilde{W}_0}\right) \simeq \mathcal{H}^0\left(p_+f_+\left((h_{\widetilde{B}})_+\mathcal{O}_{\widetilde{T}}\,\underline{t}^{(\beta_0,\beta)}\right)_{|\widetilde{W}_0}\right)\\ \simeq &\mathcal{H}^0\left(p_+f_+(\check{\mathcal{M}}^{(\beta_0,\beta)}_{\widetilde{B}})_{|\widetilde{W}_0}\right).\end{aligned}$$

The $\mathcal{D}$-module $\mathcal{H}^0 f_+(\check{\mathcal{M}}^{(\beta_0,\beta)}_{\widetilde{B}})_{|\widetilde{W}_0}$ is isomorphic to $\mathcal{D}_{W\times\mathbb{C}^*_{w_0}}/\mathcal{I}'_0$ where $\mathcal{I}'_0$ is generated by

$$\check{\Box}_{\underline{m}\in\mathbb{L}_B} = \prod_{i:m_i>0,} w_i^{m_i} - \prod_{i:m_i<0,} w_i^{-m_i}$$

and

$$Z_0 = \partial_{w_0}w_0 + \beta_0 \qquad \text{and} \qquad E_k = \sum_{i=1}^r b_{ki}\partial_{w_i}w_i + \beta_k$$

Hence $\mathcal{H}^0 f_+(\check{\mathcal{M}}^{(\beta_0,\beta)})_{|\widetilde{W}_0}$ is isomorphic to $\check{\mathcal{M}}^\beta_B \boxtimes \mathcal{D}_{\mathbb{C}^*_{w_0}}/(\partial_{w_0}w_0+\beta_0)$ as a $\mathcal{D}$-module. We therefore have

$$\mathcal{H}^0\left(p_+f_+(\check{\mathcal{M}}^{(\beta_0,\beta)})_{|\widetilde{W}_0}\right) \simeq \mathcal{H}^0\left(p_+\mathcal{H}^0f_+(\check{\mathcal{M}}^{(\beta_0,\beta)})_{|\widetilde{W}_0}\right) \simeq \mathcal{H}^0p_+\left(\check{\mathcal{M}}^\beta_B \boxtimes \mathcal{D}_{\mathbb{C}^*_{w_0}}/(\partial_{w_0}w_0+\beta_0)\right) \simeq \check{\mathcal{M}}^\beta_B\,.$$

□

2.2 V-filtration

As above let B be a $r \times s$ integer matrix s.t. $\mathbb{Z}B = \mathbb{Z}^r$. In this section we additionally assume that the matrix B satisfies the following conditions:

$$\mathbb{N}B = \mathbb{Z}^r \cap \mathbb{R}_{\geq 0}B\,, \tag{5}$$

where $\mathbb{R}_{\geq 0}B$ is the cone generated by the columns of B and that the interior $int(\mathbb{N}B) = \mathbb{Z}^r \cap (\mathbb{R}_{\geq 0}B)^\circ$, where $(\mathbb{R}_{\geq 0}B)^\circ$ is the topological interior of $\mathbb{R}_{\geq 0}B$, is given by

$$int(\mathbb{N}B) = \mathbb{N}B + c \quad \text{for some} \quad c \in \mathbb{N}B\,. \tag{6}$$

The condition (5) is equivalent to the fact that the semigroup ring $\mathbb{C}[\mathbb{N}B]$ is normal, whereas the condition (6) is equivalent to $\mathbb{C}[\mathbb{N}B]$ being Gorenstein (cf. e.g. [BH93, Theorem 6.3.5]). Notice that in this case $0 \in \mathbb{N}B \subset \mathbb{Z}^r \setminus sRes(B)$ (cf. [Rei14, Lemma 1.11]). The semi-group $\mathbb{N}B$ can be decomposed into a positive semi-group P (i.e. a semi-group with no invertible elements except 0) and a group $G \simeq \mathbb{Z}^{r'}$ with $r' \leq r$ (cf. [BH93, Proposition 6.1.3])

$$\mathbb{N}B = P \oplus G\,.$$

It is easy to see that we can choose c to lie in P, which we will do from now on.
By [GKZ94, Chap.5, Proposition 2.3] , the spectrum $Y_B = \mathrm{Spec}\,(\mathbb{C}[\mathbb{N}B])$ is the closure of the locally closed embedding

$$\begin{aligned} h_B : T &\longrightarrow W\\ (t_1,\ldots t_d) &\mapsto (\underline{t}^{\underline{b}_1},\ldots,\underline{t}^{\underline{b}_s})\,.\end{aligned}$$

The affine variety Y_B carries a stratification by tori. The strata are in one-to-one correspondence with the faces Γ of the cone $\mathbb{R}_{\geq 0}B$. For a face $\Gamma \subset \mathbb{R}_{\geq 0}B$ the stratum $Y^0_B(\Gamma)$ is given by $\underline{t}^{\underline{b}} = 0$ for $\underline{b} \notin \Gamma \cap \mathbb{N}B$ and $\underline{t}^{\underline{b}} \neq 0$ for $\underline{b} \in \Gamma \cap \mathbb{N}B$ (cf. [GKZ94, Chap. 5 , Proposition 2.5]). The torus $T \subset Y_B$ is the open dense stratum given by the cone itself.
Denote by $D_B = div(\underline{t}^c)$ the principal divisor which has a zero precisely along $Y_B \setminus T$. Recall the isomorphism

$$\mathbb{C}[\underline{w}]/\left((\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_B}\right) \simeq \mathbb{C}[\mathbb{N}B]$$

from above.

Lemma 2.8. *1. The divisor D_B is equal to $D_1 + \ldots + D_m$ where the D_i are the torus-invariant divisors of the affine toric variety $Y_B = \mathrm{Spec}\,(\mathbb{C}[\mathbb{N}B])$. In particular D_B is an anti-canonical divisor.*

2. There exist an element $c' \in \mathbb{N}B$ satisfying $div(\underline{t}^{c'}) = div(\underline{t}^c)$ which can be represented by

$$w_{i_1} \cdot \ldots \cdot w_{i_l} \in \mathbb{C}[w_1, \ldots, w_s]$$

such that for each facet τ of $\mathbb{R}_{\geq 0}B$ exactly one of the generators $\underline{b}_{i_1}, \ldots, \underline{b}_{i_l}$ does not lie in τ. We denote the corresponding principal divisor of W by

$$D = div(w_{i_1} \ldots w_{i_l}).$$

In particular, if $\mathbb{N}B = \mathbb{Z}^r$, then $c' = 0$, hence $D = 0$.

Proof. Denote by σ the dual cone of $\mathbb{R}_{\geq 0}B$. This gives rise to a fan Σ (with only one maximal cone σ of dimension $r - r'$) corresponding to the toric variety $Y_B = \mathrm{Spec}\,(\mathbb{C}[\mathbb{N}B]) = \mathrm{Spec}\,(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^r])$. Let $v_1, \ldots, v_m$ be the primitive generators of the one-dimensional faces of σ. Notice that the v_a are in one-to-one correspondence with the facets of $\mathbb{R}_{\geq 0}B$ and that the direct summand G of $\mathbb{N}B \subset \mathbb{Z}^r$ is characterized by

$$G = \{x \in \mathbb{Z}^r \mid \langle x, v_a \rangle = 0 \text{ for all } a\}.$$

Let $D_1, \ldots, D_m$ be the torus-invariant divisors corresponding to the v_a. An anti-canonical divisor of Y_B is then given by $-K = D_1 + \ldots + D_m$ (cf. [Ful93, Chapter 4.4 p.89]). Since $\mathbb{C}[\mathbb{N}B]$ is Gorenstein the divisor $-K$ is Cartier. Because Y_B is toric and affine this divisor is principal, i.e. $-K = div(\underline{t}^k)$ for some $k \in \mathbb{Z}^r$ (cf. Chapter 3.3 in loc. cit.). Using the formula

$$-K = \sum_{a=1}^{m} \langle k, v_a \rangle D_a = D_1 + \ldots + D_m$$

we see that we can choose k to lie in $P \cap int(\mathbb{N}B)$. Since $int(\mathbb{N}B) = \mathbb{N}B + c$ we have $\langle c, v_a \rangle \in \mathbb{Z}_{>0}$ and $\langle c, v_a \rangle \leq \langle k, v_a \rangle = 1$. This shows the first claim.
Since $c \in \mathbb{N}B$, we have a presentation

$$c = \sum_{j \in J_1} c_j \underline{b}_j + \sum_{j \in J_2} d_k \underline{b}_j$$

with $c_k, d_k \in \mathbb{Z}_{>0}$, $J_1 \subset \{j \mid \underline{b}_j \notin G\}$ and $J_2 \subset \{j \mid \underline{b}_j \in G\}$. This gives

$$1 = \langle c, v_a \rangle = \sum_{j \in J_1} c_j \langle \underline{b}_j, v_a \rangle + 0$$

(with $\langle \underline{b}_j, v_a \rangle \in \mathbb{Z}_{\geq 0}$). Hence for each $a \in \{1, \ldots, m\}$ we have a unique $j_a \subset J_1$ such that $c_{j_a} = 1$, $\langle \underline{b}_{j_a}, v_a \rangle = 1$ and $\langle \underline{b}_j, v_a \rangle = 0$ for $j \in (J_1 \cup J_2) \setminus \{j_a\}$. Since for each $\underline{b}_j$ with $j \in J_1$ there exists at least one v_a with $\langle \underline{b}_j, v_a \rangle \neq 0$, we conclude that $c_j = 1$ for all $j \in J_1$. Define

$$c' := \sum_{j \in J_1} 1 \cdot \underline{b}_j + \sum_{j \in J_2} 1 \cdot \underline{b}_j \,.$$

We clearly have $div(\underline{t}^c) = div(\underline{t}^{c'})$ and $\underline{t}^{c'}$ can be represented by $w_{i_1} \cdot \ldots \cdot w_{i_l}$ where $\{i_1, \ldots, i_l\} = J_1 \cup J_2$. If $\mathbb{N}B = \mathbb{Z}^r$, then $c = 0$ and therefore $c' = 0$ by the definition of c' above. □

Example 2.9. *1. Consider the matrix B with columns $(1,0,0), (1,1,0), (1,0,1), (1,1,1)$. The element $c = (2,1,1)$ is the unique element which generates the interior $Int(\mathbb{N}B)$. It can be represented by $(1,0,0) + (1,1,1)$ as well as $(1,1,0) + (1,0,1)$.*

2. Consider the matrix B with columns $(1,0), (2,1), (-1,0), (-2,1)$. The element $c \in P$ is given by $(0,1)$. It can be represented by $(2,1) + 2 \cdot (0,-1)$. In this case, the element c' is given by $(1,1) = (2,1) + (0,-1)$.

We can factorize h_B into the morphism

$$\begin{aligned} k_B : T &\to W \setminus D \\ (t_1, \dots, t_r) &\mapsto (\underline{t}^{\underline{b}_1}, \dots, \underline{t}^{\underline{b}_s}) \end{aligned} \tag{7}$$

and the canonical open embedding $l_B : W \setminus D \to W$.

Lemma 2.10. *The morphism $k_B : T \to W \setminus D$ is a closed embedding.*

Proof. Recall the stratification of the affine variety $Y_B = \overline{h_B(T)}$. The boundary components $Y_B^0(\Gamma)$ for $\Gamma \subsetneq \mathbb{R}_{\geq 0}B$ all lie in $D_B = Y_B \cap D$, hence the claim follows.

□

The aim of this section is to compute the canonical (descending) V-filtration of $\check{\mathcal{M}}_B := \check{\mathcal{M}}_B^0 \simeq h_{B+}\mathcal{O}_T$ (or Kashiwara-Malgrange filtration) along the normal crossing divisor D. By Lemma 2.8 and Lemma 2.10 we can assume $\mathbb{N}B \neq \mathbb{Z}^r$, i.e. $c' \neq 0$.

We review very briefly some facts about the V-filtration for D-modules. Let $X = \operatorname{Spec}(R)$ be a smooth affine variety and $Y = div(t)$ be a smooth reduced principal divisor. Denote by $I = (t)$ the corresponding ideal. The V-filtration on D_X is defined by

$$V^k D_X = \{P \in D_X \mid P I^j \subset I^{j+k} \quad \text{for any} \quad j \in \mathbb{Z}\}\,,$$

where $I^j = R$ for $j \leq 0$. One has

$$\begin{aligned} V^k D_X &= t^k V^0 D_X\,, \\ V^{-k} D_X &= \sum_{0 \leq j \leq k} \partial_t^j V^0 D_X\,. \end{aligned}$$

Choose a total ordering $<$ on $\mathbb{C}$ such that, for any $\alpha, \beta \in \mathbb{C}$, the following conditions hold:

1. $\alpha < \alpha + 1$
2. $\alpha < \beta \quad$ if and only if $\quad \alpha + 1 < \beta + 1$
3. $\alpha < \beta + m \quad$ for some $m \in \mathbb{Z}$

Let N be a coherent D_X-module. The canonical V-filtration (or Kashiwara-Malgrange filtration) is an exhaustive filtration on N indexed discretely by $\mathbb{C}$ with total order as above and is uniquely determined by the following conditions

1. $(V^k D_X)(V^\alpha N) \subset V^{\alpha+k} N$ for all k, α
2. $V^\alpha N$ is coherent over $V^0 D_X$ for any α
3. $t(V^\alpha N) = V^{\alpha+1} N$ for $\alpha >> 0$
4. the action of $\partial_t t - \alpha$ on $Gr_V^\alpha N = V^\alpha N / V^{>\alpha} N$ is nilpotent

where $V^{>\alpha} N := \bigcup_{\beta > \alpha} V^\beta$.

We reduce the computation of the V-filtration on $\check{M}_B$ along the possibly singular divisor D to the computation of a V-filtration along a smooth divisor by considering the following graph embedding:

$$\begin{aligned} i_g : W &\longrightarrow W \times \mathbb{C}_t \\ (w_1, \dots, w_s) &\mapsto (w_1, \dots, w_s, w_{i_1} \cdot \ldots \cdot w_{i_l})\,. \end{aligned}$$

Instead of computing the V-filtration on $\check{M}_B$, we will compute it on $\Gamma(W \times \mathbb{C}_t, \mathcal{H}^0(i_{g+}\check{\mathcal{M}}_A))$ along $t = 0$ (notice that i_g is an affine embedding hence i_{g+} is exact). In order to compute the direct image we consider the composed map

$$\begin{aligned} i_g \circ h_B : T &\longrightarrow W \times \mathbb{C}_t \\ (t_1, \dots, t_r) &\mapsto (\underline{t}^{\underline{b}_1}, \dots, \underline{t}^{\underline{b}_s}, \underline{t}^{\underline{b}_{i_1} + \ldots + \underline{b}_{i_l}})\,. \end{aligned} \tag{8}$$

Notice that the matrix B' (built from the columns $\underline{b}_1, \ldots, \underline{b}_s, \underline{b}_{i_1} + \ldots + \underline{b}_{i_l}$) gives a saturated semigroup $\mathbb{N}B' = \mathbb{N}B$. Hence we can apply again Proposition 2.7 to compute

$$\check{\mathcal{M}}_{B'} \simeq \mathcal{H}^0 i_{g+} \check{\mathcal{M}}_B \simeq \mathcal{H}^0 (i_g \circ h_B)_+ \mathcal{O}_T \, .$$

This means that $\mathcal{H}^0 i_{g+} \check{\mathcal{M}}_B$ is a cyclic $\mathcal{D}_{W\times\mathbb{C}_t}$-module $\mathcal{D}_{W\times\mathbb{C}_t}/\mathcal{I}'$, where $\mathcal{I}'$ is generated by

$$\check{E}'_k := \sum_{i=1}^{s} b_{ki} \partial_{w_i} w_i + c_k \partial_t t \quad \text{for } \; k = 1, \ldots, r \, , \tag{9}$$

where $c_k = b_{ki_1} + \ldots + b_{ki_l}$ is the k -th component of $c \in \mathbb{Z}^r$ and

$$\check{\Box}_{\underline{m}\in\mathbb{L}_{B'}} := \prod_{m_i > 0} w_i^{m_i} - \prod_{m_i<0} w_i^{-m_i} \, , \tag{10}$$

where $\mathbb{L}_{B'}$ is the $\mathbb{Z}$-module of relations among the columns of B'.

We are going to use the following characterization of the canonical V-filtration along $t = 0$.

Proposition 2.11. *[MM04, Definition 4.3-3, Proposition 4.3-9] Let $n \in N$ and set $E := \partial_t t$. The Bernstein-Sato polynomial of n is the unitary polynomial of smallest degree, satisfying*

$$b(E)n \in V^1(D_X)n \, .$$

We denote it by $b_n(x) \in \mathbb{C}[x]$ and the set of roots of $b_n(x)$ by $ord(n)$. The canonical V-filtration on N is then given by

$$V^\alpha N = \{ n \in N \mid ord(n) \subset [\alpha, \infty) \} \, .$$

We will use this characterization to compute the canonical V-filtration on $\check{M}_{B'} := \check{M}^0_{B'}$ along $t = 0$.

Key-Lemma 2.12. *The Bernstein-Sato polynomial of $[1] \in \check{M}_{B'}$ is equal to x^m for some $m \in \mathbb{N}$.*

Proof. In order to prove the claim it is enough to show that

$$(\partial_t t)^m \in V^1(D_{W\times\mathbb{C}_t}) + I' \, . \tag{11}$$

Notice that for each $u \in \mathbb{Z}^r$ the following element lies in I':

$$\sum_{i=1}^{s} \langle u, \underline{b}_i \rangle \partial_{w_i} w_i + \langle u, c' \rangle \partial_t t \, .$$

Hence for each u with $\langle u, c' \rangle \neq 0$ (recall that we assume $c' \neq 0$) we have the following expression for $\partial_t t$ modulo I':

$$\partial_t t \equiv \frac{1}{\langle u, c' \rangle} \sum_{i=1}^{s} \langle u, \underline{b}_i \rangle \partial_{w_i} w_i \, .$$

We claim that for $k \leq r$ we can write $(\partial_t t)^k$ modulo $V^1(D_V) + I'$ as a linear combination of monomials

$$\partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_k}} w_{j_k}$$

such that $\underline{b}_{j_1}, \ldots, \underline{b}_{j_k}$ lie in a common face of $\mathbb{R}_{\geq 0} B$ and span a k-dimensional subspace. We prove this by induction, the case $k = 1$ being clear. Now assume that we have proven this for $k - 1 < r$. Let

$$\partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} w_{j_{k-1}}$$

be a monomial occurring in the expression of $(\partial_t t)^{k-1}$ which we obtained by the inductive assumption. Choose an $u \in \mathbb{Z}^r$ such that $\langle u, \underline{b}_{j_1} \rangle = \ldots = \langle u, \underline{b}_{j_{k-1}} \rangle = 0$ and $\langle u, c' \rangle \neq 0$. This is possible since the $\underline{b}_{j_1}, \ldots, \underline{b}_{j_{k-1}}$ lie on a common face of $\mathbb{R}_{\geq 0} B$ and c' lies in the interior of $\mathbb{R}_{\geq 0} B$. Then

$$(\partial_t t) \cdot \left(\partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} w_{j_{k-1}} \right) \equiv \left(\sum_{i=1}^{s} \frac{\langle u, \underline{b}_i \rangle}{\langle u, c \rangle} \partial_{w_i} w_i \right) \cdot \left(\partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} w_{j_{k-1}} \right) .$$

Notice that the first bracket on the right hand side does not contain any monomial $\partial_{w_j} w_j$ such that $\underline{b}_j$ lies on the face $\{x \in \mathbb{R}_{\geq 0} \mid \langle u, x\rangle = 0\}$. This shows that for the monomials $\left(\partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} w_{j_{k-1}} \cdot \partial_{w_{j_k}} w_{j_k}\right)$ occurring on the right hand side the $\underline{b}_{j_1}, \ldots, \underline{b}_{j_{k-1}}, \underline{b}_{j_k}$ span a k-dimensional cone. Now, there are two possible cases. Either, the $\underline{b}_{j_1}, \ldots, \underline{b}_{j_{k-1}}, \underline{b}_{j_k}$ lie on a common face of $\mathbb{R}_{\geq 0}$, in this case we have shown the inductive step, or they do not lie on a common face in which case the sum $\underline{b}_{j_1} + \ldots + \underline{b}_{j_{k-1}} + \underline{b}_{j_k}$ does lie in the interior of $\mathbb{R}_{\geq 0} B$, i.e.

$$\underline{b}_{j_1} + \ldots + \underline{b}_{j_{k-1}} + \underline{b}_{j_k} = c' + b'$$

for some $b' \in \mathbb{N}B$. Notice that this gives rise a relation in $\mathbb{L}_B \subset \mathbb{L}_{B'}$, which in turn gives the following equivalences

$$\begin{aligned} \partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} w_{j_{k-1}} \cdot \partial_{w_{j_k}} w_{j_k} &= \partial_{w_{j_1}} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} \cdot \partial_{w_{j_k}} \cdot w_{j_1} \cdot \ldots \cdot w_{j_{k-1}} \cdot w_{j_k} \\ &\equiv \partial_{w_{j_1}} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} \cdot \partial_{w_{j_k}} \cdot w^{c'} \cdot w^{b'} . \end{aligned}$$

Recall that the matrix B' has the columns $\underline{b}_1, \ldots, \underline{b}_s, c'$ with $c' = \underline{b}_{i_1} + \ldots \underline{b}_{i_l}$ which gives the relation

$$w^{c'} \simeq t .$$

Hence, the monomial in question is equivalent to

$$\partial_{w_{j_1}} w_{j_1} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} w_{j_{k-1}} \cdot \partial_{w_{j_k}} w_{j_k} \equiv \partial_{w_{j_1}} \cdot \ldots \cdot \partial_{w_{j_{k-1}}} \cdot \partial_{w_{j_k}} \cdot w^{b'} \cdot t \in V^1(D_{W \times \mathbb{C}_t}) .$$

Notice that in the case $k = r$, only the second case can occur, since the $\underline{b}_{j_1}, \ldots, \underline{b}_{j_r}$ have to span an r-dimensional cone and therefore can not lie on a common face. But this shows the claim.

□

We are now able to compute the full canonical filtration with respect to $t = 0$ on $\mathcal{H}^0 i_{g+} \check{\mathcal{M}}_B$. For this consider the induced V-filtration on $\check{M}_{B'} = \Gamma(W \times \mathbb{C}_t, \mathcal{H}^0 i_{g+} \check{\mathcal{M}}_B)$

$$V^{\alpha}_{ind} \check{M}_{B'} := \{[P] \in \check{M}_{B'} \mid P \in V^{\alpha} D\} .$$

The next proposition shows that this induced filtration is the canonical V-filtration.

Proposition 2.13. *The canonical V-filtration of $(i_{g+} \check{\mathcal{M}}_B)$ along $t = 0$ is equal to the induced one, i.e.:*

$$V^{\alpha} \check{M}_{B'} = V^{\alpha}_{ind} \check{M}_{B'} .$$

Proof. We will show that $V^{\alpha}_{ind} \check{M}_{B'}$ satisfies the defining property of the canonical V-filtration.

An element $[P]$ of $V^{k}_{ind} \check{M}_{B'}$ for $k \geq 0$ can be written as

$$[P] = [\sum_{i=0}^{l} t^k (\partial_t t)^i P_i] + [R] ,$$

where $[R] \in V^{k+1}_{ind} \check{M}_{B'}$ and $P_i \in \mathbb{C}[w_1, \ldots, w_s]\langle \partial_{w_1}, \ldots, \partial_{w_s}\rangle$. We have

$$\begin{aligned} (\partial_t t - k)^m \cdot [P] &= [\sum_{i=0}^{l} t^k (\partial_t t)^i P_i \cdot (\partial_t t)^m] + (\partial_t t - k)^m \cdot [R] \\ &= \sum_{i=0}^{l} t^k (\partial_t t)^i P_i \cdot (\partial_t t)^m \cdot [1] + (\partial_t t - k)^m \cdot [R] . \end{aligned}$$

But $\sum_{i=0}^{l} t^k (\partial_t t)^i P_i \cdot (\partial_t t)^m \cdot [1] \in V^{k+1}_{ind} \check{M}_{B'}$ because $\sum_{i=0}^{l} t^k (\partial_t t)^i P_i \in V^k D$ and $(\partial_t t)^m \cdot [1] \in V^1_{ind} \check{M}_{B'}$ which follows from Lemma 2.12 or, more precisely, from formula (11). Therefore

$$(\partial_t t - k)^m \cdot [P] \in V^{k+1}_{ind} \check{M}_{B'} .$$

Now let $[P] \in V_{ind}^{-k}\check{M}_{B'}$ with $k > 0$. It can be written as

$$[P] = [\sum_{i=0}^{l} \partial_t^k (\partial_t t)^i P_i] + [R]\,,$$

where $[R] \in V_{ind}^{k+1}\check{M}_{B'}$. By a similar argument we have

$$(\partial_t t + k)^m \cdot [P] \in V_{ind}^{-k+1}\check{M}_{B'}\,.$$

Hence $(\partial_t t - k)^m$ is nilpotent on $Gr^k_{V_{ind}}\check{M}_{B'}$, which is the characterizing property of the canonical V-filtration by Proposition 2.11. $\square$

2.3 Compatibility of filtrations

Let $V = \mathbb{C}^{s+1}$ and denote by D_V the Weyl-algebra $\mathbb{C}[w_0, \ldots, w_s]\langle \partial_{w_0}, \ldots, \partial_{w_s}\rangle$. We will denote the ascending order filtration on D_V by $F_\bullet$. The descending V-filtration on D_V with respect to $w_0 = 0$ is denoted by $V^\bullet D_V$. We have

$$V^0 D_V = \sum_{i\geq 0} (\partial_{w_0} w_0)^i P_i$$

for $P_i \in \mathbb{C}[w_0, \ldots, w_s]\langle \partial_{w_0}, \ldots, \partial_{w_s}\rangle$ and

$$V^k D_V = w_0^k V^0 D_V \quad \text{and} \quad V^{-k} D_V = \sum_{j\geq 0} \partial^j_{w_0} V^0 D_V$$

for $k > 0$. Now let $I \subset D_V$ be a left ideal and denote by $M := D_V/I$ the corresponding left D_V-module. The filtrations $V^\bullet$ and $F_\bullet$ induce filtration $V^\bullet_{ind}$ and $F^{ord}_\bullet$ on M:

$$V^k_{ind} M := \frac{V^k D_V + I}{I} \quad \text{and} \quad F^{ord}_p M := \frac{F_p D_V + I}{I}\,.$$

We take an element $P \in D_V$ and denote by $P = \sum_{\gamma,\delta} c_{\gamma\delta} w^\gamma \partial_w^\delta$ its standard form. The element $P \in D_V$ is called pure of order k if each monomial $c_{\gamma\delta} w^\gamma \partial_w^\delta \in V^k D_V$ and $0 \neq [c_{\gamma\delta} w^\gamma \partial_w^\delta] \in Gr_V^k D_V$. Notice that the product $P_1 \cdot P_2$ of two elements P_1 and P_2 of pure order k_1 resp. k_2 is of pure order $k_1 + k_2$.

We now want to recall Buchberger's algorithm in the Weyl algebra D_V. As a weight vector $\vec{w}$ we choose $(0, \ldots, 0, 1, \ldots, 1)$ such that the w_i's have weight zero and the ∂_{w_i}'s have weight 1. If an operator P has standard form $\sum_{(\gamma,\delta)} c_{\gamma\delta} w^\gamma \partial_w^\delta$ and $m = \max\{\mid \delta \mid : c_{\gamma,\delta} \neq 0\}$, then the initial form of P with respect to $\vec{w}$ is defined as

$$in_w(P) := \sum_{(\gamma,\delta), |\delta| = m} c_{\gamma\delta} w^\gamma \partial_w^\delta\,.$$

We also need a total order $\prec$ on D which refines the order induced by the weight $\vec{w}$. We say

$$w^\gamma \partial_w^\delta \prec w^c \partial_w^d$$

if $\mid \delta \mid < \mid d \mid$ or $\mid \delta \mid = \mid d \mid$ and $\delta_n = d_n, \ldots, \delta_{i+1} = d_{i+1}, \delta_i < d_i$ for some $i \in \{0, \ldots, n\}$ or $\mid \delta \mid = \mid d \mid$ and $\delta_n = d_n, \ldots, \delta_0 = d_0$ and $\gamma_n = c_n, \ldots, \gamma_{i+1} = c_{i+1}, \gamma_i < c_i$ for some $i \in \{0, \ldots, n\}$. The order $\prec$ is a term order (cf. [SST00][Chapter 1.1]) and refines the order given by the weight $\vec{w}$.

Let P, Q be two normally order elements with

$$P = p_{\gamma\delta} w^\gamma \partial_w^\delta + \text{ lower order terms w.r.t. } \prec$$
$$Q = q_{cd} w^c \partial_w^d + \text{ lower order terms w.r.t. } \prec$$

so that $in_\prec(P) = p_{\gamma\delta} w^\gamma \partial_w^\delta$ and $in_\prec(Q) = q_{cd} w^c \partial_w^d$. The S-pair of P and Q is given by

$$sp(P, Q) := w^{\gamma'} \partial_w^{\delta'} P - (p_{\gamma\delta}/q_{cd}) w^{c'} \partial_w^{d'} Q\,,$$

where $\gamma'_i := \max(\gamma_i, c_i) - \gamma_i$, $\delta'_i := \max(\delta_i, d_i) - \delta_i$, $c'_i := \max(\gamma_i, c_i) - c_i$ and $d'_i := \max(\delta_i, d_i) - d_i$. Notice that if $sp(P,Q) \neq 0$, P is of pure order p and Q is of pure order q, then $sp(P,Q)$ is of pure order $p + |\delta'| = q + |d'|$.
First notice that, if P is pure of order p its normal form (cf . [SST00, Chapter 1.1] w.r.t. to pure elements $G_1, \ldots, G_m$ is also pure of order p. Now let I be an ideal which is generated by elements $F_1, \ldots, F_M \in D_V$. Buchberger's algorithm gives back a set G of elements in D_V which is a Gröbner basis for I.
From the algorithm 1.1.9 in loc. cit. we see that if $F_1, \ldots, F_m$ are generators for I which are pure, then the Gröbner basis G will also consist of pure elements. It follows form [SST00, Theorem 1.1.6] that this is also a Gröbner basis with respect to the order coming from the weight vector w.

Example 2.14. *Let $F_1 = \partial_{w_0} w_0 + \partial_{w_1} w_1 + \partial_{w_2} w_2$ and $F_2 = \partial_{w_1} - \partial_{w_2}$ be elements of pure order* 1 *and $F_3 = w_1 w_2 - w_0^2$ be of pure order* 0*. Denote by I the ideal which is generated by F_1, F_2 and F_3. The Gröbner basis $\{G_1, \ldots, G_5\}$ given by Macaulay2 is $G_i = F_i$ for $i \in \{1,2,3\}$ and the elements*

$$\begin{aligned} G_4 &= w_2 w_3 \partial_{w_1} + 2 w_1 w_3 \partial_{w_3} + w_1 \\ G_5 &= w_3^2 \partial_{w_1} \partial_{w_3} + 2 w_1 w_3 \partial_{w_2} \partial_{w_3} + w_3 \partial_{w_1} + w_1 \partial_{w_2} \end{aligned}$$

are pure of order 0*.*

Lemma 2.15. *Let I be a left ideal in D_V that can be generated by finitely many elements of pure order, then the following map is surjective:*

$$V^k D_V \cap F_p D_V \longrightarrow V^k_{ind} M \cap F^{ord}_p M \,.$$

Proof. Let $m \in V^k_{ind} M \cap F^{ord}_p M$. We can find $P, Q \in D_V$ such that $P \in F_p D_V$, $Q \in V^k D_V$ and $[P] = m = [Q]$, i.e. $P = Q - i$ for some $i \in I$. We have to find a Q' with $Q' \in V^k D_V \cap F_p D_V$ with $P = Q' - i'$ for $i' \in I$. We will construct this element Q' by decreasing induction on the order of Q by killing its leading term in each step. Let $t_Q := \operatorname{ord} Q$, $t_i := \operatorname{ord} i$ and set $t := \max(t_Q, t_i)$. Obviously we have $t \geq p$. If $t = p$ we are done. Hence, we assume $t > p$, thus we have

$$0 = \sigma_t(P) = \sigma_t(Q - i)$$

and therefore $t = t_Q = t_i$ and therefore $\sigma_t(Q) = \sigma_t(i) \neq 0$.
Since I is generated by elements which have pure order, we can find a Gröbner basis $G = \{G_1, \ldots, G_m\}$ which is also of pure order. We can write

$$i = \sum_{l=1}^{m} i_l G_l$$

with $i_l \in F_{t - ord\, G_l}$ by [SST00, Theorem 1.2.5]. We have

$$\sigma_t(i) = \sum_{i=1}^{m} \sigma_{t - ord\, G_l}(i_l) \sigma_{ord\, G_l}(G_l) \,.$$

Now let $\tilde{i}_l$ be of pure order s.t. $\sigma_{t - ord\, G_l}(i_l) = \sigma_{t - \mathrm{ord}\ G_l}(\tilde{i}_l)$ and put

$$\tilde{i} = \sum_{l=1}^{m} \tilde{i}_l G_l$$

which is again of pure order t. Notice that each monomial in $\sigma_t(Q) \in \mathbb{C}[w_0, \ldots, w_s, \xi_0, \ldots, \xi_s]$ is of the following form $w_0^{i_0} \cdots w_n^{i_n} \xi_0^{j_0} \cdots \xi_n^{j_n}$ with $\sum_{l=1}^{0} j_l = t$ and $i_0 - j_0 \geq k$, where the last inequality follows from $Q \in V^k D_V$. Since $\sigma_t(Q) = \sigma_t(\tilde{i})$ and $\tilde{i}$ is of pure order, we can conclude that $\tilde{i}$ is in $V^k D_V$, too. We therefore have

$$P = Q - \tilde{i} - (i - \tilde{i})$$

with $Q - \tilde{i} \in F_{t-1} D_V \cap V^k D_V$. The claim follows now by descending induction on the order t. $\square$

Example 2.16. *Consider the D-module $M = \mathbb{C}[w]\langle\partial_w\rangle/(w^2\partial_w - 1)$. Notice that the generator $w^2\partial_w - 1$ is not pure. The element $1 \equiv w^2\partial_w$ lies in $V^1_{ind}M \cap F^{ord}_0 M$ but not in the image of*

$$V^1 D \cap F^{ord}_0 D = (w \cdot \mathbb{C}[w\partial_w]) \cap \mathbb{C}[w] \longrightarrow M\,.$$

Proposition 2.17. *The Lemma above applies to $\check{M}_{B'} = D_{\mathbb{C}_t \times W}/\left((\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}} + (\check{E}'_k + \beta_k)_{k=1,\dots,r}\right)$.*

Proof. First notice that the generators $(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}}$ and $(\check{E}_k + \beta_k)_{k=1,\dots,r}$ of the ideal I' are pure. It remains to show that finitely many of them suffice to generate I'. But this follows by taking a finite Gröbner basis of the ideal $((\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}})$ in the commutative ring $\mathbb{C}[w_1,\dots,w_s,t]$ and the elements $(\check{E}_k + \beta_k)_{k=1,\dots,r}$. □

2.4 Calculation of the Hodge filtration

In this section we want to compute the Hodge filtration on the mixed Hodge module

$$\mathcal{H}^0(h_{B*}{}^p\mathbb{Q}^H_T)\,.$$

We will need the following formula which describes the extension of a mixed Hodge-module over a smooth hypersurface. Let X be a smooth variety, let $t, x_1, \dots x_n$ be local coordinates on X and $j : Y \hookrightarrow X$ be a smooth hypersurface given by $t = 0$. Let $\mathcal{N}^H$ be a mixed Hodge module on $X \setminus Y$ with underlying filtered $\mathcal{D}$-module $(\mathcal{N}, F^H_\bullet\mathcal{N})$, then

$$F^H_p\mathcal{H}^0 j_+\mathcal{N} = \sum_{i\geq 0} \partial^i_t F^H_{p-i} V^0\mathcal{H}^0 j_+\mathcal{N}, \quad \text{where} \quad F^H_q V^0\mathcal{H}^0 j_+\mathcal{N} := V^0\mathcal{H}^0 j_+\mathcal{N} \cap j_*(F^H_q\mathcal{N})\,, \tag{12}$$

where $V^0\mathcal{H}^0 j_+\mathcal{N}$ is the canonical V-filtration on the $\mathcal{D}$-module $\mathcal{H}^0 j_+\mathcal{N} \simeq j_*\mathcal{N}$.

If Y is a non-smooth hypersurface locally given by $f = 0$, we consider (locally) the graph embedding

$$\begin{aligned} i_f : X &\longrightarrow X \times \mathbb{C}_t \\ x &\mapsto (x, f(x)) \end{aligned}$$

together with its restriction $i^*_f : X \setminus Y \to X \times \mathbb{C}^*_t$. Notice that i^*_f is a closed embedding. Given a mixed Hodge module $\mathcal{N}$ on $X \setminus Y$ we proceed as follows. We first extend the Hodge filtration of $(i^*_f)_+\mathcal{N}$ over the smooth divisor given by $\{t = 0\}$ as explained above. Afterwards we restrict the mixed Hodge module which we obtained to the smooth divisor given by $\{t = f\}$.

Recall from section 1 that ${}^p\mathbb{Q}^H_T$ has the underlying filtered $\mathcal{D}$-module $(\mathcal{O}_T, F^H_\bullet\mathcal{O}_T)$, where the Hodge filtration is given by

$$F^H_k\mathcal{O}_T = \begin{cases} \mathcal{O}_T & \text{for } k \geq 0 \\ 0 & \text{else}\,. \end{cases}$$

We will use several different presentations of $\mathcal{O}_T$ as a $\mathcal{D}_T$-module. For each $\alpha = (\alpha_k)_{k=1,\dots,r} \in \mathbb{Z}^r$ we have a D_T-linear isomorphism

$$\Gamma(T, \mathcal{O}_T) \simeq D_T/(t_k\partial_{t_k} + \alpha_k)_{k=1,\dots,r}$$

such that the Hodge filtration is simply the order filtration on the right hand side.

Consider the following commutative diagram

$$\begin{tikzcd} T \arrow[r, "k_B"] \arrow[rr, bend left, "h_B"] & W^* \arrow[r, "l"] \arrow[rrd, "i^*_f"] & W \arrow[r, "i_f"] & W \times \mathbb{C}_t \\ & & & W \times \mathbb{C}^*_t \arrow[u, "j_t"'] \end{tikzcd}$$

where $W^* := W \setminus D = W \setminus \{w_{i_1} \dots w_{i_l} = 0\} \simeq \mathbb{C}^{s-l} \times (\mathbb{C}^*)^l$ (cf. Lemma 2.8) and i_f is the graph embedding

$$\begin{aligned} i_f : W &\longrightarrow W \times \mathbb{C}_t \\ w &\mapsto (w, w_{i_1} \cdot \ldots \cdot w_{i_l}) \end{aligned} \tag{13}$$

associated to the function $f : W \to \mathbb{C}_t, \underline{w} \mapsto w_{i_1} \cdot \ldots \cdot w_{i_l}$. Notice that $i_f \circ l$ factors over $W \times \mathbb{C}_t^*$. We have the following isomorphisms

$$i_{f+} h_+ \mathcal{O}_T \simeq i_{f+} l_+ k_{B+} \mathcal{O}_T \simeq j_{t+} i^*_{f+} k_{B+} \mathcal{O}_T \,.$$

Lemma 2.18. *The direct image $\mathcal{H}^0 k_{B+} \mathcal{O}_T$ is isomorphic to the cyclic $\mathcal{D}_{W^*}$ module $\mathcal{D}_{W^*}/\check{\mathcal{I}}^*$ where $\check{\mathcal{I}}^*$ is the left ideal generated by $(\check{E}_k + \beta_k)_{k=1,\dots,r}$ for $\beta = (\beta_k)_{k=1,\dots,r} \in \mathbb{Z}^r$ and $(\check{\Box}_{\underline{m}})_{\underline{m} \in \mathbb{L}_B}$. Furthermore, the Hodge-filtration on $\mathcal{D}_{W^*}/\check{\mathcal{I}}^*$ shifted by $s-r$ is equal to the induced order filtration, i.e.*

$$F^H_{p+(s-r)} \mathcal{D}_{W^*}/\check{\mathcal{I}}^* = F^{ord}_p \mathcal{D}_{W^*}/\check{\mathcal{I}}^* \,.$$

Proof. We factorize the map k_B from above in the following way. Let $B = C \cdot E \cdot F$ be the Smith normal form of B, i.e. $C = (c_{pq}) \in GL(r, \mathbb{Z})$, $F = (f_{uv}) \in GL(s, \mathbb{Z})$ and $E = (I_r, 0_{r,s-r})$. This gives rise to the maps

$$\begin{aligned} k_C : T &\longrightarrow T \\ (t_1, \dots, t_r) &\mapsto (\tilde{t}_1, \dots, \tilde{t}_r) = (\underline{t}^{\underline{c}_1}, \dots, \underline{t}^{\underline{c}_r}) \\ k_E : T &\longrightarrow (\mathbb{C}^*)^s \\ (\tilde{t}_1, \dots, \tilde{t}_r) &\mapsto (\tilde{w}_1, \dots, \tilde{w}_s) = (\tilde{t}_1, \dots, \tilde{t}_r, 1, \dots, 1) \\ k_F : (\mathbb{C}^*)^s &\longrightarrow (\mathbb{C}^*)^s \\ (\tilde{w}_1, \dots, \tilde{w}_s) &\mapsto (w_1, \dots, w_s) = (\underline{\tilde{w}}^{\underline{f}_1}, \dots, \underline{\tilde{w}}^{\underline{f}_s}) \,. \end{aligned}$$

We denote by $j_k : (\mathbb{C}^*)^s \to W^*$ the canonical embedding, then

$$k_{B+} \mathcal{O}_T \simeq (j_k \circ k_F \circ k_E \circ k_C)_+ \mathcal{O}_T \simeq j_{k+} k_{F+} k_{E+} k_{C+} \mathcal{O}_T \,.$$

Since all maps and all spaces involved are affine, we will work on the level of global sections.
Since k_C is a simple change of coordinates we have $\Gamma(T, \mathcal{H}^0 (k_C)_+ \mathcal{O}_T) \simeq D_T/(\sum_{i=1}^r c_{ki} \tilde{t}_i \partial_{\tilde{t}_i} + \alpha_k)_{k=1,\dots,r}$ for all $\alpha \in \mathbb{Z}^r$ and again the Hodge filtration is equal to the order filtration. We now calculate $\Gamma((\mathbb{C}^*)^s, k_{E+} k_{C+} \mathcal{O}_T)$. Since k_E is a closed embedding of a hyperplane, we have

$$\begin{aligned} \Gamma((\mathbb{C}^*)^s, \mathcal{H}^0 k_{E+} k_{C+} \mathcal{O}_T) &\simeq \Gamma(T, \mathcal{H}^0 k_{C+} \mathcal{O}_T)[\partial_{\tilde{w}_{r+1}}, \dots, \partial_{\tilde{w}_s}] \\ &\simeq D_{(\mathbb{C}^*)^s}/(\sum_{i=1}^r c_{ki} \tilde{w}_i \partial_{\tilde{w}_i} + \alpha_k)_{k=1,\dots,r}, (\tilde{w}_i - 1)_{i=r+1,\dots,s} \,. \end{aligned} \tag{14}$$

The Hodge-filtration is (cf. [Sai93, Formula (1.8.6)])

$$\begin{aligned} F^H_{p+(s-r)} \left(\Gamma((\mathbb{C}^*)^s, \mathcal{H}^0 k_{E+} k_{C+} \mathcal{O}_T) \right) &= \sum_{p_1+p_2=p} F^H_{p_1} \Gamma(T, \mathcal{H}^0 k_{C+} \mathcal{O}_T) \otimes \underline{\partial}^{p_2} \\ = \sum_{p_1+p_2=p} F^{ord}_{p_1} \Gamma(T, \mathcal{H}^0 k_{C+} \mathcal{O}_T) \otimes \underline{\partial}^{p_2} &= F^{ord}_p \left(\Gamma((\mathbb{C}^*)^s, \mathcal{H}^0 k_{E+} k_{C+} \mathcal{O}_T) \right) \,. \end{aligned} \tag{15}$$

Hence we see that the Hodge filtration on the presentation (14) shifted by $(s-r)$ is equal to the order filtration, i.e. $F^H_{p+(s-r)} = F^{ord}_p$.
The map k_F is again a change of coordinates, so we have

$$\begin{aligned} \Gamma((\mathbb{C}^*)^s, \mathcal{H}^0 k_{F+} k_{E_+} k_{C+} \mathcal{O}_T) &\simeq D_{(\mathbb{C}^*)^s}/ \left((\sum_{j=1}^s b_{kj} w_j \partial_{w_j} + \alpha_k)_{k=1,\dots,r}, (\underline{w}^{\underline{m}_i} - 1)_{i=r+1,\dots,s} \right) \\ &\simeq D_{(\mathbb{C}^*)^s}/ \left((\sum_{j=1}^s b_{kj} w_j \partial_{w_j} + \alpha_k)_{k=1,\dots,r}, (\check{\Box}_{\underline{m}})_{\underline{m} \in \mathbb{L}_B} \right) \,, \end{aligned} \tag{16}$$

where $\underline{m}_i$ are the columns of the inverse matrix $M = F^{-1}$. The first isomorphism follows from the equality $B = C \cdot E \cdot F$. The second isomorphism follows from the fact that an element $\underline{m} \in \mathbb{Z}^s$ is a relation between the columns of B if and only if it is a relation between the columns of $E \cdot F$. So the Hodge filtration on the presentation (16) shifted by $(s-r)$ is again the order filtration.
Now consider the open embedding j_k. By Lemma 2.10 the closure of $\mathrm{im}(j_k)$ has an empty intersection with the reduced normal crossing divisor $D = \{f = 0\} = \{w_{i_1} \cdots w_{i_l} = 0\}$. Therefore we have

$$\begin{aligned}\Gamma(W^*, \mathcal{H}^0 j_{k+} k_{F+} k_{E_+} k_{C+} \mathcal{O}_T) &\simeq \Gamma((\mathbb{C}^*)^s, \mathcal{H}^0 k_{F+} k_{E_+} k_{C+} \mathcal{O}_T) \\ &\simeq D_{(\mathbb{C}^*)^s} / \left((\sum_{j=1}^s b_{kj} w_j \partial_{w_j} + \alpha_k)_{k=1,\dots,r}, (\check{\Box}_{\underline{m}})_{\underline{m} \in \mathbb{L}_B} \right) \\ &\simeq D_{W^*} / \left((\sum_{j=1}^s b_{kj} w_j \partial_{w_j} + \alpha_k)_{k=1,\dots,r}, (\check{\Box}_{\underline{m}})_{\underline{m} \in \mathbb{L}_B} \right), \end{aligned} \tag{17}$$

where the first and last isomorphism follows from the fact that the $\mathcal{D}$-module $\mathcal{H}^0 k_{B+} \mathcal{O}_T$ has no support on $W^* \setminus (\mathbb{C}^*)^s$. Using the graph construction to extend the Hodge filtration as explained at the beginning of this section, we see that the extensions is simply given by

$$F_p^H \mathcal{H}^0 j_{k+} (k_F \circ k_E \circ k_C)_+ \mathcal{O}_T = j_{k*} F_p^H \mathcal{H}^0 (k_F \circ k_E \circ k_c)_+ \mathcal{O}_T$$

hence the Hodge filtration on $\Gamma(W^*, \mathcal{H}^0 j_{k+} k_{F+} k_{E_+} k_{C+} \mathcal{O}_T) \simeq \Gamma(W^*, \mathcal{H}^0 k_{B+} \mathcal{O}_T)$ shifted by $(s-r)$ is equal to the order filtration. We have

$$\sum_{j=1}^s b_{kj} w_j \partial_{w_j} + \alpha_k = \sum_{j=1}^s b_{kj} \partial_{w_j} w_j - \sum_{j=1}^s b_{kj} + \alpha_k \,.$$

Setting $\beta_k := -\sum_{j=1}^s b_{kj} + \alpha_k$, proves the claim. □

We now would like to compute the Hodge filtration of $l_+ k_{B+} \mathcal{O}_T \simeq (h_B)_+ \mathcal{O}_T$. As mentioned at the beginning of this section, we will consider the graph embedding i_f with respect to the function $f = w_{i_1} \cdots w_{i_l}$ and extend the module $\mathcal{H}^0 k_{B+} \mathcal{O}_T \cong \mathcal{D}_{W^*} / \check{\mathcal{I}}^*$ over the smooth divisor $\{t = 0\}$.

Lemma 2.19. *The direct image $i^*_{f+} k_{B+} \mathcal{O}_T$ is isomorphic to the cyclic $\mathcal{D}_{W \times \mathbb{C}^*_t}$-module $\mathcal{D}_{W \times \mathbb{C}^*_t} / \mathcal{I}^\circ$ where $\mathcal{I}^\circ$ is the left ideal generated by $(\check{E}'_k + \beta_k)_{k=1,\dots r}$ for $\beta = (\beta_k)_{k=1,\dots,r} \in \mathbb{Z}^r$ and $(\check{\Box}_{\underline{m}})_{\underline{m} \in \mathbb{L}_{B'}}$. Furthermore, the Hodge filtration shifted by $s - r + 1$ is equal to the induced order filtration, i.e., we have*

$$F^H_{p+(s-r+1)} i^*_{f+} k_{B+} \mathcal{O}_T / \mathcal{I}^\circ \simeq F_p^{ord} \mathcal{D}_{W \times \mathbb{C}^*_t} / \mathcal{I}^\circ \,.$$

Proof. We define

$$\tilde{W} := (W^* \times \mathbb{C}_{\tilde{t}}) \setminus \{\tilde{t} + f(\underline{w}) = 0\}$$

and factor the map i^*_g in the following way. Set

$$\begin{aligned} l_1 : W^* &\longrightarrow \tilde{W} \\ \underline{w} &\mapsto (\underline{w}, 0) \\ l_2 : \tilde{W} &\longrightarrow W^* \times \mathbb{C}^*_t \\ (\underline{w}, \tilde{t}) &\mapsto (\underline{w}, \tilde{t} + f(\underline{w})) \end{aligned}$$

and let $l_3 : W^* \times \mathbb{C}^*_t \to W \times \mathbb{C}^*_t$ be the canonical inclusion. We have $i^*_f = l_3 \circ l_2 \circ l_1$. Notice again that all spaces involved are affine, hence we will work with the modules of global sections. Since l_1 is just the inclusion of a coordinate hyperplane we have

$$\Gamma(\tilde{W}, \mathcal{H}^0 l_{1+} k_{B+} \mathcal{O}_T) \simeq \Gamma(W^*, \mathcal{H}^0 k_{B+} \mathcal{O}_T)[\partial_{\tilde{t}}] \,.$$

The Hodge-filtration is given by

$$\Gamma(\tilde{W}, F^H_{p+1}(\mathcal{H}^0 l_{1+} k_{B+} \mathcal{O}_T)) \simeq \sum_{p_1 + p_2 = p} \Gamma(W^*, F^H_{p_1} \mathcal{H}^0 k_{B+} \mathcal{O}_T) \otimes \partial_{\tilde{t}}^{p_2} \,. \tag{18}$$

Notice that $\Gamma(\tilde{W}, \mathcal{H}^0 l_{1+} k_+ \mathcal{O}_T) \simeq D_{\tilde{W}}/I_1^\circ$ where I_1° is the left ideal generated by $(\check{E}_k + \beta_k)_{k=1,\dots,r}$, $(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_B}$ and $\tilde{t}$.

Under this isomorphism the Hodge filtration on $\Gamma(\tilde{W}, \mathcal{H}^0 l_{1+} k_+ \mathcal{O}_T)$ shifted by $(s-r)+1$ is equal to the order filtration by Lemma 2.18 and (18). The map l_2 is just a change of coordinates, hence under the substitution $\tilde{t} \mapsto t = \tilde{t} + f(\underline{w})$ and

$$w_i \partial_{w_i} \mapsto \begin{cases} w_i \partial_{w_i} & \text{for } i \in \{1, \dots, s\} \setminus \{i_1, \dots, i_l\} \\ w_i \partial_{w_i} + f(\underline{w})\partial_t \equiv w_i \partial_{w_i} + \partial_t t & \text{for } i \in \{i_1, \dots, i_l\} \end{cases}$$

and using the presentation of $\mathcal{H}^0(k_{B+}\mathcal{O}_T)$ as acyclic $\mathcal{D}$-module , we get that

$$\begin{aligned} \Gamma(W^* \times \mathbb{C}_t^*, \mathcal{H}^0 l_{2+} l_{1+} k_{B+} \mathcal{O}_T) &\simeq D_{W^* \times \mathbb{C}_t^*} / \big((\check{E}'_k + \beta_k)_{k=1,\dots,r} + (\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_B} + (t - w_{i_1} \cdots w_{i_l}) \big) \\ &\simeq D_{W^* \times \mathbb{C}_t^*} / \big((\check{E}'_k \beta_k)_{k=1,\dots,r} + (\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}} \big) \ , \end{aligned} \tag{19}$$

where $\check{E}'_k$ was defined in formula (9). Notice that the Hodge filtration shifted by $(s-r)+1$ is again equal to the order filtration.

Since the support of $\mathcal{H}^0 l_{2+} l_{1+} k_+ \mathcal{O}_T$ lies in the subvariety $\{t = f(\underline{w})\}$, the closure of the support in $W \times \mathbb{C}_t^*$ does not meet $(W \setminus W^*) \times \mathbb{C}_t^*$. We conclude that

$$\begin{aligned} \Gamma(W \times \mathbb{C}_t^*, \mathcal{H}^0 i^*_{f+} k_{B+} \mathcal{O}_T) &\simeq \Gamma(W \times \mathbb{C}_t^*, \mathcal{H}^0 l_{3+} l_{2+} l_{1+} k_{B+} \mathcal{O}_T) \\ &\simeq \Gamma(W^* \times \mathbb{C}_t^*, \mathcal{H}^0 l_{2+} l_{1+} k_+ \mathcal{O}_T) \\ &\simeq D_{W^* \times \mathbb{C}_t^*} / \big((\check{E}'_k)_{k=1,\dots,r} + (\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}} \big) \\ &\simeq D_{W \times \mathbb{C}_t^*} / \big((E'_k)_{k=1,\dots,r} + (\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}} \big) \ . \end{aligned}$$

The Hodge filtration is then simply extended by using the following formula

$$F_p^H \mathcal{H}^0 l_{t+} k_{B+} \mathcal{O}_T \simeq F_p^H \mathcal{H}^0 l_{3+} l_{2+} l_{1+} k_{B+} \mathcal{O}_T \simeq l_{3*} F_p^H \mathcal{H}^0 l_{2+} l_{1+} k_{B+} \mathcal{O}_T \ .$$

□

Proposition 2.20. *The direct image $\mathcal{H}^0 j_{t+} i^*_{f+} k_{B+} \mathcal{O}_T$ is isomorphic to the quotient $\mathcal{D}_{W\times\mathbb{C}_t}/\mathcal{I}'_\beta$, where $\mathcal{I}'_\beta$ is the left ideal which is generated by $(\check{E}'_k + \beta_k)_{k=1,\dots,r}$ for $\beta = (\beta_k)_{k=1,\dots,r} \in \mathbb{N}B'$ and $(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{B'}}$. Furthermore, the Hodge-filtration on $\mathcal{D}_{W\times\mathbb{C}_t}/\mathcal{I}' = \mathcal{D}_{W\times\mathbb{C}_t}/\mathcal{I}'_0$ shifted by $(s-r)+1$ is equal to the induced order filtration, that is,*

$$F^H_{p+(s-r+1)} \mathcal{H}^0 j_{t+} i^*_{f+} k_{B+} \mathcal{O}_T = F_p^{ord} \mathcal{D}_{W\times\mathbb{C}_t}/\mathcal{I}' \ .$$

Proof. First recall that we have an isomorphism $\mathcal{H}^0 j_{t+} i^*_{f+} k_{B+} \mathcal{O}_T \simeq \mathcal{H}^0 i_{f+} h_{B+} \mathcal{O}_T$. The composed map $i_f \circ h_B$ is a torus embedding given by the matrix B'. Hence, we have an isomorphism $j_{t+} l_{t+} k_+ \mathcal{O}_T \simeq \check{\mathcal{M}}^\beta_{B'}$ for $\beta \in \mathbb{Z}^r$ and $\beta \notin sRes(B')$. Since $\mathbb{N}B' = \mathbb{N}B$ the semigroup $\mathbb{N}B'$ is saturated. Therefore the set $\mathbb{N}B' \subset \mathbb{Z}^r \setminus sRes(B')$ by [Rei14, Lemma 1.11], which shows the first claim.
We will show the second claim for the case $\beta = 0$. The formula for extending the Hodge filtration over the smooth divisor $\{t = 0\}$ is

$$F_p^H \check{\mathcal{M}}_{B'} = \sum_{i\geq 0} \partial_t^i (V^0 \check{\mathcal{M}}_{B'} \cap j_{t*} j_t^{-1} F^H_{p-i} \check{\mathcal{M}}_{B'}) \ . \tag{20}$$

On the level of global sections the adjunction morphism $\check{\mathcal{M}}_{B'} \longrightarrow j_{t+} j_t^+ \check{\mathcal{M}}_{B'}$ is given by the inclusion $\check{M}_{B'} \to {}^*\check{M}_{B'}$, where ${}^*\check{M}_{B'}$ is the $D_{W\times\mathbb{C}_t^*}$-module from Lemma 2.19 seen as a $D_{W\times\mathbb{C}_t}$-module . Hence on the level of global section formula (20) gets

$$F_p^H \check{M}_{B'} = \sum_{i\geq 0} \partial_t^i (V^0 \check{M}_{B'} \cap F^H_{p-i} {}^*\check{M}_{B'}) \ .$$

Since we have $F^H_{p+(s-r+1)} {}^*\check{M}_{B'} = F_p^{ord*} \check{M}_{B'}$ by the same lemma, we conclude that $F^H_{s-r} \check{M}_{B'} = 0$. The element $1 \in \check{M}_{B'}$ is in $V^0 \check{M}_{B'}$ by Proposition 2.13 and $1 \in F^H_{(s-r)+1} {}^*\check{M}_{B'} = F_0^{ord*} \check{M}_{B'}$ and therefore

$1 \in F^H_{(s-r)+1}\check{M}_{B'}$. Notice that both $(\check{M}_{B'}, F^H_\bullet)$ and $(\check{M}_{B'}, F^{ord}_\bullet)$ are cyclic, well-filtered $D_{W\times\mathbb{C}_t}$-modules (see e.g. [Sai88, Section 2.1.1], therefore we can conclude

$$F^{ord}_p \check{M}_{B'} \subset F^H_{p+(s-r+1)}\check{M}_{B'}\,.$$

In order to show the converse inclusion, we have to show

$$F^{ord}_p \check{M}_{B'} \supset F^H_{p+(s-r+1)}\check{M}_{B'} = \sum_{i\geq 0} \partial_t^i (V^0 \check{M}_{B'} \cap F^H_{p+(s-r+1)-i}{}^*\check{M}_{B'}) = \sum_{i\geq 0}\partial_t^i (V^0_{ind}\check{M}_{B'} \cap F^{ord*}_{p-i}\check{M}_{B'})$$

for all $p \geq 0$, where the last equality follows from Proposition 2.13 and Lemma 2.19. Since we have

$$F^{ord}_p \check{M}_{B'} \supset \partial_t^i F^{ord}_{p-i}\check{M}_{B'} \text{ for } 0 \leq i \leq p \text{ and } p \geq 0$$

it remains to show

$$F^{ord}_{p-i}\check{M}_{B'} \supset V^0_{ind}\check{M}_{B'} \cap F^{ord*}_{p-i}\check{M}_{B'} \text{ for } 0 \leq i \leq p \text{ and } p \geq 0$$

resp.

$$F^{ord}_p \check{M}_{B'} \supset V^0_{ind}\check{M}_{B'} \cap F^{ord*}_p \check{M}_{B'} \text{ for } p \geq 0\,.$$

Now let $[P] \in V^0_{ind}\check{M}_{B'} \cap F^{ord*}_p\check{M}_{B'}$, then $P \in D_{W\times\mathbb{C}_t}$ can be written as

$$P = t^{-k}P_k + t^{-k+1}P_{k-1} + \dots$$

with $P_i \in \mathbb{C}[w_1,\dots,w_s]\langle\partial_t, \partial_{w_1},\dots,\partial_{w_s}\rangle$ and $P_k \neq 0$. Since $t^k \cdot [P] \in V^k_{ind}\check{M}_{B'} \cap F^{ord}_p\check{M}_{B'}$ it is enough to prove

$$t^k F^{ord}_p\check{M}_{B'} \supset V^k_{ind}\check{M}_{B'} \cap F^{ord}_p\check{M}_{B'} \text{ for } p \geq 0\,.$$

Given an element $[Q] \in V^k_{ind}\check{M}_{B'} \cap F^{ord}_p\check{M}_{B'}$ we can find, using Lemma 2.15, a $Q' \in V^k D_{W\times\mathbb{C}_t} \cap F_p D_{W\times\mathbb{C}_t}$ with $[Q] = [Q']$. But this element Q' can be written as a linear combination of $t^{l_0}w_1^{l_1}\dots w_s^{l_s}\partial_t^{p_0}\partial_{w_1}^{p_1}\dots\partial_{w_s}^{p_s}$ with $p_0 + \dots p_s \leq p$ and $l_0 - p_0 \geq k$, hence $[Q'] \in t^k F^{ord}_p\check{M}_{B'}$.

□

Now we want to deduce the Hodge filtration on $h_{B+}\mathcal{O}_T$ from the proposition above.

Theorem 2.21. *The direct image $h_+\mathcal{O}_T$ is isomorphic to the cyclic D_V-module $\check{\mathcal{M}}_B := \mathcal{D}_W/\check{\mathcal{I}}$, where $\check{\mathcal{I}}$ is the left ideal generated by $(\check{E}_k)_{k=1,\dots,r}$ and $(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_B}$. The Hodge filtration on $\check{\mathcal{M}}_B$ is equal to the order filtration shifted by $s-r$, i.e.*

$$F^H_{p+(s-r)}\check{\mathcal{M}}_B = F^{ord}_p\check{\mathcal{M}}_B\,.$$

Proof. Recall that we have $j_t \circ i_f^* \circ k_B = i_f \circ h_B$ where i_f is the graph embedding from (13). The map i_f can be factored by

$$\begin{aligned} i_0 : W &\longrightarrow W \times \mathbb{C}_{\widetilde{t}} \\ \underline{w} &\mapsto (\underline{w}, 0) \\ l_f : W \times \mathbb{C}_{\widetilde{t}} &\longrightarrow W \times \mathbb{C}_t \\ (\underline{w}, \widetilde{t}) &\mapsto (\underline{w}, \widetilde{t} + f(\underline{w}))\,. \end{aligned}$$

We first compute $\mathcal{H}^0(l_f^{-1})_+\check{\mathcal{M}}_{B'}$ with its corresponding Hodge filtration. Since $(l_f)^{-1}$ is just a coordinate change we get similarly to formula (19)

$$\Gamma(W \times \mathbb{C}_{\widetilde{t}}, \mathcal{H}^0(l_f^{-1})_+\check{\mathcal{M}}^0_{B'}) \simeq D_{W\times\mathbb{C}_{\widetilde{t}}}/\left((\check{E}_k)_{k=1,\dots,r}, (\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_B}, (\widetilde{t})\right)\,, \tag{21}$$

where the Hodge filtration on the right hand side is the induced order filtration shifted by $(s-r+1)$. Notice that the right hand side of (21) is simply $\check{M}^\beta_B[\partial_{\widetilde{t}}]$, hence the Hodge filtration on

$$\check{M}_B = \Gamma(W, \check{\mathcal{M}}_B) = \Gamma(W \times \mathbb{C}_{\widetilde{t}}, \mathcal{H}^0 i_0^+(l_f^{-1})_+\check{\mathcal{M}}_{B'})$$

is simply the order filtration shifted by $(s-r)$ by [Sai88, Proposition 3.2.2 (iii)]. □

3 Radon transforms of torus embeddings

3.1 Hypergeometric modules, Gauß-Manin systems and the Radon transformation

In this section we want to give a brief reminder on the relationship between GKZ-hypergeometric systems, Gauß-Manin systems of families of Laurent polynomials developed in [Rei14].

Definition 3.1. *Let $A = (a_{ki})$ be a $d \times n$ integer matrix. We assume that the columns $\underline{a}_1, \ldots, \underline{a}_n$ generate $\mathbb{Z}^d$ as a $\mathbb{Z}$-module. Moreover, let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$. Write $\mathbb{L}_A$ for the $\mathbb{Z}$-module of integer relations among the columns of A and write $\mathcal{D}_{\mathbb{C}^n}$ for the sheaf of rings of differential operators on $\mathbb{C}^n$ (with coordinates $\lambda_1, \ldots, \lambda_n$). Define*

$$\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{C}^s}/\mathcal{I}_A\,,$$

where $\mathcal{I}_A$ is the sheaf of left ideals generated by

$$\Box_{\underline{l}} := \prod_{i:l_i<0} \partial_{\lambda_i}^{-l_i} - \prod_{i:l_i>0} \partial_{\lambda_i}^{l_i}$$

for all $\underline{l} \in \mathbb{L}_A$ and

$$E_k := \sum_{i=1}^{n} a_{ki}\lambda_i\partial_{\lambda_i}$$

for $i = 1, \ldots, d$.

Since GKZ-systems are defined on the affine space $\mathbb{C}^n$, we will often work with the D-modules of global sections $M_A^\beta := \Gamma(\mathbb{C}^n, \mathcal{M}_A^\beta)$ rather than with the sheaves themselves.
As in [RS15, RS12], we will consider a homogenization of the above systems. Namely, given the matrix $A = (a_{ki})$, we consider the system $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$, where $\widetilde{A}$ is the $(d+1) \times (n+1)$ integer matrix

$$\widetilde{A} := (\underline{\widetilde{a}}_0, \ldots, \underline{\widetilde{a}}_n) := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{d1} & \ldots & a_{dn} \end{pmatrix} \tag{22}$$

and $\widetilde{\beta} \in \mathbb{C}^{d+1}$.
In order to show that such a homogenized GKZ-system comes from geometry we have to review briefly the so-called Radon transformation for $\mathcal{D}$-modules which was introduced by Brylinski [Bry86] and variants were later added by d'Agnolo and Eastwood [DE03].

Let W be the dual vector space of V with coordinates $w_0, \ldots, w_n$ and $\lambda_0, \ldots, \lambda_n$, respectively. We will denote by $Z \subset \mathbb{P}(W) \times V$ the universal hyperplane given by $Z := \{\sum_{i=0}^n \lambda_i w_i = 0\}$ and by $U := (\mathbb{P}(W) \times V) \setminus Z$ its complement. Consider the following diagram

$$\begin{tikzcd} & U \arrow[dl, "\pi_1^U"'] \arrow[d, hook, "j_U"] \arrow[dr, "\pi_2^U"] & \\ \mathbb{P}(W) & \mathbb{P}(V') \times V \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & V\,. \\ & Z \arrow[ul, "\pi_1^Z"] \arrow[u, hook, "i_Z"'] \arrow[ur, "\pi_2^Z"'] & \end{tikzcd}$$

We will use in the sequel several variants of the so-called Radon transformation in the derived category of mixed Hodge modules. These are functors from $D^bMHM(\mathbb{P}(W))$ to $D^bMHM(\mathcal{D}_V)$ given by

$$
\begin{aligned}
{}^*\mathcal{R}(M) &:= \pi^Z_{2*}(\pi^Z_1)^* M \simeq \pi_{2*} i_{Z*} i^*_Z \pi^*_1 M\,, \\
{}^!\mathcal{R}(M) &:= \mathbb{D} \circ {}^*\mathcal{R} \circ \mathbb{D}\,(M) \simeq \pi^Z_{2*}(\pi^Z_1)^! M \simeq \pi_{2*} i_{Z*} i^!_Z \pi^!_1 M \\
{}^*\mathcal{R}_{cst}(M) &:= \pi_{2*}\pi^*_1 M\,. \\
{}^!\mathcal{R}_{cst}(M) &:= \mathbb{D} \circ {}^*\mathcal{R}_{cst} \circ \mathbb{D}\,(M) \simeq \pi_{2*}\pi^!_1 M\,. \\
{}^*\mathcal{R}^\circ_c(M) &:= \pi^U_{2!}(\pi^U_1)^*(M) \simeq \pi_{2*} j_{U!} j^*_U \pi^*_1(M) \\
{}^!\mathcal{R}^\circ(M) &:= \mathbb{D} \circ {}^*\mathcal{R}^\circ_c \circ \mathbb{D}\,(M) \simeq \pi^U_{2*}(\pi^U_1)^!(M) \simeq \pi_{2*} j_{U*} j^!_U \pi^!_1(M)\,.
\end{aligned}
$$

The adjunction triangle corresponding to the open embedding j_U and the closed embedding i_Z gives rise to the following triangles of Radon transformations

$$
{}^!\mathcal{R}(M) \longrightarrow {}^!\mathcal{R}_{cst}(M) \longrightarrow {}^!\mathcal{R}^\circ(M) \xrightarrow{+1}\,, \tag{23}
$$

$$
{}^*\mathcal{R}^\circ_c(M) \longrightarrow {}^*\mathcal{R}_{cst}(M) \longrightarrow {}^*\mathcal{R}(M) \xrightarrow{+1}\,, \tag{24}
$$

where the second triangle is dual to the first.

We now introduce a family of Laurent polynomials defined on $T \times \Lambda := (\mathbb{C}^*)^d \times \mathbb{C}^n$ using the columns of the matrix A, more precisely, we put

$$
\begin{aligned}
\varphi_A : T \times \Lambda &\longrightarrow V = \mathbb{C}_{\lambda_0} \times \Lambda\,, \\
(t_1, \ldots, t_d, \lambda_1, \ldots, \lambda_n) &\mapsto (-\sum_{i=1}^n \lambda_i \underline{t}^{\underline{a}_i}, \lambda_1, \ldots, \lambda_n)\,.
\end{aligned} \tag{25}
$$

The following theorem of [Rei14] constructs a morphism between the Gauß-Manin system $\mathcal{H}^0(\varphi_{A,+}\mathcal{O}_{S\times W})$ resp. its proper version $\mathcal{H}^0(\varphi_{A,\dagger}\mathcal{O}_{S\times W})$ and certain GKZ-hypergeometric systems and identify both with a corresponding Radon transformation.
For this we apply the triangle (23) to $M = g_!\,\mathbb{D}\,{}^p\mathbb{Q}^H_T$ and the triangle (24) to $M = g_*{}^p\mathbb{Q}^H_T$, where the map g was defined by

$$
\begin{aligned}
g : T &\longrightarrow \mathbb{P}(W) \\
(t_1, \ldots, t_d) &\mapsto (1 : \underline{t}^{\underline{a}_1} : \ldots : \underline{t}^{\underline{a}_n})\,.
\end{aligned}
$$

Assume that the matrix $\widetilde{A}$ satisfies

1. $\mathbb{Z}\widetilde{A} = \mathbb{Z}^{d+1}$
2. $\mathbb{N}\widetilde{A} = \mathbb{R}_{\geq 0}\widetilde{A} \cap \mathbb{Z}^{d+1}$
3. $int(\mathbb{N}\widetilde{A}) = \mathbb{N}\widetilde{A} + c$ for some $c \in \mathbb{N}\widetilde{A}$.

Theorem 3.2. *[Rei14, Lemma 1.11, Proposition 3.4] For every $\beta \in \mathbb{N}\widetilde{A}$ and every $\beta' \in int(\mathbb{N}\widetilde{A})$, the following sequences of mixed Hodge-modules are exact and dual to each other:*

$$
\begin{tikzcd}[column sep=small]
 & H^{d-1}(T,\mathbb{C}) \otimes {}^p\mathbb{Q}^H_V & \mathcal{H}^0(\varphi_{A*}{}^p\mathbb{Q}^H_{T\times\Lambda}) & \mathcal{M}^\beta_{\widetilde{A}} & H^d(T,\mathbb{C}) \otimes {}^p\mathbb{Q}^H_V & \\
0 \arrow[r] & \mathcal{H}^n({}^*\mathcal{R}_{cst}(g_*{}^p\mathbb{Q}^H_T)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^n({}^*\mathcal{R}(g_*{}^p\mathbb{Q}^H_T)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{n+1}({}^*\mathcal{R}^\circ_c(g_*{}^p\mathbb{Q}^H_T)) \arrow[r] \arrow[u, "\simeq"] & \mathcal{H}^{n+1}({}^*\mathcal{R}_{cst}(g_*{}^p\mathbb{Q}^H_T)) \arrow[r] \arrow[u, "\simeq"] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
0 & \mathcal{H}^{-n}({}^!\mathcal{R}_{cst}(g_!\mathbb{D}\,{}^p\mathbb{Q}^H_T)) \arrow[l] \arrow[d, "\simeq"] & \mathcal{H}^{-n}({}^!\mathcal{R}(g_!\mathbb{D}\,{}^p\mathbb{Q}^H_T)) \arrow[l] \arrow[d, "\simeq"] & \mathcal{H}^{-n-1}({}^!\mathcal{R}^\circ(g_!\mathbb{D}\,{}^p\mathbb{Q}^H_T)) \arrow[l] \arrow[d, "\simeq"] & \mathcal{H}^{-n-1}({}^!\mathcal{R}_{cst}(g_!{}^p\mathbb{Q}^H_T)) \arrow[l] \arrow[d, "\simeq"] & 0\,. \arrow[l] \\
 & H_{d+1}(T,\mathbb{C}) \otimes \mathbb{D}\,{}^p\mathbb{Q}^H_V & \mathcal{H}^0(\varphi_{A!}\mathbb{D}\,{}^p\mathbb{Q}^H_{T\times\Lambda}) & \mathcal{M}^{-\beta'}_{\widetilde{A}} & H_d(T,\mathbb{C}) \otimes \mathbb{D}\,{}^p\mathbb{Q}^H_V &
\end{tikzcd}
$$

Notice that the isomorphisms in the third column induce a mixed Hodge-module structure on the GKZ-systems $\mathcal{M}_{\widetilde{A}}^{\widetilde{\beta}}$ and $\mathcal{M}_{\widetilde{A}}^{-\widetilde{\beta}'}$ for $\widetilde{\beta} \in \mathbb{N}\widetilde{A}$ resp. $\widetilde{\beta}' \in int(\mathbb{N}\widetilde{A})$.

Proposition 3.3. *Let $\beta \in \mathbb{N}\widetilde{A}$ and $\beta' \in int(\mathbb{N}\widetilde{A})$. There exists a unique (up to multiplication with a constant) morphism of mixed Hodge modules, given by*

$$\begin{aligned} \mathcal{M}_{\widetilde{A}}^{-\beta'}(-d-n) &\longrightarrow \mathcal{M}_{\widetilde{A}}^{\beta} \\ P &\mapsto P \cdot \partial^{\beta+\beta'}, \end{aligned}$$

where $\partial^{\beta+\beta'} := \prod_{i=0}^{n} \partial_{\lambda_i}^{k_i}$ for any $\underline{k} = (k_0, \ldots, k_n)$ with $\widetilde{A} \cdot \underline{k} = \beta + \beta'$.

Proof. First notice that there is a natural morphism of mixed Hodge modules

$$\mathcal{H}^0(\varphi_{A!}\mathbb{D}\,{}^p\mathbb{Q}^H_{T\times\Lambda})(-d-n) \longrightarrow \mathcal{H}^0(\varphi_{A*}{}^p\mathbb{Q}^H_{T\times\Lambda})$$

which is induced by the morphism $\mathbb{D}\,{}^p\mathbb{Q}^H_{T\times\Lambda}(-d-n) \to {}^p\mathbb{Q}^H_{T\times\Lambda}$. Using the isomorphisms in the second column, this gives a morphism

$$\mathcal{H}^{-n}({}^!\mathcal{R}(g_!\mathbb{D}\,{}^p\mathbb{Q}^H_T))(-n-d) \longrightarrow \mathcal{H}^n({}^*\mathcal{R}(g_*{}^p\mathbb{Q}^H_T)).$$

Now we can concatenate this with the following morphisms

$$\begin{tikzcd} \mathcal{H}^n({}^*\mathcal{R}(g_*{}^p\mathbb{Q}^H_T)) \arrow[rr] & & \mathcal{H}^{n+1}({}^*\mathcal{R}^\circ_c(g_*{}^p\mathbb{Q}^H_T)) \\ \mathcal{H}^{-n}({}^!\mathcal{R}(g_!\mathbb{D}\,{}^p\mathbb{Q}^H_T))(-n-d) \arrow[u] & & \mathcal{H}^{-n-1}({}^!\mathcal{R}^\circ(g_!\mathbb{D}\,{}^p\mathbb{Q}^H_T))(-n-d). \arrow[ll] \arrow[u, dotted] \end{tikzcd}$$

This gives the desired morphism of mixed Hodge modules between $\mathcal{M}_{\widetilde{A}}^{-\beta'}(-d-n)$ and $\mathcal{M}_{\widetilde{A}}^{\beta}$. The uniqueness follows from [Rei14, Lemma 1.15]. □

3.2 Calculation in charts

Let A be a $d \times n$-integer matrix with columns $(\underline{a}_1, \ldots, \underline{a}_n)$ and assume that $\widetilde{A}$ satisfies $\mathbb{Z}\widetilde{A} = \mathbb{Z}^{d+1}$, $\mathbb{N}\widetilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\widetilde{A}$ and $int(\mathbb{N}\widetilde{A}) = \mathbb{N}\widetilde{A} + c$ for some $c \in \mathbb{N}\widetilde{A}$. Consider the locally closed embedding from above, i.e.,

$$\begin{aligned} g : T &\longrightarrow \mathbb{P}(W) \\ (t_1, \ldots, t_d) &\mapsto (\underline{t}^{\underline{a}_0} : \ldots : \underline{t}^{\underline{a}_n}), \end{aligned}$$

where $T = (\mathbb{C}^*)^d$, $W = \mathbb{C}^{n+1}$ and $\underline{t}^{\underline{a}_i} := \prod_{k=1}^{n} t_k^{a_{ki}}$. Let $(w_0 : \ldots : w_n)$ be the homogeneous coordinates on $\mathbb{P}(W)$ and denote by $j_u : W_u \hookrightarrow \mathbb{P}(W)$ the chart $w_u \neq 0$ with coordinates $w_{iu} := \frac{w_i}{w_u}$ for $i \neq u$. The map g factors over the chart W_u and gives rise to the map

$$\begin{aligned} g_u : T &\longrightarrow W_u \\ (t_1, \ldots, t_n) &\mapsto (\underline{t}^{\underline{a}_0 - \underline{a}_u}, \ldots, \underline{t}^{\underline{a}_n - \underline{a}_u}). \end{aligned}$$

Let $A_u = (a^u_{ki})$ be the $d \times n$-matrix with columns $(\underline{a}_i - \underline{a}_u)$ for $i \in \{0, \ldots, n\} \setminus \{u\}$ Notice that $A_0 = A$.

Lemma 3.4. *The matrices A_u satisfy the following conditions*

1. $\mathbb{Z}A_u = \mathbb{Z}^d$
2. $\mathbb{N}A_u = \mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A_u$
3. $int(\mathbb{N}A_u) = \mathbb{N}A_u + c$ *for some* $c \in \mathbb{N}A_u$

Proof. Denote by $\widetilde{A}_u$ the $(d+1)\times(n+1)$-matrix with columns $(1, \underline{a}_i - \underline{a}_u)$ for $i \in \{0,\dots,n\}$. We will first show the corresponding properties for the matrix $\widetilde{A}_u$. Denote by $C_u \in GL(d+1,\mathbb{Z})$ the matrix

$$C_u := \begin{pmatrix} 1 & & & \\ -a_{1u} & 1 & & \\ \vdots & & \ddots & \\ -a_{du} & & & 1 \end{pmatrix}.$$

Notice that we have $C_u \cdot \widetilde{A} = C_u \cdot \widetilde{A}_0 = \widetilde{A}_u$. Since C_u is a linear map, it is in particular a homeomorphism. Hence $C_u(int(\mathbb{N}\widetilde{A})) = int(\mathbb{N}\widetilde{A}_u)$ and similarly for $\mathbb{Z}\widetilde{A}_u$, $\mathbb{N}\widetilde{A}_u$ and $\mathbb{R}_{\geq 0}\widetilde{A}_u$. Therefore the three properties hold for $\widetilde{A}_u$.

Denote by $p : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$ the projection to the last d-coordinates. Notice that p is an open map and $p(1, \underline{a}_i - \underline{a}_u) = \underline{a}_i - \underline{a}_u$. Now it is easy to show that the three properties also hold for A_u.

□

Denote by $\mathbb{L}_{A_u}$ the $\mathbb{Z}$-module of relations among the columns of A_u. In order to calculate the direct image of $\mathcal{O}_T$ under the map g_u, we use Theorem 2.21 where A_u takes the role of the matrix B in loc.cit.

Proposition 3.5. *Consider the $\mathcal{D}_{W_u}$-module $\check{\mathcal{M}}_{A_u} := \check{\mathcal{M}}^0_{A_u}$ as defined in Definition 2.1, that is, $\check{\mathcal{M}}_{A_u} = \mathcal{D}_{W_u}/\check{\mathcal{I}}_{A_u}$ where the left ideal $\check{\mathcal{I}}_{A_u}$ is generated by*

$$\check{\Box}_{\underline{m}\in\mathbb{L}_{A_u}} = \prod_{i\neq u: m_i>0} w_{iu}^{m_i} - \prod_{i\neq u: m_i<0} w_{iu}^{-m_i}$$

and the Euler vector fields:

$$\begin{aligned} \check{E}^u_k &:= \sum_{i\neq u} a^u_{ki}\partial_{w_{iu}} w_{iu} \\ &= \sum_{i\neq u}(a_{ki}-a_{ku})\partial_{w_{iu}} w_{iu}\,. \end{aligned}$$

Then the direct image $g_{u+}\mathcal{O}_T$ is isomorphic to $\check{\mathcal{M}}_{A_u}$. Moreover, the Hodge filtration on $\check{\mathcal{M}}_{A_u}$ is the order filtration shifted by $(n-d)$, i.e.

$$F^H_{p+(n-d)}\check{\mathcal{M}}_{A_u} = F^{ord}_p\check{\mathcal{M}}_{A_u}\,.$$

Proof. The statement follows from Theorem 2.21 and Lemma 3.4.

□

We now want to compute how the the $\mathcal{D}$-modules $g_{u+}\mathcal{O}_T$ glue on their common domain of definition. Let $u_1, u_2 \in \{0,\dots,n\}$ and denote by $W_{u_1u_2}$ the intersection $W_{u_1}\cap W_{u_2}$. We fix $u_1, u_2 \in \{0,\dots,n\}$ with $u_1 < u_2$. We have the following change of coordinates between the charts W_{u_1} and W_{u_2}

$$w_{iu_1} = w_{iu_2}w^{-1}_{u_1u_2} \quad \text{for } i\neq u_2 \quad \text{and} \quad w_{u_2u_1} = w^{-1}_{u_1u_2}$$

which gives the following transformation rules for vector field:

$$w_{iu_1}\partial_{w_{iu_1}} = w_{iu_2}\partial_{w_{iu_2}} \quad \text{for } i\neq u_2 \quad w_{u_2u_1}\partial_{w_{u_2u_1}} = -\sum_{i\neq u_2} w_{iu_2}\partial_{w_{iu_2}}\,. \tag{26}$$

The module of global sections $\Gamma(W_{u_1u_2}, g_{u_1+}\mathcal{O}_T)$ can be expressed as the quotient $D_{W_{u_1}}[w^{-1}_{u_2u_1}]/\check{I}'_{A_{u_1}}$, where $\check{I}'_{A_{u_1}} \subset D_{W_{u_1}}[w^{-1}_{u_2u_1}] := \mathbb{C}[(w_{iu_1})_{i\neq u_1}][w^{-1}_{u_2u_1}]\otimes_{\mathbb{C}[(w_{iu_1})_{i\neq u_1}]} D_{W_{u_1}}$ is the left ideal generated by

1. $\check{E}^u_k = \sum_{\substack{i=0\\ i\neq u_1}}^n a^{u_1}_{ki}\partial_{w_{iu_1}} w_{iu_1} = \sum_{\substack{i=0\\ i\neq u_1}}^n a^{u_1}_{ki} w_{iu_1}\partial_{w_{iu_1}} + \sum_{\substack{i=0\\ i\neq u_1}}^n a^{u_1}_{ki} \qquad k = 1,\dots,d$
2. $\check{\Box}_{\underline{m}} = \prod_{\substack{m_i>0\\ i\neq u_1}} w^{m_i}_{iu_1} - \prod_{\substack{m_i<0\\ i\neq u_1}} w^{-m_i}_{iu_1} \qquad \underline{m}\in\mathbb{L}_{A_{u_1}}\,.$

The above mentioned transformation rules define an algebra isomorphism

$$\iota_{u_1u_2} : D_{W_{u_1}}[w_{u_2u_1}^{-1}] \longrightarrow D_{W_{u_2}}[w_{u_1u_2}^{-1}].$$

We can now give an explicit expression for the gluing map between the various charts of the module $g_+\mathcal{O}_T$.

Lemma 3.6. *The isomorphism between $g_{u_1+}\mathcal{O}_T$ and $g_{u_2+}\mathcal{O}_T$ on their common domain of definition $W_{u_1u_2} = W_{u_1} \cap W_{u_2}$ is given by*

$$\begin{aligned} \Gamma(W_{u_1u_2}, g_{u_1+}\mathcal{O}_T) \simeq D_{W_{u_1}}[w_{u_2u_1}^{-1}]/\check{I}'_{A_{u_1}} &\longrightarrow D_{W_{u_2}}[w_{u_1u_2}^{-1}]/\check{I}'_{A_{u_2}} \simeq \Gamma(W_{u_1u_2}, g_{u_2+}\mathcal{O}_T) \\ P &\longmapsto \iota_{u_1u_2}(P) w_{u_1u_2}^{n+1}. \end{aligned}$$

Proof. The well-definedness follows from the following calculations:

$$\begin{aligned}
&\iota_{u_1u_2}\left(\sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1} \partial_{w_{iu_1}} w_{iu_1}\right) w_{u_1u_2}^{n+1} \\
&= \iota_{u_1u_2}\left(\sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1} w_{iu_1} \partial_{w_{iu_1}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1}\right) w_{u_1u_2}^{n+1} \\
&= \iota_{u_1u_2}\left(\sum_{\substack{i=0\\ i\neq u_1,u_2}}^{n} a_{ki}^{u_1} w_{iu_1} \partial_{w_{iu_1}} + a_{ku_2}^{u_1} w_{u_2u_1} \partial_{w_{u_2u_1}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1}\right) w_{u_1u_2}^{n+1} \\
&= \iota_{u_1u_2}\left(\sum_{\substack{i=0\\ i\neq u_1,u_2}}^{n} (a_{ki} - a_{ku_1}) w_{iu_1} \partial_{w_{iu_1}} + (a_{ku_2} - a_{ku_1}) w_{u_2u_1} \partial_{w_{u_2u_1}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1}\right) w_{u_1u_2}^{n+1} \\
&= \left(\sum_{\substack{i=0\\ i\neq u_1,u_2}}^{n} (a_{ki} - a_{ku_1}) w_{iu_2} \partial_{w_{iu_2}} - (a_{ku_2} - a_{ku_1}) \sum_{\substack{i=0\\ i\neq u_2}}^{n} w_{iu_2} \partial_{w_{iu_2}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1}\right) w_{u_1u_2}^{n+1} \\
&= \left(\sum_{\substack{i=0\\ i\neq u_2}}^{n} a_{ki}^{u_2} w_{iu_2} \partial_{w_{iu_2}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1}\right) w_{u_1u_2}^{n+1} \\
&= w_{u_1u_2}^{n+1}\left(\sum_{\substack{i=0\\ i\neq u_2}}^{n} a_{ki}^{u_2} w_{iu_2} \partial_{w_{iu_2}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} a_{ki}^{u_1} + (n+1) a_{ku_1}^{u_2}\right) \\
&= w_{u_1u_2}^{n+1}\left(\sum_{\substack{i=0\\ i\neq u_2}}^{n} a_{ki}^{u_2} w_{iu_2} \partial_{w_{iu_2}} + \sum_{\substack{i=0\\ i\neq u_1}}^{n} (a_{ki} - a_{ku_1}) + (n+1)(a_{ku_1} - a_{ku_2})\right) \\
&= w_{u_1u_2}^{n+1}\left(\sum_{\substack{i=0\\ i\neq u_2}}^{n} a_{ki}^{u_2} w_{iu_2} \partial_{w_{iu_2}} + \sum_{\substack{i=0\\ i\neq u_2}}^{n} a_{ki}^{u_2}\right) = w_{u_1u_2}^{n+1}\left(\sum_{\substack{i=0\\ i\neq u_2}}^{n} a_{ki}^{u_2} \partial_{w_{iu_2}} w_{iu_2}\right),
\end{aligned} \tag{27}$$

and, for $\underline{m} \in \mathbb{L}_{A_{u_1}}$ with $m_{u_2} \geq 0$,

$$\begin{aligned}
&\iota_{u_1 u_2}\left(\prod_{\substack{m_i>0\\ i\neq u_1}} w_{iu_1}^{m_i} - \prod_{\substack{m_i<0\\ i\neq u_1}} w_{iu_1}^{-m_i}\right) w_{u_1u_2}^{n+1}\\
&= w_{u_1u_2}^{n+1}\left(\left(\prod_{\substack{m_i>0\\ i\neq u_1,u_2}} w_{iu_2}^{m_i}(w_{u_1u_2})^{-m_i}\right) w_{u_1u_2}^{-m_{u_2}} - \left(\prod_{\substack{m_i<0\\ i\neq u_1,u_2}} w_{iu_2}^{-m_i}(w_{u_1u_2})^{m_i}\right)\right)\\
&= w_{u_1u_2}^{n+1}\left(\prod_{\substack{m_i>0\\ i\neq u_1,u_2}} w_{iu_2}^{m_i}\right) w_{u_1u_2}^{-\sum_{i\neq u_1} m_i} - w_{u_1u_2}^{n+1}\left(\prod_{\substack{m_i<0\\ i\neq u_1,u_2}} w_{iu_2}^{-m_i}\right)\\
&= w_{u_1u_2}^{n+1}\cdot \check{\Box}_{\underline{m}'}
\end{aligned}$$

for some $\underline{m}' \in \mathbb{L}_{A_{u_2}}$. The proof for $m_{u_2} < 0$ is similar.

□

3.3 Tensoring the kernel

Our final aim is the computation of the Hodge filtration on the modules $\mathcal{M}_{\widetilde{A}}^{\beta}$ for sufficiently well chosen parameters β. Recall the definition of the Radon transformation ${}^*\mathcal{R}_c^\circ$. We had the following diagram

$$\begin{tikzcd}
 & U \arrow[dl, "\pi_1^U"'] \arrow[d, hook, "j_U"] \arrow[dr, "\pi_2^U"] & \\
\mathbb{P}(W) & \mathbb{P}(W)\times V \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & V\,,
\end{tikzcd}$$

where $U = \{\sum_{i=0}^n \lambda_i w_i \neq 0\}$ is the complement of the universal hyperplane. The Radon transformation ${}^*\mathcal{R}_c^\circ$ was defined by

$$ {}^*\mathcal{R}_c^\circ(M) = \pi_{2!}^U\,(\pi_1^U)^* M \simeq \pi_{2*}\, j_{U!} j_U^* \pi_1^* M\,, $$

where $M \in D^b(MHM(\mathbb{P}(W))$. Notice that the underlying complex of $\mathcal{D}$-modules is equal to $\pi_{2+}\, j_{U\dagger} j_U^\dagger \pi_1^\dagger M$. As a first step we compute $\mathcal{H}^{n+1}(j_{U\dagger} j_U^\dagger \pi_1^\dagger g_{B+}\mathcal{O}_T)$ together with its Hodge filtration on a chart $W_u \times V$. In order to compute the restriction $\mathcal{H}^{n+1}\left((j_u\times id)^+ j_{U\dagger} j_U^\dagger \pi_1^\dagger g_+\mathcal{O}_T\right)$ consider the following diagram

$$\begin{tikzcd}
 & U_u \arrow[r, "(j_u\times id)^r"] \arrow[d, "j_U^r"] & U \arrow[d, "j_U"] \\
 & W_u\times V \arrow[r, "(j_u\times id)"] \arrow[d, "\pi_1^r"] & \mathbb{P}(W)\times V \arrow[d, "\pi_1"] \\
T \arrow[r, "g_u"] \arrow[rr, bend right, "g"'] & W_u \arrow[r, "j_u"] & \mathbb{P}(W)
\end{tikzcd}$$

where all squares are cartesian. Then the following holds.

Lemma 3.7. *There is an isomorphism in $D^b_{rh}(\mathcal{D}_{W_u\times V})$*

$$ (j_u\times id)^+ j_{U\dagger} j_U^\dagger \pi_1^\dagger \mathcal{H}^0 g_+\mathcal{O}_T \simeq (j_U^r)_\dagger (j_U^r)^\dagger (\pi_1^r)^\dagger \mathcal{H}^0 (g_u)_+\mathcal{O}_T. $$

Moreover, these complexes have only cohomology in degree $n+1$.

Proof. First notice that the functors $\pi_1^\dagger[-n-1]$, $j_U^\dagger$, $(\pi_1^r)^\dagger[-n-1]$ and $(j_U^r)^\dagger$ are exact as the corresponding maps are smooth. Similarly, the exceptional direct images $(j_U)_\dagger$ and $(j_U^r)_\dagger$ are exact because j_U and j_U^r

are affine. Hence the complexes $(j_u\times id)^+ j_{U\dagger}j_U^\dagger\pi_1^\dagger\mathcal{H}^0(g_+\mathcal{O}_T)$ resp. $(j_U^r)_\dagger(j_U^r)^\dagger(\pi_1^r)^\dagger\mathcal{H}^0((g_u)_+\mathcal{O}_T)$ have only cohomology in degree $n+1$. In order to show that they are the same, we consider the following isomorphisms in $D^b_{rh}(\mathcal{D}_{W_u\times V})$

$$\begin{aligned}
(j_u\times id)^+(j_U)_\dagger j_U^\dagger\pi_1^\dagger\mathcal{H}^0 g_+\mathcal{O}_T &\simeq (j_u\times id)^\dagger(j_U)_\dagger j_U^\dagger\pi_1^\dagger\mathcal{H}^0 g_+\mathcal{O}_T \\
&\simeq (j_U^r)_\dagger((j_u\times id)^r)^\dagger j_U^\dagger\pi_1^\dagger\mathcal{H}^0 g_+\mathcal{O}_T \qquad \text{base change}\\
&\simeq (j_U^r)_\dagger(j_U^r)^\dagger(j_u\times id)^\dagger\pi_1^\dagger\mathcal{H}^0 g_+\mathcal{O}_T \qquad j_U\circ(j_u\times id)^r=(j_u\times id)\circ j_U^r\\
&\simeq (j_U^r)_\dagger(j_U^r)^\dagger(\pi_1^r)^\dagger j_u^\dagger\mathcal{H}^0 g_+\mathcal{O}_T \qquad \text{base change}\\
&\simeq (j_U^r)_\dagger(j_U^r)^\dagger(\pi_1^r)^\dagger j_u^\dagger j_{u+}\mathcal{H}^0 g_{u+}\mathcal{O}_T \qquad g=j_u\circ g_u,\ j_{u+}\ \text{exact}\\
&\simeq (j_U^r)_\dagger(j_U^r)^\dagger(\pi_1^r)^\dagger\mathcal{H}^0(g_u)_+\mathcal{O}_T \qquad j_u^\dagger\circ j_{u+}\simeq j_u^+\circ j_{u+}\simeq id\,.
\end{aligned}$$

□

The subvariety U_u in $W_u\times V$ is given by

$$U_u=\{\lambda_u+\sum_{\substack{i=0\\ i\neq u}}^n\lambda_i w_i\neq 0\}\,.$$

Consider the following change of coordinates

$$\widetilde{\lambda}_u=\lambda_u+\sum_{\substack{j=0\\ j\neq u}}^n\lambda_j w_{ju}\quad,\quad\widetilde{\lambda}_i=\lambda_i\quad\text{and}\quad\widetilde{w}_{iu}=w_{iu}\tag{28}$$

for $i=0,\ldots,n$ and $i\neq u$. Using these coordinates we can identify U_u with $W_u\times\mathbb{C}^n\times\mathbb{C}^*$ where we have the coordinates $(\widetilde{\lambda}_i)_{i\neq u}$ on $\mathbb{C}^n$ and the coordinate $\widetilde{\lambda}_u$ on $\mathbb{C}^*$. The map $(\pi_1^r\circ j_U^r)$ is then simply given by the projection to the first factor. The exceptional inverse image $(j_U^r)^\dagger(\pi_1^r)^\dagger(g_u)_+\mathcal{O}_T$ of $(g_u)_+\mathcal{O}_T$ is then given by $(g_u)_+\mathcal{O}_T\boxtimes\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}[-n-1]$, where $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}$ carries the Hodge filtration $Gr_i^F\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}=0$ for $i\neq 0$.
Notice that we have the following isomorphisms

$$\mathcal{H}^{n+1}((g_u)_+\mathcal{O}_T\boxtimes\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}[-n-1])\simeq\mathcal{H}^0((g_u)_+\mathcal{O}_T)\boxtimes\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}\simeq\check{\mathcal{M}}_{A_u}\boxtimes\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\Big((\partial_{\widetilde{\lambda}_j})_{j\neq u},(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\Big)\,,\tag{29}$$

where we have used the isomorphisms $\mathcal{H}^0((g_u)_+\mathcal{O}_T)\simeq\check{\mathcal{M}}_{A_u}$ and

$$\begin{aligned}
\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\Big((\partial_{\widetilde{\lambda}_j})_{j\neq u},(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\Big)&\longrightarrow\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}\\
1&\mapsto\widetilde{\lambda}_u^{-1},.
\end{aligned}$$

Hence we can formulate the following result.

Lemma 3.8. *We have the following isomorphism of $\mathcal{D}_{U_u}$-modules:*

$$\mathcal{H}^{n+1}((j_U^r)^\dagger(\pi_1^r)^\dagger(g_u)_+\mathcal{O}_T)\simeq\mathcal{H}^{n+1}((g_u)_+\mathcal{O}_T\boxtimes\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*}[-n-1])\simeq\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}_u^*,\tag{30}$$

where $\widetilde{\mathcal{K}}_u^*$ is the left $\mathcal{D}_{U_u}$-ideal generated by $(\check{E}_k^u)_{k=1,\ldots,d}$, $(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{A_u}}$, $(\partial_{\widetilde{\lambda}_j})_{j\neq u}$ and $(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)$.
The Hodge filtration on this module, underlying the mixed Hodge module

$$\mathcal{H}^{n+1}((j_U^r)^*(\pi_1^r)^*(g_u)_*{}^p\mathbb{Q}_T^H)\simeq\mathcal{H}^{n+1}((g_u)_*{}^p\mathbb{Q}_T^H\boxtimes\mathbb{Q}^H_{\mathbb{C}^n\times\mathbb{C}^*})$$

is the order filtration shifted by $n-d$, that is,

$$F^H_{p+(n-d)}\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}_u^*\simeq F_p^{ord}\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}_u^*\,.\tag{31}$$

Proof. We only have to show formula (31), since formula (30) follows directly from formula (29). Since the equality $\partial^n_{\widetilde{\lambda}_u} = (-1)^n\, n!\, \widetilde{\lambda}_u^{-n}$ holds in $\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}, (\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)$ we easily see that the Hodge filtration on this cyclic $\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}$-module, underlying the mixed Hodge module $\mathcal{H}^{n+1}(\mathbb{Q}^H_{\mathbb{C}^n\times\mathbb{C}^*})$, is equal to the order filtration, i.e.

$$F^H_\bullet \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^*} \simeq F^{ord}_\bullet \mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}, (\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right).$$

Using the following isomorphisms

$$\begin{aligned}
F^H_{p+(n-d)}\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}^*_u &\simeq F^H_{p+(n-d)}\left(\check{\mathcal{M}}_{A_u}\boxtimes \mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}, (\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)\right)\\
&\simeq \bigoplus_{p+(n-d)=q+r} F^H_q\check{\mathcal{M}}_{A_u}\boxtimes F^H_r\left(\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}, (\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)\right)\\
&\simeq \bigoplus_{p+(n-d)=q+r} F^{ord}_{q-(n-d)}\check{\mathcal{M}}_{A_u}\boxtimes F^{ord}_r\left(\mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}, (\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)\right)\\
&\simeq F^{ord}_p\left(\check{\mathcal{M}}_{A_u}\boxtimes \mathcal{D}_{\mathbb{C}^n\times\mathbb{C}^*}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}, (\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)\right)\\
&\simeq F^{ord}_p\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}^*_u,
\end{aligned}$$

this shows that the Hodge filtration on $\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}^*_u$ is the order filtration shifted by $(n-d)$. □

The next step is to compute the proper direct image under the open embedding $j^r_U : U_u \to W_u \times V$ of the module $\mathcal{D}_U/\widetilde{\mathcal{K}}^*_u$ together with its Hodge filtration. The map j^r_U is simply given by the inclusion

$$(id_{W_u}\times id_{\mathbb{C}^n}\times j) : W_u\times\mathbb{C}^n\times\mathbb{C}^*\to W_u\times\mathbb{C}^n\times\mathbb{C} = W_u\times V\,.$$

where $j:\mathbb{C}^*\to\mathbb{C}$ is the canonical inclusion. Notice that the functor $(j^r_u)_\dagger = (id_{W_u}\times id_{\mathbb{C}^n}\times j)_\dagger$ is exact, since j^r_U is an affine embedding. Therefore the direct image is given by

$$\begin{aligned}
\mathcal{H}^0(j^r_u)_\dagger\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}^*_u &\simeq \mathcal{H}^0(id_{W_u}\times id_{\mathbb{C}^n}\times j)_\dagger\left(\check{\mathcal{M}}_{A_u}\boxtimes\mathcal{D}_{\mathbb{C}^n}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}\right)\boxtimes\mathcal{D}_{\mathbb{C}^*}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)\\
&\simeq \check{\mathcal{M}}_{A_u}\boxtimes\mathcal{D}_{\mathbb{C}^n}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}\right)\boxtimes\mathcal{H}^0 j_\dagger\left(\mathcal{D}_{\mathbb{C}^*}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right)\\
&\simeq \check{\mathcal{M}}_{A_u}\boxtimes\mathcal{D}_{\mathbb{C}^n}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}\right)\boxtimes\mathcal{D}_{\mathbb{C}}/(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u})\,.
\end{aligned}$$

The last line follows from the fact that $\mathcal{D}_{\mathbb{C}^*}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\simeq\mathcal{O}_{\mathbb{C}^*}$ and that

$$\mathcal{D}_{\mathbb{C}} \overset{\cdot\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u}{\longrightarrow} \mathcal{D}_{\mathbb{C}} \tag{32}$$

is a free resolution of $\mathcal{D}_{\mathbb{C}}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)$. Applying $\mathcal{H}om(-,\mathcal{D}_{\mathbb{C}})$ and a right-left transformation yields

$$\mathcal{D}_{\mathbb{C}} \overset{\cdot(-\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u})}{\longrightarrow} \mathcal{D}_{\mathbb{C}}\,,$$

which gives the desired result.

Lemma 3.9. *We have the following isomorphisms of $\mathcal{D}_{W_u\times V}$-modules*

$$\mathcal{H}^{n+1}\left((j^r_U)_\dagger(j^r_U)^\dagger(\pi^r_1)^\dagger(g_u)_+\mathcal{O}_T\right)\simeq\mathcal{H}^0(j^r_U)_\dagger\left(\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}^*_u\right)\simeq\mathcal{D}_{W_u\times V}/\widetilde{\mathcal{K}}_u\,,$$

where $\widetilde{\mathcal{K}}_u$ is the left ideal in $\mathcal{D}_{W_u\times V}$ generated by $(\check{E}^u_k)_{k=1,\ldots,d}$, $(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{A_u}}$, $(\partial_{\widetilde{\lambda}_j})_{j\neq u}$ and $(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u})$. Moreover, the Hodge filtration on this $\mathcal{D}$-module underlying the mixed Hodge module

$$(j^r_U)_!(\pi^r_1)^*(g_u)_*{}^p\mathbb{Q}^H_T$$

is the order filtration, shifted by $n-d$, that is,

$$F^H_{p+(n-d)}(\mathcal{H}^0(j^r_u)_\dagger\mathcal{D}_{U_u}/\widetilde{\mathcal{K}}^*_u)\simeq F^{ord}_p\mathcal{D}_{W_u\times V}/\widetilde{\mathcal{K}}_u\,.$$

Proof. The first statement has already been shown above. In order to compute the Hodge filtration on $\mathcal{H}^0(j_U^r)_\dagger \mathcal{D}_{U_u}/\widetilde{\mathcal{K}}_u^*$, we first have to compute the Hodge filtration on

$$\mathcal{H}^0 j_\dagger \left(\mathcal{D}_{\mathbb{C}^*}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u)\right) \simeq \mathcal{H}^1 j_\dagger \mathcal{O}_{\mathbb{C}^*}[-1] \simeq \mathcal{H}^1 \mathbb{D} j_+ \mathbb{D}\mathcal{O}_{\mathbb{C}^*}[-1]$$

underlying the mixed Hodge module

$$\mathcal{H}^1(j_! \mathbb{Q}^H_{\mathbb{C}^*}) \simeq \mathcal{H}^1(\mathbb{D} j_* \mathbb{D}\mathbb{Q}^H_{\mathbb{C}^*}).$$

First recall that we have $Gr_i^{F^H}\mathcal{H}^1(\mathcal{O}_{\mathbb{C}^*}[-1]) = 0$ for $i \neq 0$. The filtration on $\mathcal{H}^{-1}\mathbb{D}(\mathcal{O}_{\mathbb{C}^*}[-1]) \simeq \mathcal{H}^{-1}\mathcal{O}_{\mathbb{C}^*}[1]$ is given by $Gr_i^{F^H}\mathcal{H}^{-1}\mathbb{D}(\mathcal{O}_{\mathbb{C}^*}[-1]) = 0$ for $i \neq 1$ since $\mathbb{D}\mathbb{Q}^H_{\mathbb{C}^*} \simeq \mathbb{Q}^H_{\mathbb{C}^*}(1)[2]$. By [Sai93, Corollary 4.3] the underlying $\mathcal{D}$-module of $\mathcal{H}^1 j_* \mathbb{Q}^H_{\mathbb{C}^*}$ is isomorphic to $\mathcal{O}_{\mathbb{C}}(*0)$ where the Hodge filtration is equal to the pole-order filtration. We get the following filtered isomorphism

$$\begin{aligned} DMod(\mathcal{H}^1 j_* \mathbb{Q}^H_{\mathbb{C}^*}) \simeq (\mathcal{D}_{\mathbb{C}}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u), F^{ord}_\bullet) &\longrightarrow (\mathcal{O}_{\mathbb{C}}(*0), P_\bullet) \\ 1 &\mapsto \widetilde{\lambda}_u^{-1}, \end{aligned}$$

where $P_\bullet$ is the pole order filtration on $\mathcal{O}_{\mathbb{C}}(*0)$ and we have used that $\partial^n_{\widetilde{\lambda}_u} \mapsto (-1)^n\, n!\, \widetilde{\lambda}_u^{-n-1}$.
Hence the underlying filtered $\mathcal{D}$-module of $\mathcal{H}^{-1} j_* \mathbb{D}\mathbb{Q}^H_{\mathbb{C}^*}$ is isomorphic to $(\mathcal{D}_{\mathbb{C}}/(\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u), F^{ord}_{\bullet-1})$.
In order to compute the Hodge filtration on $\mathcal{H}^1\mathbb{D} j_+ \mathbb{D}\mathcal{O}_{\mathbb{C}^*}[-1] \simeq \mathcal{D}_{\mathbb{C}}/(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u})$ we remark that the resolution (32) gives rise to a strictly filtered resolution

$$(\mathcal{D}_{\mathbb{C}}, F^{ord}_{\bullet-2}) \xrightarrow{\cdot\partial_{\widetilde{\lambda}_u}\widetilde{\lambda}_u} (\mathcal{D}_{\mathbb{C}}, F^{ord}_{\bullet-1}).$$

Applying $\mathcal{H}om(-, (\mathcal{D}_{\mathbb{C}}, F^{ord}_{\bullet-2}) \otimes \omega_X^\vee)$ (cf. [Sai94, page 55] for the choice of filtration on $\mathcal{D}_{\mathbb{C}} \otimes \omega_X^\vee$) one easily sees that

$$(\mathcal{H}^1\mathbb{D} j_+\mathbb{D}\mathcal{O}_{\mathbb{C}^*}[-1], F^H_\bullet) \simeq (\mathcal{D}_{\mathbb{C}}/(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u}), F^{ord}_\bullet).$$

Using the formula

$$\begin{aligned} F^H_{p+(n-d+)}(\mathcal{H}^0(j_u^r)_\dagger \mathcal{D}_{U_u}/\widetilde{\mathcal{K}}_u^*) &\simeq \bigoplus_{p+(n-d)=q+r+s} F^H_q \check{\mathcal{M}}_{A_u} \boxtimes F^H_r \mathcal{D}_{\mathbb{C}^n}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}\right) \boxtimes F^H_s \mathcal{D}_{\mathbb{C}}/(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u}) \\ &= \bigoplus_{p+(n-d)=q+r+s} F^{ord}_{q-(n-d)} \check{\mathcal{M}}_{A_u} \boxtimes F^{ord}_r \mathcal{D}_{\mathbb{C}^n}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}\right) \boxtimes F^{ord}_s \mathcal{D}_{\mathbb{C}}/(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u}) \\ &= F^{ord}_p\left(\check{\mathcal{M}}_{A_u} \boxtimes \mathcal{D}_{\mathbb{C}^n}/\left((\partial_{\widetilde{\lambda}_j})_{j\neq u}\right) \boxtimes \mathcal{D}_{\mathbb{C}}/(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u})\right) \end{aligned}$$

we see that

$$F^H_{p+(n-d)}(\mathcal{H}^0(j_u^r)_\dagger \mathcal{D}_{U_u}/\widetilde{\mathcal{K}}_u^*) \simeq F^{ord}_p \mathcal{D}_{W_u\times V}/\widetilde{\mathcal{K}}_u.$$

□

The final step in this section is to compute a presentation of

$$\mathcal{N} := \mathcal{H}^{n+1}(j_U)_\dagger j_U^\dagger \pi_1^\dagger g_{B+}\mathcal{O}_T \tag{33}$$

in each chart $W_u \times V$ of $\mathbb{P}(W) \times V$. Recall from lemma 3.7 that the restriction

$$\mathcal{N}_{|W_u\times V} = \mathcal{H}^{n+1}(j_u \times id)^+(j_U)_\dagger j_U^\dagger \pi_1^\dagger g_{B+}\mathcal{O}_T$$

is given by the module $\mathcal{N}_u := \mathcal{H}^{n+1}(j_U^r)_\dagger (j_U^r)^\dagger (\pi_1^r)^\dagger (g_u)_+\mathcal{O}_T$.

Proposition 3.10. *Consider the original coordinates* $((w_{iu})_{i\neq u},(\lambda_0,\ldots,\lambda_n))$ *of* $W_u\times V$*. Then there is an isomorphism of* $\mathcal{D}_{W_u\times V}$*-modules* $\mathcal{N}_u\simeq\mathcal{D}_{W_u\times V}/\mathcal{K}_u$*, where* $\mathcal{K}_u$ *is the left* $\mathcal{D}_{W_u\times V}$*-ideal generated by the following classes of operators*

1. $\displaystyle\sum_{\substack{i=0\\ i\neq u}}^{n} a^u_{ki}\partial_{w_{iu}}w_{iu}-\sum_{i=1}^{n}a_{ki}\lambda_i\partial_{\lambda_i}$
2. $\displaystyle\check{\Box}_{\underline{m}}=\prod_{\substack{m_i>0\\ i\neq u}}w_{iu}^{m_i}-\prod_{\substack{m_i<0\\ i\neq u}}w_{iu}^{-m_i}\qquad \underline{m}\in\mathbb{L}_{A_u}$
3. $\partial_{\lambda_i}-w_{iu}\partial_{\lambda_u}\qquad$ *for* $\quad i=0,\ldots,n$ *and* $i\neq u$
4. $\displaystyle\sum_{j=0}^{n}\lambda_j\partial_{\lambda_j}\,.$

Moreover, we have

$$F^H_{p+(n-d)}\mathcal{N}_u\simeq F^{ord}_p\mathcal{D}_{W_u\times V}/\mathcal{K}_u\,.$$

Proof. Recall that $\mathcal{N}_u=\mathcal{D}_{W_u\times V}/\widetilde{\mathcal{K}}_u$, where $\widetilde{\mathcal{K}}_u=(\check{E}^u_k)_{k=1,\ldots,d}+(\check{\Box}_{\underline{m}})_{\underline{m}\in\mathbb{L}_{A_u}}+(\partial_{\widetilde{\lambda}_j})_{j\neq u}+(\widetilde{\lambda}_u\partial_{\widetilde{\lambda}_u})$. Using the coordinate transformation (28) we see that $\widetilde{\mathcal{K}}_u$ is transformed into the ideal $\mathcal{K}_u$ generated by the operators

$$\overset{(\vee)}{E}{}^u_k=\sum_{\substack{i=0\\ i\neq u}}^{n}a^u_{ki}(\partial_{w_{iu}}-\lambda_i\partial_{\lambda_u})w_{iu}\qquad k=1,\ldots,d$$

$$\check{\Box}_{\underline{m}}=\prod_{\substack{m_i>0\\ i\neq u}}w_{iu}^{m_i}-\prod_{\substack{m_i<0\\ i\neq u}}w_{iu}^{-m_i}\qquad \underline{m}\in\mathbb{L}_{A_u}$$

$$\partial_{\lambda_i}-w_{iu}\partial_{\lambda_u}\qquad \text{for}\quad i=0,\ldots,n\text{ and }i\neq u$$

$$(\lambda_u+\sum_{\substack{j=0\\ j\neq u}}^{n}\lambda_j w_{ju})\partial_{\lambda_u}\,.$$

The last operator can be rewritten (using the relations $\partial_{\lambda_i}-w_{iu}\partial_{\lambda_u}$, i.e., the third class of operators)

$$\sum_{j=0}^{n}\lambda_j\partial_{\lambda_j}\equiv(\lambda_u+\sum_{\substack{j=0\\ j\neq u}}^{n}\lambda_j w_{ju})\partial_{\lambda_u}\,.$$

The operators $\overset{(\vee)}{E}{}^u_k$ can be further simplified by writing

$$\begin{aligned}\sum_{\substack{i=0\\ i\neq u}}^{n}a^u_{ki}(\partial_{w_{iu}}w_{iu}-\lambda_i\partial_{\lambda_i})&=\sum_{\substack{i=0\\ i\neq u}}^{n}a^u_{ki}\partial_{w_{iu}}w_{iu}-\sum_{i=1}^{n}a_{ki}\lambda_i\partial_{\lambda_i}+a_{ku}\sum_{i=0}^{n}\lambda_i\partial_{\lambda_i}\\&\equiv\sum_{\substack{i=0\\ i\neq u}}^{n}a^u_{ki}\partial_{w_{iu}}w_{iu}-\sum_{i=1}^{n}a_{ki}\lambda_i\partial_{\lambda_i}\,,\end{aligned}$$

where the last equivalence follows by using the relation $\sum_{j=0}^{n}\lambda_j\partial_{\lambda_j}\equiv(\lambda_u+\sum_{\substack{j=0\\ j\neq u}}^{n}\lambda_j w_{ju})\partial_{\lambda_u}$ from above. Hence we obtain the presentation $\mathcal{N}_u\simeq\mathcal{D}_{W_u\times V}/\mathcal{K}_u$, and the statement on the Hodge filtration follows directly from Lemma 3.9. $\square$

3.4 A Koszul complex

In this section, we will construct a strict resolution of the filtered module $(\mathcal{N}_u, F^H)$. For this purpose, we first describe an alternative presentation of the ideal $\mathcal{K}_u \subset \mathcal{D}_{W_u \times v}$. Let A^s_u be the $(d+1)\times(2n+1)$-matrix with columns $(0,\underline{a}_0-\underline{a}_u),\ldots,\widehat{(0,\underline{a}_u-\underline{a}_u)},\ldots,(0,\underline{a}_n-\underline{a}_u),(1,\underline{a}_0),\ldots,(1,\underline{a}_n)$ (here the symbol $\widehat{\ }$ means that the zero column $(0,\underline{a}_u-\underline{a}_u)$ is omitted). In other words, we have

$$A^s_u = \left(\begin{array}{ccc|c|ccc} 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ \hline & & & 0 & & & \\ & A_u & & \vdots & & A & \\ & & & 0 & & & \end{array}\right).$$

Consider the GKZ system $\mathcal{M}_{A^s_u}$ on $\hat{W}_u \times V$ with coordinates $(\hat{w}_{iu})_{i\neq u}, \lambda_0,\ldots,\lambda_n$. Let $FL_{\hat{W}_u}$ be the partial Fourier-Laplace transformation which interchanges $\partial_{\hat{w}_{iu}}$ with $(w_{iu})_{i\neq u}$ and $\hat{w}_{iu}$ with $-\partial_{w_{iu}}$.

Lemma 3.11. *Let $\overset{(\vee)}{\mathcal{I}_u}$ be the left $\mathcal{D}_{W_u\times V}$ ideal generated by the operators*

$$\overset{(\vee)}{\Box}_{(\underline{m},\underline{l})} := \prod_{\substack{m_i>0\\ i\neq u}} w_{iu}^{m_i} \prod_{\substack{l_i>0\\ 0\leq i\leq n}} \partial_{\lambda_i}^{l_i} - \prod_{\substack{m_i<0\\ i\neq u}} w_{iu}^{-m_i} \prod_{\substack{l_i>0\\ 0\leq i\leq n}} \partial_{\lambda_i}^{-l_i},$$

where $(\underline{m},\underline{l}) = ((m_i)_{i\neq u}, l_0,\ldots,l_n) \in \mathbb{L}_{A^s_u}$,

$$\overset{(\vee)}{E^u_k} := -\sum_{\substack{i=0\\ i\neq u}}^n a^u_{ki}\partial_{w_{iu}} w_{iu} + \sum_{i=1}^n a_{ki}\lambda_i\partial_{\lambda_i} \qquad \textit{for} \quad k=1,\ldots,d$$

and

$$\overset{(\vee)}{E^u_0} := \sum_{i=0}^n \lambda_i\partial_{\lambda_i}.$$

Then we have $\overset{(\vee)}{\mathcal{I}_u} = \mathcal{K}_u$, and hence the $\mathcal{D}_{W_u\times V}$-module $\mathcal{N}_u$ is isomorphic to $\mathcal{D}_{W_u\times V}/\overset{(\vee)}{I_u}$, in other words, we have an isomorphism

$$\mathcal{N}_u \simeq FL_{\hat{W}_u}\mathcal{M}_{A^s_u}.$$

Proof. For the first statement, notice that $\overset{(\vee)}{\Box}_{(\underline{m},0)}$ equals the operator $\check{\Box}_{\underline{m}}$ from the definition of the ideal $\mathcal{K}_u$. On the other hand, one can obtain all operators $\overset{(\vee)}{\Box}_{(\underline{m},\underline{l})}$ from the operators $\check{\Box}_{\underline{m}'}$ using the relations $\partial_{\lambda_i} - w_{iu}\partial_{\lambda_u}$. The last statement follows by exchanging $\partial_{w_{iu}}$ with $-\hat{w}_{iu}$ and w_{iu} with $\partial_{\hat{w}_{iu}}$ in the classes of operators of type 1., 2., 3. and 4. in the definition of the ideal $\mathcal{K}_u$.

□

Now we will construct a Koszul-type resolution of $\mathcal{N}_u$. It is related (though not equal) to the Euler-Koszul complex of GKZ-systems (see [MMW05]). We work on the level of global sections. Let $J_{A^s_u}$ be the ideal in $D_{W_u\times V}$ generated by the box operators $\overset{(\vee)}{\Box}_{(\underline{m},\underline{l})}$ for $(\underline{m},\underline{l}) \in \mathbb{L}_{A^s_u}$.
A simple computation, which uses the fact that $\sum_{i\neq u} m_i a^u_{ki} + \sum_{i=0}^n l_i a_{ki} = 0$, shows that the maps

$$\begin{aligned} D_{W_u\times V}/J_{A^s_u} &\longrightarrow D_{W_u\times V}/J_{A^s_u} \\ P &\mapsto P\cdot \check{E}^u_k \qquad \text{for} \quad k=0,\ldots,d \end{aligned} \tag{34}$$

are well-defined. Since $[\check{E}^u_{k_1}, \check{E}^u_{k_2}] = 0$ for $k_1,k_2 \in \{0,\ldots,d\}$ we can build a Koszul complex

$$K^\bullet_u = (\ldots \to K^1_u \to K^0_u \to 0) := Kos(D_{W_u\times V}/J_{A^s_u}, (\check{E}^u_k)_{k=0,\ldots,d}),$$

where the terms K^l_u are given by

$$K^l_u = \bigoplus_{0\leq i_1<\ldots<i_p\leq d} D_{W_u\times V}/J_{A^s_u}.$$

Define a filtration $\{F_\bullet K_u^\bullet\}$ of the complex $K_u^\bullet$ by

$$F_p K_u^l := \bigoplus_{0\leq i_1<\ldots<i_p\leq d} F^{ord}_{p-l+(n-d)} D_{W_u\times V}/J_{A_u^s}$$

which makes $K_u^\bullet$ into a filtered complex.

Proposition 3.12. *The Koszul complex $(K_u^\bullet, F_\bullet)$ is a resolution of $(N_u, F_\bullet^H)$ in the category of filtered $D_{W_u\times V}$-modules, i.e., we have a quasi-isomorphism $K_u^\bullet \twoheadrightarrow N_u$ and the complex $K_u^\bullet$ is strictly filtered by $F_\bullet$.*

Proof. Let FL_{W_u} be the partial Fourier-Laplace transformation defined on $D_{W_u\times V}$-modules which interchanges w_{iu} with $\partial_{\hat{w}_{iu}}$ and $\partial_{w_{iu}}$ with $-\hat{w}_{iu}$. By Lemma 3.11 the Fourier-Laplace transform $\mathrm{FL}_{W_u\times V} N_u$ is isomorphic to $M_{A_u^s}$. Consider the Fourier-Laplace transform

$$\hat{K}_u^\bullet := FL_{W_u} K_u^\bullet = Kos(D_{\hat{W}_u\times V}/\hat{J}_{A_u^s}, (\hat{E}_k^u)_{k=0,\ldots,d})\,,$$

where $\hat{K}_u^l$ is given by

$$\hat{K}_u^l = \bigoplus_{0\leq i_1<\ldots<i_l\leq d} D_{\hat{W}_u\times V}/\hat{J}_{A_u^s}\,,$$

where the left ideal $\hat{J}_{A_u^s}\subset D_{\hat{W}_u\times V}$ is generated by

$$\hat{\Box}_{(\underline{m},\underline{l})} := \prod_{\substack{m_i>0\\ i\neq u}} \partial_{\hat{w}_{iu}}^{m_i} \prod_{\substack{l_i>0\\ 0\leq i\leq n}} \partial_{\lambda_i}^{l_i} - \prod_{\substack{m_i<0\\ i\neq u}} \partial_{\hat{w}_{iu}}^{-m_i} \prod_{\substack{l_i<0\\ 0\leq i\leq n}} \partial_{\lambda_i}^{-l_i}$$

together with the Euler operators

$$\hat{E}_k^u := \sum_{i\neq u} a_{ki}^u \hat{w}_{iu}\partial_{\hat{w}_{iu}} + \sum_{i=1}^n a_{ki}\lambda_i\partial_{\lambda_i} \qquad \text{for} \quad k=1,\ldots,d$$

and

$$\hat{E}_0^u := \sum_{i=0}^n \lambda_i\partial_{\lambda_i}\,.$$

Let $F_\bullet^{\hat{\omega}} D_{\hat{W}_u\times V}$ the filtration on $D_{\hat{W}_u\times V}$ corresponding to the weight vector

$$\begin{aligned}\hat{\omega} &= ((weight(\hat{w}_{iu}))_{i\neq u}, (weight(\partial_{\hat{w}_{iu}}))_{i\neq u}, weight(\lambda_1),\ldots, weight(\lambda_n), weight(\partial_{\lambda_1}),\ldots, weight(\partial_{\lambda_n}))\\ &= (1,\ldots,1,0,\ldots,0,0,\ldots,0,1,\ldots,1)\,.\end{aligned}$$

Notice that this filtration corresponds to the order filtration $F_\bullet^{ord} D_{W_u\times V}$ under the Fourier-Laplace transform FL_{W_u}. If we endow the complex $\hat{K}_u^\bullet$ with the filtration

$$F_p\hat{K}_u^l := \bigoplus_{0\leq i_1<\ldots<i_l\leq d} F^{\hat{\omega}}_{p-l} D_{\hat{W}_u\times V}/\hat{J}_{A_u^s}$$

one easily sees that $(K_u^\bullet, F_\bullet)$ is a strict resolution of $(N_u, F_\bullet^H)$ iff $(\hat{K}_u^\bullet, F_\bullet)$ is a strict resolution of $(M_{A_u^s}, F_\bullet^{\hat{\omega}})$.

Notice that the complex $\hat{K}_u^\bullet$ is just the Euler-Koszul complex $\mathcal{K}_\bullet(E;\mathbb{C}[\mathbb{N}A_u^s])$ of [MMW05, Definition 4.2]. It follows that this Euler-Koszul complex is a resolution of $M_{A_u^s}$ by Definition 4.10, Remark 6.4 of loc. cit. and the fact that $\mathbb{C}[\mathbb{N}A_u^s]$ is normal, hence Cohen-Macaulay.

It remains to prove that the filtered complex $(\hat{K}_u^\bullet, F_\bullet)$ is strict. By Lemma 3.13 it is enough to show that $H_i(gr_\bullet^F \hat{K}_u) = 0$ for $i\geq 1$ and $H_0(gr_\bullet^F \hat{K}_u)\simeq gr_\bullet^{F^{\hat{\omega}}} M_{A_u^s}$. Denote by $GD_{\hat{W}_u\times V} = gr_\bullet^{\hat{\omega}} D_{\hat{W}_u\times V}$ the associated graded object of $D_{\hat{W}_u\times V}$, by $(\hat{v}_{iu})_{i\neq u}$ the symbol of $(\partial_{\hat{w}_{iu}})_{i\neq u}$ and by μ_j the symbol of ∂_{λ_j} in $G\hat{D}_u$. Since $\hat{\Box}_{(k,l)}$ is homogeneous in (∂_{λ_j}) we have

$$gr^{\hat{\omega}}(D_{\hat{W}_u\times V}/D_{\hat{W}_u\times V}\hat{J}_{A_u^s}) = G\hat{D}_u/\hat{J}^g_{A_u^s}\,,$$

where $\hat{J}^g_{A^s_u}$ is generated by

$$^g\hat{\Box}_{(\underline{m},\underline{l})} := \prod_{\substack{m_i>0\\ i\neq u}} \hat{v}_{iu}^{m_i} \prod_{\substack{l_i>0\\ 0\leq i\leq n}} \mu_i^{l_i} - \prod_{\substack{m_i<0\\ i\neq u}} \hat{v}_{iu}^{-k_i} \prod_{\substack{l_i>0\\ 0\leq i\leq n}} \mu_i^{-l_i}\,.$$

Notice that

$$GD_{\hat{W}_u\times V}/\hat{J}^g_{A^s_u} \simeq \mathbb{C}[(\hat{w}_{iu})_{i\neq u}, \lambda_0, \ldots, \lambda_n] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{N}A^s_u]\,.$$

The associated graded complex $gr^F\hat{K}_u$ is isomorphic to a Koszul complex

$$gr^F\hat{K}_u \simeq Kos(GD_{\hat{W}_u\times V}/\hat{J}^g_{A^s_u}, ({}^g\hat{E}^u_k)_{k=0,\ldots,d})\,,$$

where ${}^g\hat{E}^u_k$ is defined by

$${}^g\hat{E}^u_k := \sum_{i\neq u} a^u_{ki}\hat{w}_{iu}\hat{v}_{iu} + \sum_{i=1}^n a_{ki}\lambda_i\mu_i \qquad \text{for} \quad k=1,\ldots,d$$

and

$${}^g\hat{E}^u_0 := \sum_{i=0}^n \lambda_i\mu_i\,.$$

It follows from the proof of [Ado94, Theorem 3.9] that the ${}^g\hat{E}^u_k$ are part of a system of parameters. Since $\mathbb{N}A^s_u$ is a normal semigroup, the ring $G\hat{D}_u/\hat{J}^g_{A^s_u}$ is Cohen-Macaulay. Hence $({}^g\hat{E}^u_k)_{k=0,\ldots,d}$ is a regular sequence in $GD_{\hat{W}_u\times V}/\hat{J}^g_{A^s_u}$. This shows $H_i(gr^F_\bullet\hat{K}_u) = 0$. From [SST00, Theorem 4.3.5] follows that $H_0(gr^F_\bullet\hat{K}_u) = GD_{\hat{W}_u\times V}/(\hat{J}^g_{A^s_u} + ({}^g\hat{E}^u_k)_{k=0,\ldots,d}) \simeq gr^{F^{\hat{\omega}}}M_{A^s_u}$ but this shows the strictness of $(\hat{K}^\bullet_u, F_\bullet)$ and therefore the claim. □

Lemma 3.13. *Let*

$$0 \longrightarrow (M_1,F), \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} (M_n,F) \to 0$$

be a sequence of filtered D-modules. The following is equivalent

1. *The map d_k is strict.*
2. *$H^k(F_lM_\bullet) \simeq F_lH^k(M_\bullet)$ for all l.*
3. *$H^k(gr^F_lM_\bullet) \simeq gr^F_lH^k(M_\bullet)$ for all l.*

Proof. First recall that the map d_k is strict iff $F_l \operatorname{im} d_k = F_lM_k \cap \operatorname{im} d_k = F_l \ker d_k \cap \operatorname{im} d_k$ is equal to $d(F_lM_{k-1})$. The two commutative squares

$$\begin{array}{ccc} d_{k-1}(F_{l-1}M_{k-1}) & \longrightarrow & F_{l-1}\ker d_k \\ \downarrow & & \downarrow \\ d_{k-1}(F_lM_{k-1}) & \longrightarrow & F_l\ker d_k \end{array} \qquad \begin{array}{ccc} F_{l-1}\operatorname{im} d_{k-1} & \longrightarrow & F_{l-1}\ker d_k \\ \downarrow & & \downarrow \\ F_l\operatorname{im} d_{k-1} & \longrightarrow & F_l\ker d_k \end{array}$$

can be extended by the "lemme des neuf" to the following diagrams with exact rows and columns

$$\begin{array}{ccccc} d_{k-1}(F_{l-1}M_{k-1}) & \longrightarrow & F_{l-1}\ker d_k & \longrightarrow & H^k(F_{l-1}M_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ d_{k-1}(F_lM_{k-1}) & \longrightarrow & F_l\ker d_k & \longrightarrow & H^k(F_lM_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ d_{k-1}(gr^F_lM_{k-1}) & \longrightarrow & gr^F_l ker d_k & \longrightarrow & H^k(gr^F_lM_\bullet) \end{array} \qquad \begin{array}{ccccc} F_{l-1}\operatorname{im} d_{k-1} & \longrightarrow & F_{l-1}\ker d_k & \longrightarrow & F_{l-1}H^k(M_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ F_l\operatorname{im} d_{k-1} & \longrightarrow & F_l\ker d_k & \longrightarrow & F_lH^k(M_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ gr^F_l\operatorname{im} d_{k-1} & \longrightarrow & gr^F_l\ker d_k & \longrightarrow & gr^F_lH^k(M_\bullet) \end{array}$$

Since the filtration is bounded below for all M_k and therefore also for $H^k(M_\bullet)$, this shows the claim. □

3.5 $\mathscr{R}$-modules

In the following the Rees construction of a filtered $\mathcal{D}$-module will be helpful, we are following [Sab05] . Let X be a smooth variety of dimension n. The order filtration of $\mathcal{D}_X$ gives rise to the Rees ring $R_F\mathcal{D}_X$. Given a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ we construct the corresponding graded $R_F\mathcal{D}_X$-module $R_F\mathcal{M} := \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}z^k$. In local coordinates the sheaf of rings $R_F\mathcal{D}_X$ is given by

$$R_F\mathcal{D}_X = \mathcal{O}_X[z]\langle z\partial_{x_1}, \ldots, z\partial_{x_n}\rangle\,.$$

Denote by $\mathscr{X}$ the product $X \times \mathbb{C}$. We will consider the sheaf

$$\mathscr{R}_{\mathscr{X}} = \mathcal{O}_{\mathscr{X}} \otimes_{\mathcal{O}_X[z]} R_F\mathcal{D}_X$$

and its ring of global sections

$$\mathrm{R}_{\mathscr{X}} := \Gamma(\mathscr{X}, \mathscr{R}_{\mathscr{X}}) = \mathcal{O}_X(X)[z]\langle z\partial_{x_1}, \ldots, z\partial_{x_n}\rangle\,.$$

Given a $R_F\mathcal{D}_X$-module $R_F\mathcal{M}$ the corresponding $\mathscr{R}_{\mathscr{X}}$-module

$$\mathscr{M} := \mathcal{O}_{\mathscr{X}} \otimes_{\mathcal{O}_X[z]} R_F\mathcal{M}\,.$$

This gives an exact functor $\mathscr{T}$ from the category of filtered $\mathcal{D}_X$-modules $MF(\mathcal{D}_X)$ to the category of $\mathscr{R}_{\mathscr{X}}$-modules $Mod(\mathscr{R}_{\mathscr{X}})$

$$\begin{aligned} \mathscr{T} : MF(\mathcal{D}_X) &\longrightarrow Mod(\mathscr{R}_{\mathscr{X}}) \\ (\mathcal{M}, F_\bullet\mathcal{M}) &\mapsto \mathscr{M}\,. \end{aligned}$$

We denote by $Mod_{qc}(\mathscr{R}_{\mathscr{X}})$ the category of $\mathscr{R}_{\mathscr{X}}$-modules which are quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules. We denote by $\Omega^1_{\mathscr{X}} = z^{-1}\Omega^1_{X\times\mathbb{C}/\mathbb{C}}$ the sheaf of algebraic 1-forms on $\mathscr{X}$ relative to the projection $\mathscr{X} \to \mathbb{C}$ having at most a pole of order one along $z = 0$. If we put $\Omega^k_{\mathscr{X}} = \wedge^k\Omega^1_{\mathscr{X}}$, we get a deRham complex

$$0 \longrightarrow \mathcal{O}_{\mathscr{X}} \xrightarrow{d} \Omega^1_{\mathscr{X}} \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_{\mathscr{X}} \longrightarrow 0\,,$$

where the differential d is induced by the relative differential $d_{X\times\mathbb{C}/\mathbb{C}}$. If X is a smooth affine variety we get the following equivalence of categories.

Lemma 3.14. *Let X be a smooth affine variety. The functor*

$$\Gamma(\mathscr{X}, \bullet) : Mod_{qc}(\mathscr{R}_{\mathscr{X}}) \longrightarrow Mod(R_{\mathscr{X}})$$

is exact and gives an equivalence of categories.

Proof. The proof is completely parallel to the $\mathcal{D}$-module case (see e.g. [HTT08, Proposition 1.4.4]). □

One can also define a notion of direct image in the category of $\mathscr{R}$-modules. Since we only need the case of a projection, we will restrict to this special situation. Let X, Y smooth algebraic varieties and $f : X \times Y \to Y$ be the projection to the to the second factor. Similarly as above we have a relative de Rham complex $\Omega^\bullet_{\mathscr{X}\times\mathscr{Y}/\mathscr{Y}} = z^{-1}\Omega^\bullet_{X\times Y\times\mathbb{C}/Y\times\mathbb{C}}$. If $\mathscr{M}$ is an $\mathscr{R}_{\mathscr{X}\times\mathscr{Y}}$-module the relative de Rham complex $DR_{\mathscr{X}\times\mathscr{Y}/\mathscr{Y}}(\mathscr{M})$ is locally given by

$$d(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^{n} (\frac{dx_i}{z} \wedge \omega) \otimes z\partial_{x_i} m\,,$$

where $(x_i)_{1\leq i\leq n}$ is a local coordinate of X. The direct image with respect to f is then defined as

$$f_+\mathscr{M} := Rf_* DR_{\mathscr{X}\times\mathscr{Y}/\mathscr{Y}}(\mathscr{M})[n]\,.$$

Recall that for a filtered $\mathcal{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ the direct image under f is given by

$$f_+\mathcal{M} = Rf_*\left(0 \to \mathcal{M} \to \Omega^1_{X\times Y/Y} \otimes \mathcal{M} \to \ldots \to \Omega^n_{X\times Y/Y} \otimes \mathcal{M} \to 0\right)[n]$$

together with its filtration

$$F_p f_+\mathcal{M} = Rf_*\left(0 \to F_p\mathcal{M} \to \Omega^1_{X\times Y/Y}\otimes F_{p+1}\mathcal{M} \to \ldots \to \Omega^n_{X\times Y/Y}\otimes F_{p+n}\mathcal{M} \to 0\right)[n]\,.$$

It is a straightforward but tedious exercise to check that the functor $\mathscr{T}$ commutes with the direct image functor f_+.

We will apply this to the filtered $\mathcal{D}$-module $(\mathcal{N}, F^H)$ as defined in equation (33) in order to compute $\pi_{2+}\mathcal{N} \simeq \mathcal{R}^\circ_c(g_+\mathcal{O}_T)$ together with its corresponding Hodge filtration. We will denote by $\mathscr{P}\times\mathscr{V}$ the space $\mathbb{P}(W)\times V\times\mathbb{C}$. The corresponding $\mathscr{R}$-module is

$$\mathscr{N} := \mathscr{T}(\mathcal{N}) = \mathcal{O}_{\mathscr{P}\times\mathscr{V}}\otimes_{\mathcal{O}_{\mathbb{P}(W)\times V}[z]} R_{F^H}\mathcal{N}\,.$$

The direct image with respect to π_2 is then given by

$$\pi_{2+}\mathscr{N} \simeq R\pi_{2*}\left(0\to\mathscr{N}\to\Omega^1_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}\to\ldots\to\Omega^n_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}\to 0\right)[n]\,. \tag{35}$$

Since this is rather hard to compute, we will replace the complex

$$0\to\mathscr{N}\to\Omega^1_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}\to\ldots\to\Omega^n_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}\to 0$$

by a quasi-isomorphic one. For this we will construct a resolution of $\mathscr{N}$. Let $\mathscr{W}_u\times\mathscr{V} := W_u\times V\times\mathbb{C}$ and denote by $\mathscr{N}_u$ the restriction of $\mathscr{N}$ to $\mathscr{W}_u\times\mathscr{V}$. We write $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}} = \Gamma(\mathscr{W}_u\times\mathscr{V}, \mathscr{R}_{\mathscr{W}_u\times\mathscr{V}})$, then the module of global sections of $\mathscr{N}_u$ is the $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}$-module

$$\mathrm{N}_u := \Gamma(\mathscr{W}_u\times\mathscr{V}, \mathscr{N}_u)\,.$$

Proposition 3.15. *The $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}$-module N_u is isomorphic to*

$$z^{n-d}\cdot\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{I}_u\,,$$

where I_u is generated by

$$\overline{\Box}_{(m,l)} := \prod_{\substack{m_i>0\\ i\neq u}} w_{iu}^{m_i}\prod_{\substack{l_i>0\\ 0\le i\le n}}(z\partial_{\lambda_i})^{l_i} - \prod_{\substack{m_i<0\\ i\neq u}} w_{iu}^{-m_i}\prod_{\substack{l_i<0\\ 0\le i\le n}}(z\partial_{\lambda_i})^{-l_i},$$

where $(m,l) = ((m_i)_{i\neq u}, l_0,\ldots,l_n)\in\mathbb{L}_{A_u^s}$,

$$\overline{E}^u_k := -\sum_{\substack{i=0\\ i\neq u}}^n a^u_{ki} z\partial_{w_{iu}} w_{iu} + \sum_{i=1}^n a_{ki}\lambda_i z\partial_{\lambda_i} \qquad \textit{for}\quad k=1,\ldots,d$$

and

$$\overline{E}^u_0 := \sum_{i=0}^n \lambda_i z\partial_{\lambda_i}\,.$$

Proof. This follows easily from Lemma 3.11 and Lemma 3.14. □

We will now define a Koszul complex $\mathrm{K}^\bullet_u$ in the category of R_u-modules which corresponds to the Koszul complex $K^\bullet_u$ alluded to above. Write J_u for the left ideal in $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}$ generated by all operators $\overline{\Box}_{(k,l)}$ for $(k,l)\in\mathbb{L}_{A_u^s}$, then a computation similar to formula (34) shows that the maps

$$\begin{aligned}\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_u &\longrightarrow \mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_u\\ P &\mapsto P\cdot\overline{E}^u_k \qquad \text{for}\quad k=0,\ldots,d\end{aligned} \tag{36}$$

are well-defined. Since $[\overline{E}^u_{k_1},\overline{E}^u_{k_2}] = 0$ for $k_1,k_2\in\{0,\ldots,d\}$ we can built a Koszul complex

$$\mathrm{K}^\bullet_u := Kos\left(z^{n-d}\cdot\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_u, (\cdot\,\overline{E}^u_k)_{k=0,\ldots,d}\right)$$

whose terms are given by

$$z^{n-2d-1}\cdot\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_u\to\ldots\to\bigoplus_{i=1}^n z^{n-d-1}\cdot\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_u\, e_1\wedge\widehat{e_i}\wedge\ldots\wedge e_d\to z^{n-d}\cdot\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_u\, e_1\wedge\ldots\wedge e_d$$

Lemma 3.16. *The Koszul complex $\mathrm{K}_u^\bullet$ is a resolution of N_u.*

Proof. In order to prove the Lemma it is enough to apply the exact Rees functor $\mathscr{T}$ to the Koszul complex $K_u^\bullet$ which is a strict resolution of N_u in the category of filtered $D: W_u \times V$-modules by Lemma 3.12. $\square$

We denote by $\mathscr{K}_u^\bullet$ the corresponding resolution of $\mathscr{N}_u = \mathscr{N}_{|W_u \times V}$. We are now able to construct a resolution of $\mathscr{N}$.

Proposition 3.17. *There exists a resolution $\mathscr{K}^\bullet$ of $\mathscr{N}$ in the category of $\mathscr{R}_{\mathscr{P}\times\mathscr{V}}$-modules which is locally given by*

$$\Gamma(\mathscr{W}_u \times \mathscr{V}, \mathscr{K}^\bullet) = \mathrm{K}_u^\bullet\,.$$

Proof. The resolution $\mathscr{K}^\bullet$ is constructed by providing glueing maps between the $\mathrm{R}_{\mathscr{W}_{u_1u_2}\times\mathscr{V}}$-modules

$$\Gamma(\mathscr{W}_{u_1u_2} \times \mathscr{V}, \mathscr{K}_{u_1}^\bullet) \simeq \mathrm{K}_{u_1}^\bullet[w_{u2u1}^{-1}] \longrightarrow \Gamma(\mathscr{W}_{u_1u_2} \times \mathscr{V}, \mathscr{K}_{u_2}^\bullet) \simeq \mathrm{K}_{u_2}^\bullet[w_{u1u2}^{-1}]$$

which are compatible with the glueing maps on

$$\Gamma(\mathscr{W}_{u_1u_2} \times \mathscr{V}, \mathscr{N}_{u_1}) \simeq \mathrm{N}_{u_1}[w_{u2u1}^{-1}] \longrightarrow \Gamma(\mathscr{W}_{u_1u_2} \times \mathscr{V}, \mathscr{N}_{u_2}) \simeq \mathrm{N}_{u_2}[w_{u1u2}^{-1}]\,.$$

Notice that the latter maps are given by

$$\begin{aligned} \mathrm{N}_{u_1}[w_{u2u1}^{-1}] &\longrightarrow \mathrm{N}_{u_2}[w_{u1u2}^{-1}] \\ P &\mapsto \iota_{u_1u_2}(P) w_{u_1u_2}^{n+1}\,, \end{aligned}$$

which follows from Lemma 3.6 and by tracing back the functors applied to $g_{u+}\mathcal{O}_T$. Using the same argument as in Lemma 3.6 shows that the maps

$$\begin{aligned} \mathrm{K}_{u_1}^\bullet[w_{u2u1}^{-1}] &\longrightarrow \mathrm{K}_{u_2}^\bullet[w_{u1u2}^{-1}] \\ P &\mapsto \iota_{u_1u_2}(P) w_{u_1u_2}^{n+1} \end{aligned}$$

are well defined. We have to check that they give rise to a morphism of complexes. But this follows from the commutativity of the diagram

$$\begin{tikzcd} P \arrow[r] & \iota_{u_1u_2}(P) w_{u_1u_2}^{n+1} \\ \mathrm{R}_{\mathscr{W}_{u_1}\times\mathscr{V}}/\mathrm{J}_{u_1} \arrow[r] & \mathrm{R}_{\mathscr{W}_{u_2}\times\mathscr{V}}/\mathrm{J}_{u_2} \\ \mathrm{R}_{\mathscr{W}_{u_1}\times\mathscr{V}}/\mathrm{J}_{u_1} \arrow[r] \arrow[u, "\cdot\overline{E}_k^{u_1}"] & \mathrm{R}_{\mathscr{W}_{u_2}\times\mathscr{V}}/\mathrm{J}_{u_2} \arrow[u, "\cdot\overline{E}_k^{u_2}"'] \\ P \arrow[r] & \iota_{u_1u_2}(P) w_{u_1u_2}^{n+1} \end{tikzcd}$$

$\square$

3.6 A quasi-isomorphism

We now apply the relative deRham functor $DR_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}$ to the resolution $\mathscr{K}^\bullet$ and get a double complex $\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}} \otimes \mathscr{K}^\bullet$:

$$\begin{tikzcd} \ldots \arrow[r] & \Omega^{n-1}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}} \otimes \mathscr{K}_0 \arrow[r, "{}_{II}d^{n,0}"] & \Omega^{n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}} \otimes \mathscr{K}_0 \\ \ldots \arrow[r] & \Omega^{n-1}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}} \otimes \mathscr{K}_{-1} \arrow[r, "{}_{II}d^{n,-1}"] \arrow[u, "{}_{I}d^{n-1,0}"] & \Omega^{n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}} \otimes \mathscr{K}_{-1} \arrow[u, "{}_{I}d^{n,0}"] \\ & \vdots \arrow[u] & \vdots \arrow[u] \end{tikzcd}$$

The corresponding total complex is denoted by $Tot\left(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}\right)$.

Proposition 3.18. *The following natural morphisms of complexes*

$$\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}\longleftarrow Tot\left(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}\right)\longrightarrow\Omega^{n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}/_{II}d\left(\Omega^{n-1}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}\right)\;=:\mathscr{L}^{\bullet}$$

are quasi-isomorphisms.

Proof. Since the double complex $\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}$ is bounded we can associate to it two spectral sequences which both converge. The first one is given by taking cohomology in the vertical direction which gives the ${}_IE_1$-page of the spectral sequence. Since $\mathscr{K}^{\bullet}$ is a resolution of $\mathscr{N}$ and $\Omega^{l}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}$ is a locally free (i.e. flat) $\mathcal{O}_{\mathscr{P}\times\mathscr{V}}$-module for every $l=1,\ldots,n$, the only terms which are non-zero are the ${}_IE_1^{0,q}$-terms which are isomorphic to $\Omega^{l}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}$. Hence the first spectral sequence degenerates at the second page which shows that $\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N}\leftarrow Tot\left(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}\right)$ is a quasi-isomorphism.

We now look at the second spectral sequence which is given by taking cohomology in the horizontal direction. We claim that ${}_{II}E_1^{p,q}=0$ for $q\neq n$. It is enough to check this locally on the charts $\mathscr{W}_u\times\mathscr{V}$ and moreover using Lemma 3.14 on the level of global sections. Notice that the complex

$$\Gamma(\mathscr{W}_u\times\mathscr{V},\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{l})$$

is isomorphic to a direct sum of Koszul complexes $Kos^{\bullet}(z^{-d-l}\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A_u^s},\frac{1}{z}(z\partial_{w_{iu}}\cdot)_{i\neq u}))$ where each summand is given by

$$z^{n-d-l}R_{\mathscr{W}_u\times\mathscr{V}}/J_{A_u^s}\longrightarrow\ldots\longrightarrow z^{-d-l}R_{\mathscr{W}_u\times\mathscr{V}}/J_{A_u^s}e_1\wedge\ldots\wedge e_n\,.$$

Since $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A_u^s}$ can be written as

$$\mathbb{C}[z,(z\partial_{w_{iu}})_{i\neq u}]\otimes_{\mathbb{C}[z]}\left(\mathbb{C}[z,\lambda_0,\ldots,\lambda_n,(w_{iu})_{i\neq u}]\langle z\partial_{\lambda_1},\ldots,z\partial_{\lambda_n}\rangle/\left(\left(\overline{\Box}_{(m,l)}\right)_{(m,l)\in\mathbb{L}_{A_u^s}}\right)\right)\,.$$

Since the operators $z\partial_{w_{iu}}\cdot$ act only on the first term in the tensor product, we immediately see that ${}_{II}E_1^{p,q}=0$ for $q\neq n$.
The fact that $Tot\left(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^{\bullet}\right)\to\mathscr{L}^{\bullet}$ is a quasi-isomorphism follows from the fact that ${}_{II}E^{p,q}=0$ for $q\neq n$, i.e. the second spectral sequence degenerates at the second page. □

The next result is an explicit local description of the complex $\mathscr{L}^{\bullet}$.

Proposition 3.19. *For any $u\in\{0,\ldots,n\}$ define the ring*

$$\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}:=\mathbb{C}[z,\lambda_0,\ldots,\lambda_n,(w_{iu})_{i\neq u}]\langle z\partial_{\lambda_1},\ldots,z\partial_{\lambda_n}\rangle$$

and denote by $\mathscr{S}$ the sheaf of rings on $\mathscr{W}\times\mathscr{V}$ which is locally given by

$$\Gamma(\mathscr{W}_u\times\mathscr{V},\mathscr{S})=\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}$$

with glueing maps

$$\begin{aligned}\mathrm{S}_{\mathscr{W}_{u_1}\times\mathscr{V}}[w_{u_2u_1}^{-1}]&\longrightarrow\mathrm{S}_{\mathscr{W}_{u_2}\times\mathscr{V}}[w_{u_1u_2}^{-1}]\\ P&\mapsto\iota_{u_1u_2}(P)\,.\end{aligned}$$

Denote by $\mathrm{J}_{A_u^s}$ the left $\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}$-ideal generated by the Box operators $\overline{\Box}_{(k,l)}$ for $(k,l)\in\mathbb{L}_{A_u^s}$. Note that this is a slight abuse of notation, as the ideal generated by the same set of operators in the ring $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}$ was also denoted by $\mathrm{J}_{A_u^s}$, but which is justified by the fact that these generators do not contain the variables $z\partial_{w_{iu}}$. Then the complex $\mathscr{L}^{\bullet}$ is given locally by

$$\Gamma(\mathscr{W}_u\times\mathscr{V},\mathscr{L}^{\bullet})\simeq Kos^{\bullet}(z^{-d}\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A_u^s},(\tilde{E}_k)_{k=0,\ldots,d})\tag{37}$$

whose terms are given by

$$z^{-2d-1}\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A^s_u} \longrightarrow \ldots \longrightarrow z^{-d}\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A^s_u} e_1\wedge\ldots\wedge e_d$$

where

$$\tilde{E}_k := \sum_{i=1}^{n} a_{ki}\lambda_i z\partial_{\lambda_i} \qquad \textit{for} \quad k=1,\ldots,d$$
$$\tilde{E}_0 := \sum_{i=0}^{n} \lambda_i z\partial_{\lambda_i}\,.$$

Proof. It follows from Proposition 3.18 that the 0-th cohomology of the complex $\left(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{K}^p, {}_{II}d^{\bullet,p}\right)$ is a direct sum of terms of the form $H^0(z^{-d}Kos^\bullet(\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A^s_u}, \frac{1}{z}(z\partial_{w_{iu}}\cdot)_{i\neq u}))$. Taking the cokernel of left multiplication on $\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A^s_u}$ by $z\partial_{w_{iu}}$ shows that we have an isomorphism of $\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}$-modules

$$H^0(z^{-d}Kos^\bullet(\mathrm{R}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A^s_u},(z\partial_{w_{iu}}\cdot)_{i\neq u}))\simeq z^{-d}\mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}/J_{A^s_u}.$$

Hence equation (37) follows. □

The ideals $\mathrm{J}_{A^s_u}$ glue to an ideal $\mathscr{J}\subset\mathscr{S}$. Notice that the Euler vector fields $(\tilde{E}_k)_{k=0,\ldots,d}$ are global sections of $\mathscr{S}$. Recall the glueing maps for $\Gamma(\mathscr{W}_u\times\mathscr{V},\Omega^n_{\mathscr{P}\times\mathscr{V}}\otimes\mathscr{K}^p)$:

$$\bigwedge_{\substack{i=0\\ i\neq u_1}}^{n} dw_{iu_1}\otimes P\mapsto \bigwedge_{\substack{i=0\\ i\neq u_2}}^{n} dw_{iu_2}\cdot(w_{u_1u_2})^{-n-1}\otimes\iota_{u_1u_2}(P)w_{u_1u_2}^{n+1}\,.$$

Since both powers of $w_{u_1u_2}$ on the right hand side cancel when considering the quotient $\mathscr{L}^p$, we see that

$$\mathscr{L}^\bullet\simeq Kos^\bullet(z^{-d}\mathscr{S}/\mathscr{J},(\tilde{E})_{k=0,\ldots,d})\,.$$

Summarizing, Proposition 3.18 and Proposition 3.19 show that instead of computing the direct image (35) we can compute

$$R\pi_{2*}(\mathscr{L}^\bullet)\simeq R\pi_{2*}(z^{-d}Kos^\bullet(\mathscr{S}/\mathscr{J},(\tilde{E})_{k=0,\ldots,d})\,.$$

3.7 Computation of the direct image

Because of Lemma 3.14 it is enough to work on the level of global sections:

$$\Gamma R\pi_{2*}(\mathscr{L}^\bullet)\simeq R\Gamma R\pi_{2*}(\mathscr{L}^\bullet)\simeq R\Gamma(\mathscr{L}^\bullet)\simeq R\Gamma(Kos^\bullet(z^{-d}\mathscr{S}/\mathscr{J},(\tilde{E})_{k=0,\ldots,d}))\,, \tag{38}$$

where the first isomorphism follows from the exactness of $\Gamma(\mathscr{V},\bullet)$.

We will show that each term of the complex $\mathscr{L}^\bullet$ is Γ-acyclic. For this it is enough to show that $\mathscr{S}/\mathscr{J}$ is Γ-acyclic. Recall that $\mathscr{P}\times\mathscr{V}=\mathbb{C}_z\times\mathbb{P}(W)\times V$. We denote by $\mathscr{W}\times\mathscr{V}$ the space $\mathbb{C}_z\times W\times V$. Let

$$\mathrm{S}:=\mathbb{C}[z,w_0,\ldots,w_n,\lambda_0,\ldots,\lambda_n]\langle z\partial_{\lambda_0},\ldots,z\partial_{\lambda_n}\rangle$$

and consider the S-module

$$\mathrm{S}/\mathrm{J}_{A^s}\,,$$

where the left ideal J_{A^s} is generated by

$$\Box_{(k,l)}=\prod_{k_i>0}w_i^{k_i}\prod_{l_i>0}(z\partial_{\lambda_i})^{l_i}-\prod_{k_i<0}w_i^{-k_i}\prod_{l_i<0}(z\partial_{\lambda_i})^{-l_i} \qquad \text{for} \quad (k,l)\in\mathbb{L}_{A^s}$$

and the matrix A^s is given by

$$A^s:=(\underline{a}^s_0,\ldots,\underline{a}^s_n,\underline{b}^s_0,\ldots,\underline{b}^s_n):=\begin{pmatrix}1&1&\ldots&1&0&0&\ldots&0\\0&0&\ldots&0&1&1&\ldots&1\\0&a_{11}&\ldots&a_{1n}&0&a_{11}&\ldots&a_{1n}\\ \vdots&\vdots&&\vdots&\vdots&\vdots&&\vdots\\0&a_{d1}&\ldots&a_{dn}&0&a_{d1}&\ldots&a_{dn}\end{pmatrix}$$

and $\mathbb{L}_{A^s}$ is the $\mathbb{Z}$-module of relations among the columns of A^s. Notice that $\mathrm{S}/\mathrm{J}_{A^s}$ is $\mathbb{Z}$-graded by the degree of the w_i. Denote by S_{w_u} the localization of S with respect to w_u, then one easily sees that the degree zero part $[\mathrm{S}_{w_u}/\mathrm{J}_{A^s}]_0$ of $\mathrm{S}_{w_u}/\mathrm{J}_{A^s}$ is equal to $\Gamma(\mathscr{W}_u \times \mathscr{V}, \mathscr{S}/\mathscr{J}) \simeq \mathrm{S}_{\mathscr{W}_u\times\mathscr{V}}/\mathrm{J}_{A^s_u}$ if we identify w_i/w_u with w_{iu}. Let $\widetilde{\mathrm{S}/\mathrm{J}_{A^s}}$ be the associated the sheaf on $\mathscr{P} \times \mathscr{V}$ having global sections $[\mathrm{S}/\mathrm{J}_{A^s}]_0$, then we obviously have

$$\widetilde{\mathrm{S}/\mathrm{J}_{A^s}} \simeq \mathscr{S}/\mathscr{J} \,.$$

Define

$$\Gamma_*(\mathscr{S}/\mathscr{J}) := \bigoplus_{a\in\mathbb{Z}} \Gamma(\mathscr{P}\times\mathscr{V}, (\mathscr{S}/\mathscr{J})(a))\,.$$

We want to use the following result applied to the graded module $\mathrm{S}/\mathrm{J}_{A^s}$

Proposition 3.20. *[Gro61, Proposition 2.1.3] There is the following exact sequence of $\mathbb{Z}$-graded* S-*modules*

$$0 \longrightarrow H^0_{(\underline{w})}(\mathrm{S}/\mathrm{J}_{A^s}) \longrightarrow \mathrm{S}/\mathrm{J}_{A^s} \longrightarrow \Gamma_*(\mathscr{S}/\mathscr{J}) \longrightarrow H^1_{(\underline{w})}(\mathrm{S}/\mathrm{J}_{A^s}) \longrightarrow 0$$

and for each $i \geq 1$ the following isomorphisms

$$\bigoplus_{a\in\mathbb{Z}} H^i(\mathscr{P}\times\mathscr{V}, (\mathscr{S}/\mathscr{J})(a)) \simeq H^{i+1}_{(\underline{w})}(\mathrm{S}/\mathrm{J}_{A^s}) \tag{39}$$

where $(\underline{w})$ is the ideal in $\mathbb{C}[z,\lambda_0,\dots,\lambda_n,w_0,\dots,w_n]$ generated by $w_0,\dots,w_n$.

Proof. In the category of $\mathbb{C}[z,\lambda_0,\dots,\lambda_n,w_0,\dots w_n]$-modules, the statement follows from [Gro61, Proposition 2.1.3]. The statement in the category of S-modules follows from the proof given there. $\square$

In order to compute the local cohomology of S/J_{A^s} we introduce a variant of the Ishida complex (see e.g. [BH93, Theorem 6.2.5]). Let $T := \mathbb{C}[w_0,\dots,w_n, z\partial_{\lambda_0},\dots,z\partial_{\lambda_n}] \subset S$ be a commutative subring and let $\mathbb{C}[\mathbb{N}A^s]$ be the affine semigroup algebra of A^s, i.e.

$$\mathbb{C}[\mathbb{N}A^s] = \{y^{\underline{c}} \in \mathbb{C}[y_0^{\pm},\dots,y_{n+1}^{\pm}] \mid \underline{c} \in \mathbb{N}A^s \subset \mathbb{Z}^{d+2}\}\,.$$

We have a map

$$\begin{aligned} \Phi_{A^s} : T &\longrightarrow \mathbb{C}[\mathbb{N}A^s] \\ w_i &\mapsto y^{\underline{a}_i^s} \\ z\partial_{\lambda_i} &\mapsto y^{\underline{b}_i^s}\,. \end{aligned}$$

Notice that the kernel K_{A^s} of Φ_{A^s} is equal to the ideal in T generated by the elements $\Box_{(k,l)}$, hence

$$T/K_{A^s} \simeq \mathbb{C}[\mathbb{N}A^s]\,.$$

Remark 3.21. *The $\mathbb{Z}$-grading of T by the degree of the w_i induces a $\mathbb{Z}$-grading on $\mathbb{C}[\mathbb{N}A^s]$ since the operators $\Box_{(k,l)}$ are homogeneous. The semi-group ring $\mathbb{C}[\mathbb{N}A^s] \subset \mathbb{C}[\mathbb{Z}^{d+2}]$ carries also a natural $\mathbb{Z}^{d+2}$-grading. Looking at the matrix A^s one sees that the $\mathbb{Z}$-grading coming from T is the first component of this $\mathbb{Z}^{d+2}$-grading.*

We regard $\mathbb{C}[\mathbb{N}A^s]$ as a T-module using the map Φ_{A^s}, which gives the isomorphisms

$$S/J_{A^s} \simeq S \otimes_T T/K_{A^s} \simeq S \otimes_T \mathbb{C}[\mathbb{N}A^s]\,.$$

We want to express the local cohomology of S/J_{A^s} by the local cohomology of the commutative ring $\mathbb{C}[\mathbb{N}A^s]$. For this, let I be the ideal in $\mathbb{C}[\mathbb{N}A^s]$ generated by $y^{\underline{a}_0^s}, \dots y^{\underline{a}_n^s}$, then we have the following change of rings formula:

Lemma 3.22. *There is the following isomorphism of $\mathbb{Z}$-graded S-modules:*

$$H^k_{(\underline{w})}(S/J_{A^s}) \simeq S \otimes_T H^k_I(\mathbb{C}[\mathbb{N}A^s])\,.$$

Proof. Notice that if S was commutative this would be a standard property of the local cohomology groups. Here we have to adapt the proof slightly. First notice that it is enough to compute $H^k_{(\underline{w})}(S/J_{A^s})$ with an injective resolution of T-modules. To see why, let $I^\bullet$ be an injective resolution (in the category of S-modules) of S/J_{A^s}. Since S is a free, hence flat, T-module, it follows from $Hom_S(S \otimes_T M, I) \simeq S \otimes_T Hom_T(M, I)$, that an injective S-module is also an injective T-module. Therefore we have

$$H^k_{(\underline{w})}(S/J_{A^s}) \simeq H^k \Gamma_{(\underline{w})}(I^\bullet) = H^k \Gamma_{I'}(I^\bullet) \simeq H^k_{I'}(S/J_{A^s}),$$

where I' is the ideal in T generated by $w_0, \ldots, w_n$ and the second isomorphism follows from the equality

$$\Gamma_{(\underline{w})}(I^k) = \{x \in I^k \mid \forall i \,\exists k_i \text{ such that } w_i^{k_i} x = 0\} = \Gamma_{I'}(I^k).$$

Let $J^\bullet$ be an injective resolution of T/K_{A^s}. In order to show the claim consider the following isomorphisms

$$\begin{aligned} S \otimes_T H^K_I(\mathbb{C}[\mathbb{N}A^s]) &\simeq S \otimes_T H^k_{I'}(T/K_{A^s}) \\ &\simeq S \otimes_T H^k \Gamma_{I'}(J^\bullet) \\ &\simeq H^k(S \otimes_T \Gamma_{I'}(J^\bullet)) \\ &\simeq H^k \Gamma_{I'}(S \otimes_T J^\bullet) \\ &\simeq H^k_{I'}(S/J_{A^s}), \end{aligned}$$

where the third isomorphism follows from the fact that S is a flat T-module and the fifth isomorphism follows from the fact that $S \otimes_T J^\bullet$ is a T-injective resolution of $S/J_{A^s} \simeq S \otimes_T T/K_{A^s}$. □

3.8 Local cohomology of semi-group rings

Let $\mathcal{F}$ be the face lattice of $\mathbb{R}_{\geq 0}A^s$ and denote by $\mathcal{F}_\sigma$ the sub-lattice of faces which lie in the face σ spanned by $\underline{a}^s_0, \ldots, \underline{a}^s_n$. For a face σ of $\mathbb{R}_{\geq 0}A^s$ consider the multiplicatively closed set

$$U_\sigma := \{y^{\underline{c}} \mid \underline{c} \in \mathbb{N}(A^s \cap \sigma)\}$$

and denote by $\mathbb{C}[\mathbb{N}A^s]_\sigma = \mathbb{C}[\mathbb{N}A^s + \mathbb{Z}(A^s \cap \sigma)]$ the localization. We put

$$L^k_\sigma = \bigoplus_{\substack{\tau \in \mathcal{F}_\sigma \\ \dim \tau = k}} \mathbb{C}[\mathbb{N}A^s]_\tau$$

and define maps $f^k : L^k_\sigma \to L^{k+1}_\sigma$ by specifying its components

$$f^k_{\tau',\tau} : \mathbb{C}[\mathbb{N}A^s]_{\tau'} \to \mathbb{C}[\mathbb{N}A^s]_\tau \quad \text{to be} \quad \begin{cases} 0 & \text{if } \tau' \not\subset \tau \\ \epsilon(\tau', \tau) nat & \text{if } \tau' \subset \tau \end{cases}$$

where ϵ is a suitable incidence function on $\mathcal{F}_\sigma$. The Ishida complex with respect to the face σ is

$$L^\bullet_\sigma : 0 \to L^0_\sigma \to L^1_\sigma \to \ldots \to L^{d+1}_\sigma \to 0.$$

The Ishida complex with respect to the face σ can be used to calculate local cohomology groups of $\mathbb{C}[\mathbb{N}A^s]$.

Proposition 3.23. *As above, denote by $I \subset \mathbb{C}[\mathbb{N}A^s]$ the ideal generated by the elements $\Phi_{A^s}(w_i) = y^{\underline{a}^s_i}$. Then for all k we have the isomorphism*

$$H^k_I(\mathbb{C}[\mathbb{N}A^s]) \simeq H^k(L^\bullet_\sigma).$$

Proof. The proof can be easily adapted from [BH93, Theorem 6.2.5]. For the convenience of the reader we sketch it here together with the necessary modifications . In order to show the claim we have to prove that the functors $N \mapsto H^k(L^\bullet_\sigma \otimes N)$ form a universal δ-functor (see e.g. [Har77]). If we can additionally show that

$$H^0_I(\mathbb{C}[\mathbb{N}A^s]) \simeq H^0(L^\bullet_\sigma) \tag{40}$$

the claim follows by [Har77, Corollary III.1.4]. Let $\mathcal{F}_\sigma(1)$ be the set of one-dimensional faces in $\mathcal{F}_\sigma$ and notice that

$$H^0_{\widetilde{I}}(\mathbb{C}[\mathbb{N}A^s]) \simeq \ker\left(\mathbb{C}[\mathbb{N}A^s] \longrightarrow \bigoplus_{\tau\in\mathcal{F}_\sigma(1)} \mathbb{C}[\mathbb{N}A^s]_\tau\right) \simeq H^0(L^\bullet_\sigma \otimes_T M)\,,$$

where $\widetilde{I} \subset \mathbb{C}[\mathbb{N}A^s]$ is the ideal generated by $\{y^{\underline{a}^s_i} \mid \mathbb{R}_{\geq 0}\underline{a}^s_i \in \mathcal{F}_\sigma(1)\}$. In order to show (40) we have to show that $rad\, I' = I$. Since $I' \subset I$ and $I = rad\, I$ (I is a prime ideal corresponding to the face spanned by $\underline{b}^s_0,\ldots,\underline{b}^s_n$), it is enough to check that a multiple of every $y^{\underline{c}} \in I$ lies in I'. But this follows easily from the fact that the elements $\{\underline{a}^s_i \mid \mathbb{R}_{\geq 0}\underline{a}^s_i \in \mathcal{F}_\sigma(1)\}$ span the same cone over $\mathbb{Q}$ as the elements $\{\underline{a}^s_0,\ldots,\underline{a}^s_n\}$. The proof that $N \mapsto H^k(L^\bullet_\sigma \otimes_T N)$ is a δ-functor is completely parallel to the proof in [BH93]. $\square$

Notice that the complex $L^\bullet_\sigma$ is $\mathbb{Z}^{d+2}$-graded since $\mathbb{C}[\mathbb{N}A^s]$ is $\mathbb{Z}^{d+2}$-graded. In order to analyze the cohomology of $L^\bullet_\sigma$ we look at its $\mathbb{Z}^{d+2}$-graded parts. For this we have to determine when $(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \neq 0$ (and therefore $(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \simeq \mathbb{C}$) for $x \in \mathbb{Z}^{d+2}$.

We are following [BH93, Chapter 6.3]. Denote by C_{A^s} the cone $\mathbb{R}_{\geq 0}A^s \subset \mathbb{R}^{d+2}$. Let $x, y \in \mathbb{R}^{d+2}$. We say that y is visible from x if $y \neq x$ and the line segment $[x,y]$ does not contain a point $y' \in C_{A^s}$ with $y' \neq y$. A subset S is visible from X if each $v \in S$ is visible from x.

Recall that the cone C_{A^s} is given by the intersection of finitely many half-spaces

$$H^+_\tau := \{x \in \mathbb{R}^{d+2} \mid \langle a_\tau, x\rangle \geq 0\} \qquad \tau \in \mathcal{F}(d+1)$$

where $\mathcal{F}(d+1)$ is the set of $d+1$-dimensional faces (facets) of C_{A^s}. We set

$$x^0 = \{\tau \mid \langle a_\tau, x\rangle = 0\}, \qquad x^+ = \{\tau \mid \langle a_\tau, x\rangle > 0\}, \qquad x^- = \{\tau \mid \langle a_\tau, x\rangle < 0\}\,.$$

Lemma 3.24. *[BH93, Lemma 6.3.2, 6.3.3]*

1. *A point $y \in C_{A^s}$ is visible from $x \in \mathbb{R}^{d+2} \setminus C_{A^s}$ if and only if $y^0 \cap x^- \neq \emptyset$.*
2. *Let $x \in \mathbb{Z}^{d+2}$ and τ be a face of C_{A^s}. The $\mathbb{C}$-vector space $(\mathbb{C}[\mathbb{N}A^s]_\tau)_x \neq 0$ if and only if τ is not visible from x.*

Recall the facet $\sigma \in \mathcal{F}(d+1)$ which is spanned by $\underline{a}^s_0,\ldots,\underline{a}^s_n$. It is the unique maximal element in the face lattice $\mathcal{F}_\sigma \subset \mathcal{F}$. Denote by H_σ its supporting hyperplane (i.e. $\sigma = C_{A^s} \cap H_\sigma$) which is given by

$$H_\sigma = \{x \in \mathbb{R}^{d+2} \mid \langle a_\sigma, x\rangle = 0\}\,,$$

where $a_\sigma = (0,1,0,\ldots,0)$. Let $\tau \in \mathcal{F}_\sigma$ be a k-dimensional face contained in σ and set $I_\tau := \{i \mid \underline{a}^s_i \in \tau\}$. Notice that the vectors $\{\underline{a}^s_i \mid i \in I_\tau\}$ span the face τ. This face τ gives rise to two other faces, namely its "shadow" τ^s which is spanned by the vectors $\{\underline{b}^s_i \mid i \in I_\tau\}$ and the unique $k+1$-dimensional face τ^c which contains both τ and τ^s. Let $\{\tau_1,\ldots,\tau_m\} = \mathcal{F}_\sigma(d)$ be the faces of dimension d contained in σ, which give rise to the facets $\tau^c_1,\ldots,\tau^c_m$.

First notice that by Lemma 3.24.1 the facet σ is visible from a point $x \in \mathbb{R}^{d+2}$ if and only if $\langle a_\sigma, x\rangle < 0$. If $\langle a_\sigma, x\rangle \geq 0$ a face τ_i is visible from x if and only if the facet τ^c_i is visible from x, i.e. $\langle a_{\tau^c_i}, x\rangle < 0$.

Notice that the set

$$S := \mathbb{Z}^{d+2} \cap (\mathbb{R}(\underline{a}^s_0,\ldots,\underline{a}^s_n) + \mathbb{R}_{\geq 0}(\underline{b}^s_0,\ldots,\underline{b}^s_n))$$

is the set of $\mathbb{Z}^{d+2}$-degrees occurring in $\mathbb{C}[\mathbb{N}A^s]_\sigma$ Given a point $x \in S$ with $\langle a_\sigma, x\rangle \geq 0$ we want to construct a point $y_x \in \mathbb{Z}^{d+2}$ which lies in H_σ so that τ_i is visible from x if and only if it is visible from y_x. Denote by z_x the projection of x to the sub-vector space generated by $\underline{b}^s_0,\ldots,\underline{b}^s_n$. Since the semi-group generated by these vectors is saturated, we can express z_x by a linear combination with positive integers

$$z_x = \sum_{i=0}^n n^x_i \underline{b}^s_i \qquad \text{with} \quad n^x_i \in \mathbb{N}\,.$$

Since τ_i^c is generated by the vectors $\{\underline{a}_i^s \mid i \in I_\tau\} \cup \{\underline{b}_i^s \mid i \in I_\tau\}$, the first two components of the vector $a_{\tau_i^c}$ are equal. Hence, if we set

$$y_x := x + \sum_{i=0}^{n} n_i^x \underline{a}_i^s - \sum_{i=0}^{n} n_i^x \underline{b}_i^s$$

we easily see that

$$\langle a_{\tau_i^c}, x\rangle = \langle a_{\tau_i^c}, y_x\rangle \qquad \text{for} \quad i = 1, \ldots, m\,. \tag{41}$$

We are now able to compute the cohomology of the Ishida complex with respect to the face σ. Set

$$H^-_{\tau_i^c} := \{x \in \mathbb{R}^{d+2} \mid \langle a_{\tau_i^c}, x\rangle < 0\} \qquad \text{for} i = 1, \ldots, m$$

and define

$$S^- := \mathbb{Z}^{d+2} \cap H_\sigma^+ \cap \bigcap_{i=1}^{m} H^-_{\tau_i^c}\,.$$

Proposition 3.25. *Let $x \in S$.*

1. *If $x \notin S$, then $(L_\sigma^\bullet)_x = 0$.*
2. *If $x \in S \setminus S^-$, then $H^i(L_\sigma^\bullet)_x = 0$ for all i*
3. *If $x \in S^-$, then $H^i(L_\sigma^\bullet)_x = 0$ for $i \neq d+1$ and $H^{d+1}(L_\sigma^\bullet)_x \simeq k$.*

Proof. The first point follows from the fact that we have $(\mathbb{C}[\mathbb{N}A^s]_\sigma)_x = 0$ for $x \notin S$, hence $(L_\sigma^i)_x = 0$ for all i. For the second and third point we can assume that x lies in H_σ, because $x \in S^-$ if and only if $y_x \in S^- \cap H_\sigma$ by formula (41). Recall the matrix $\widetilde{A}$. The semigroup ring $\mathbb{C}[\mathbb{N}\widetilde{A}]$ embeds in $\mathbb{C}[\mathbb{N}A^s]$ via the map

$$\begin{aligned} \mathbb{C}[\mathbb{N}\widetilde{A}] &\longrightarrow \mathbb{C}[\mathbb{N}A^s] \\ y^{\underline{\widetilde{a}}_i} &\mapsto y^{\underline{a}_i^s} \end{aligned}$$

Denote by p the projection

$$\begin{aligned} p : \mathbb{Z}^{d+2} &\longrightarrow \mathbb{Z}^{d+1} \\ (x_1, \ldots, x_{d+2}) &\mapsto (x_1, x_3, \ldots, x_{d+2}) \end{aligned}$$

then we have

$$\mathbb{C}[\mathbb{N}A^s]_x \simeq \mathbb{C}[\mathbb{N}\widetilde{A}]_{p(x)} \qquad \text{for} \quad x \in H_\sigma\,.$$

Under this isomorphism the $\mathbb{Z}^{d+2}$-graded part $(L_\sigma^\bullet)_x$ of the Ishida complex with respect to the face σ goes over to the $\mathbb{Z}^{d+1}$-graded part $(L^\bullet)_{p(x)}$ of the Ishida complex considered in [BH93]. Hence the proposition follows from Theorem 6.3.4 in loc. cit. . ☐

3.9 Proof of the main theorem

Corollary 3.26. *The $\mathbb{Z}$-graded local cohomology S-modules $H^*_{(w)}(S/J_{A_s})$ have strictly negative degree.*

Proof. Proposition 3.25 shows that $H^i(L_\sigma^\bullet) = 0$ for $i \neq d+1$ and $H^{d+1}(L_\sigma^\bullet)_x = 0$ if $x \notin S^-$. Now let $x \in S^-$, then $\deg_{\mathbb{Z}}(x) \leq \deg_{\mathbb{Z}}(y_x)$. Since all elements in $S^- \cap H_\sigma$ have negative $\mathbb{Z}$-degree (notice that $S^- \cap H_\sigma$ is equal to $\mathbb{R}_{<0}(\underline{a}_0^s, \ldots, \underline{a}_n^s)$), the claim follows from Proposition 3.23 and Lemma 3.22. ☐

Corollary 3.27. *The $\mathscr{S}$-modules $\mathscr{S}/\mathscr{J}$ are Γ-acyclic.*

Proof. This follows from the fact that the local cohomology groups $H^i(S/J_{A^s})$ are concentrated in strictly negative degrees and the degree zero part of formula (39). ☐

Proposition 3.28. *There is the following isomorphism in $D^b(\mathscr{R}_V)$:*

$$\Gamma\pi_{2+}\mathscr{N} \simeq \Gamma(Kos^\bullet(z^{-d}\mathscr{S}/\mathscr{J}, (\widetilde{E})_{k=0,\ldots,d})\,.$$

Proof. By formula (35), Proposition 3.18 and Proposition 3.19 we have the isomorphisms

$$\begin{aligned}\Gamma\pi_{2+}\mathscr{N} &\simeq \Gamma R\pi_{2*}(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N})\\ &\simeq R\Gamma R\pi_{2*}(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N})\\ &\simeq R\Gamma(\Omega^{\bullet+n}_{\mathscr{P}\times\mathscr{V}/\mathscr{V}}\otimes\mathscr{N})\\ &\simeq R\Gamma(\mathscr{L}^\bullet)\,. \end{aligned} \tag{42}$$

Using the last isomorphism in (38) and Corollary 3.27 we get

$$R\Gamma(\mathscr{L}^\bullet)\simeq R\Gamma(Kos^\bullet(z^{-d}\mathscr{S}/\mathscr{J},(\tilde{E})_{k=0,\dots,d}))\simeq \Gamma(Kos^\bullet(z^{-d}\mathscr{S}/\mathscr{J},(\tilde{E})_{k=0,\dots,d}))\,.$$

□

Denote by S_λ the ring

$$S_\lambda := \mathbb{C}[z,\lambda_0,\dots,\lambda_n]\langle z\partial_{\lambda_0},\dots,z\partial_{\lambda_n}\rangle\,,$$

let $J^\lambda_{\widetilde{A}}\subset S_\lambda$ be the left ideal generated by

$$\Box^\lambda_{\underline{l}} = \prod_{l_i>0}(z\partial_{\lambda_i})^{l_i} - \prod_{l_i<0}(z\partial_{\lambda_i})^{-l_i} \qquad \text{for} \quad \underline{l}\in\mathbb{L}_{\widetilde{A}}$$

and let $I^\lambda_{\widetilde{A}}\subset S_\lambda$ be the left ideal generated by $J^\lambda_{\widetilde{A}}$ and the operators

$$\begin{aligned}\tilde{E}_k &= \sum_{i=1}^n a_{ki}\lambda_i z\partial_{\lambda_i} \qquad \text{for} \quad k=1,\dots,d\\ \tilde{E}_0 &= \sum_{i=0}^n \lambda_i z\partial_{\lambda_i}\,.\end{aligned}$$

Lemma 3.29. *There is the following isomorphism of $\mathscr{R}_V$-modules*

$$\Gamma\,\mathcal{H}^0(\pi_{2+}\mathscr{N})\simeq\mathcal{H}^0(\Gamma\pi_{2+}\mathscr{N})\simeq z^{-d}S^\lambda/I^\lambda_{\widetilde{A}}\,.$$

Proof. The first isomorphism follows from Lemma 3.14. The second isomorphism follows from Proposition 3.28, the isomorphism

$$\Gamma(\mathscr{S}/\mathscr{J})\simeq S^\lambda/J^\lambda_{\widetilde{A}}$$

and the isomorphism

$$z^{-d}S^\lambda/I^\lambda_{\widetilde{A}}\simeq H^0\left(Kos^\bullet\left(z^{-d}S^\lambda/J^\lambda_{\widetilde{A}},(\tilde{E})_{k=0,\dots,d}\right)\right)\,.$$

□

We are now able to prove the main theorem of this paper. Let $\widetilde{A}$ be the $(d+1)\times(n+1)$ integer matrix

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & \dots & 1\\ 0 & a_{11} & \dots & a_{1n}\\ \vdots & \vdots & & \vdots\\ 0 & a_{d1} & \dots & a_{dn}\end{pmatrix}$$

given by a matrix $A=(a_{jk})$ such that $\mathbb{Z}\widetilde{A}=\mathbb{Z}^{d+1}$, such that $\mathbb{N}\widetilde{A}=\mathbb{Z}^{d+1}\cap\mathbb{R}_{\geq 0}\widetilde{A}$ and such that

$$int(\mathbb{N}\widetilde{A}) = \mathbb{N}\widetilde{A}+\widetilde{c} \qquad \text{for some} \quad \widetilde{c}\in\mathbb{N}\widetilde{A}\,.$$

Theorem 3.30. *Let $\widetilde{A}$ be an integer matrix as above. The GKZ-system $\mathcal{M}^0_{\widetilde{A}}$ carries the structure of a mixed Hodge module whose Hodge-filtration is given by the shifted order filtration, i.e.*

$$(\mathcal{M}^0_{\widetilde{A}},F^H_\bullet)\simeq(\mathcal{M}^0_{\widetilde{A}},F^{ord}_{\bullet+d})\,.$$

Proof. Recall the isomorphism

$$\mathcal{M}^0_{\widetilde{A}} \simeq \mathcal{H}^0(\mathcal{R}^\circ_c(g_+\mathcal{O}_T)) \simeq \mathcal{H}^0(\pi_{2+}j_{U\dagger}j_U^\dagger \pi_1^\dagger g_+\mathcal{O}_T) \simeq \mathcal{H}^0(\pi_{2+}\mathcal{N}) \,.$$

We have already computed the Hodge filtration of $\mathcal{N}$. In order to compute the Hodge filtration under the direct image of π_2, we will use the results obtained above and read off the Hodge filtration from the corresponding $\mathscr{R}_V$-module $\mathscr{T}(\mathcal{M}^0_{\widetilde{A}}, F^H)$. We have the following isomorphisms

$$\Gamma\mathscr{T}(\mathcal{M}^0_{\widetilde{A}}, F^H) \simeq \Gamma\mathscr{T}(\mathcal{H}^0(\pi_{2+}\mathcal{N}, F^H)) \simeq \Gamma\,\mathcal{H}^0(\pi_{2+}\mathscr{N}) \simeq z^{-d}S^\lambda / I^\lambda_{\widetilde{A}} \,.$$

Using these isomorphisms the claim follows easily. □

3.10 Duality

For applications like the one presented in the next section, it will be useful to extend the computation of the Hodge filtration on $\mathcal{M}^0_{\widetilde{A}}$ to the dual Hodge-module $\mathbb{D}\mathcal{M}^0_{\widetilde{A}}$. This is possible under the assumption made in the above main theorem (Theorem 3.30). More precisely, it follows from [Wal07], that under these assumptions, the $\mathcal{D}_V$-module $\mathbb{D}\mathcal{M}^0_{\widetilde{A}}$ is still a GKZ-system. Hence it is reasonable to expect that its Hodge filtration will also be the order filtration up to a suitable shift.

Theorem 3.31. *Suppose that $A \in M(d \times n, \mathbb{Z})$ is such that $\mathbb{Z}\widetilde{A} = \mathbb{Z}^{d+1}$, $\mathbb{N}\widetilde{A} = \mathbb{Z}^{d+1} \cap \mathbb{R}_{\geq 0}\widetilde{A}$ and such that there is $int(\mathbb{N}\widetilde{A}) = \widetilde{c} + \mathbb{N}\widetilde{A}$ for some $\widetilde{c} = (c_0, c) \in \mathbb{Z}^{d+1}$. Then we have*

$$\mathbb{D}\mathcal{M}^0_{\widetilde{A}} \simeq \mathcal{M}^{-\widetilde{c}}_{\widetilde{A}},$$

and the Hodge filtration on $\mathbb{D}\mathcal{M}^0_{\widetilde{A}}$ is the order filtration, shifted by $c_0 + n$, i.e., we have

$$F^H_p \mathbb{D}\mathcal{M}^0_{\widetilde{A}} \simeq F^{ord}_{p-n-c_0}\mathcal{M}^{-(c_0,c)}_{\widetilde{A}} \,.$$

Proof. The proof is very much parallel to [RS15, Proposition 2.19] resp. [RS12, Theorem 5.4], we will give the main ideas here once again for the convenience of the reader. We work again with the modules of global sections, and write $D := \mathbb{C}[\lambda_0, \ldots, \lambda_n]\langle \partial_{\lambda_0}, \ldots, \partial_{\lambda_n}\rangle$ and P for the commutative ring $\mathbb{C}[\partial_{\lambda_0}, \ldots, \partial_{\lambda_n}]/(\Box_{\underline{l}})_{\underline{l}\in\mathbb{L}_{\widetilde{A}}}$ These rings are $\mathbb{Z}^{d+1}$-graded by $\deg(\lambda_i) = \widetilde{a}_i$, $\deg(\partial_{\lambda_i}) = -\widetilde{a}_i$.
In order to calculate $\mathbb{D}M^0_{\widetilde{A}}$ together with its Hodge filtration, we need to find a strictly filtered free resolution $(L_\bullet, F_\bullet) \overset{\simeq}{\twoheadrightarrow} (M^0_{\widetilde{A}}, F^H_\bullet) = (M^0_{\widetilde{A}}, F^{ord}_{\bullet+d})$. We have already used in the previous sections of this paper resolutions of "Koszul"-type for various (filtered) $\mathcal{D}$-modules. Here we consider the Euler-Koszul complex

$$K_\bullet := K_\bullet\left(E, D \otimes_{\mathbb{C}[\underline{\partial}_\lambda]} P\right),$$

as defined in [MMW05]. We have $K_{-i} := \bigwedge^i \left(D \otimes_{\mathbb{C}[\underline{\partial}]} P\right)^n$ (notice that these modules are not D-free), and the differentials are given by a suitably twisted left action of the vector fields E_i making them *left* D-linear. Then a free resolution of $M^0_{\widetilde{A}}$ is constructed as follows: Take a $\mathbb{C}[\underline{\partial}]$-free graded resolution of $T_\bullet \twoheadrightarrow P$ (placed in negative homological degrees), and define $L_\bullet$ to be the total complex $\mathrm{Tot}\left(K_\bullet(E, D \otimes_{\mathbb{C}[\underline{\partial}_\lambda]} T_\bullet)\right)$. Notice that the double complex $K_\bullet(E, D \otimes_{\mathbb{C}[\underline{\partial}_\lambda]} T_\bullet)$ exists since $K_\bullet(E, D \otimes_{\mathbb{C}[\underline{\partial}]} -)$ is a functor from the category of $\mathbb{Z}^{d+1}$-graded $\mathbb{C}[\underline{\partial}]$-modules to the category of (bounded complexes of) $\mathbb{Z}^d$-graded D-modules. Then we have $L_{-k} = 0$ for all $k > n+1$ (notice that the length of the Euler-Koszul complexes is $d+1$, and the length of the resolution $T_\bullet \twoheadrightarrow P$ is $n-d+1$, hence the total complex has length $(d+1)+(n-d+1)-1 = n+1$). Moreover, the last term $L_{-(n+1)}$ of this complex is simply equal to D (and so is the first one L_0).
As we have $int(\mathbb{N}\widetilde{A}) = \widetilde{c} + \mathbb{N}\widetilde{A}$, the ring $\mathbb{C}[\mathbb{N}\widetilde{A}] \simeq P$ is Gorenstein, more precisely, we have $\omega_P \cong P(\widetilde{c})$, where ω_P is the canonical module of P. Then a spectral sequence argument (see also [Wal07, Proposition 4.1]), using

$$Ext^i_{\mathbb{C}[\underline{\partial}]}\left(P, \omega_{\mathbb{C}[\underline{\partial}]}\right) \simeq \begin{cases} 0 & \text{if } i < n-d \\ P\left(\widetilde{c}\right) & \text{if } i = n-d \end{cases}$$

shows that

$$\mathbb{D}M^0_{\widetilde{A}} \cong M^{-\widetilde{c}}_{\widetilde{A}}.$$

In order to calculate the Hodge filtration on $M_{\widetilde{A}}^{-\widetilde{c}}$, we remark that the Euler-Koszul complex is naturally filtered by putting $F_p K_{-i} := \bigwedge^i F_{p+d-i}^{ord}\left(D \otimes_{\mathbb{C}[\underline{\partial}]} P\right)$. Notice that $D \otimes_{\mathbb{C}[\underline{\partial}]} P \simeq D/(\Box_{\underline{l}})_{\underline{l}\in\mathbb{L}_{\widetilde{A}}}$, so that this D-module has an order filtration induced from $F_\bullet^{ord} D$. In order to show that $(F_\bullet, K_\bullet) \twoheadrightarrow (M_{\widetilde{A}}^0, F^H)$ is a filtered quasi-isomorphism, it suffices (by Lemma 3.13) to show that $gr_\bullet^F K_\bullet \twoheadrightarrow gr_\bullet^{F^H} M_{\widetilde{A}}^0$ is a quasi-isomorphism. This follows from [SST00, Formula 4.32, Lemma 4.3.7], as $\mathbb{C}[\mathbb{N}\widetilde{A}]$ is Cohen-Macaulay due to he normality assumption on $\widetilde{A}$. The final step is to endow the free resolution $L_\bullet = \mathrm{Tot}\left(K_\bullet(E, D \otimes_{\mathbb{C}[\underline{\partial}_\lambda]} T_\bullet)\right)$ with a strict filtration $F_\bullet$ and to show that $(L_\bullet, F_\bullet) \overset{\simeq}{\twoheadrightarrow} (M_{\widetilde{A}}^0, F_\bullet^H)$. As the resolution $T_\bullet \twoheadrightarrow P$ is taken in the category of $\mathbb{Z}^{d+1}$-graded $\mathbb{C}[\underline{\partial}]$-modules, the morphisms of this resolution are homogeneous for the ($\mathbb{Z}$-)grading $\deg(\lambda_i) = -1$ and $\deg(\partial_{\lambda_i}) = 1$ (notice that this is the grading opposite to the $\mathbb{Z}$-grading given by the first component of the $\mathbb{Z}^{d+1}$-grading of the ring $D \otimes_{\mathbb{C}[\underline{\partial}]} P$ and its powers). Hence these morphisms are naturally filtered for the order filtration $F_\bullet^{ord}(D \otimes_{\mathbb{C}[\underline{\partial}]} P)$ and they are even strict: for a map given by homogeneous operators from $\mathbb{C}[\partial]$ taking the symbols has simply no effect, so that $gr_\bullet^F(D \otimes_{\mathbb{C}[\underline{\partial}]} T_\bullet) \twoheadrightarrow gr_\bullet^{F^{ord}}(D \otimes_{\mathbb{C}[\underline{\partial}]} P)$ is a filtered quasi-isomorphism (and similarly for the exterior powers occurring in the terms K_{-i}). However, we have to determine the $\mathbb{Z}$-degree (for the grading $\deg(\partial_{\lambda_i}) = 1$) of the highest (actually, the only nonzero) cohomology module $Ext_{\mathbb{C}[\underline{\partial}]}^{n-d}(P, \omega_{\mathbb{C}[\underline{\partial}]})$: it is the first component of the difference of the degree of $\omega_{\mathbb{C}[\underline{\partial}]}$ (i.e, the first component of the sum of the columns of $\widetilde{A}$), which is $n+1$, and the first component of the degree of ω_P, which is c_0. Now the shift of the filtration between $M_{\widetilde{A}}^0$ and the dual module module $M_{\widetilde{A}}^{-(c_0,c)}$ is the sum of the length of the complex $K_\bullet(E, D \otimes_{\mathbb{C}[\underline{\partial}]} P)$, i.e., $d+1$, and the above $\mathbb{Z}$-degree of $Ext_{\mathbb{C}[\underline{\partial}]}^{n-d}(P, \omega_{\mathbb{C}[\underline{\partial}]})$, i.e. $n+1-c_0$. Hence the filtration $F_\bullet L_{-(m+1)}$ is again the shifted order filtration, more precisely, we have

$$F_p L_{-(n+1)} = F_{p+d-(d+1)-(n+1-c_0)}^{ord} D = F_{p-n-2+c_0}^{ord} D\,.$$

Now it follows from [Sai94, page 55] that

$$\mathbb{D}(M_{\widetilde{A}}^0, F^H) \simeq Hom_D\left((L_\bullet, F_\bullet), \left((D \otimes \Omega_V^{n+1})^\vee, F_{\bullet-2(n+1)} D \otimes (\Omega_V^{n+1})^\vee\right)\right)$$

so that finally we obtain

$$F_p^H \mathbb{D} M_{\widetilde{A}}^0 = F_{p-n-c_0}^{ord} M_{\widetilde{A}}^{-(c_0,c)}.$$

□

From Proposition 3.3, we get the morphism

$$\begin{aligned}\phi : F_{p+d-c_0}^{ord} M_{\widetilde{A}}^{-(c_0,c)} = F_{p+n+d}^H \mathbb{D}(M_{\widetilde{A}}^0) = F_p^H \mathbb{D}(M_{\widetilde{A}}^0)(-n-d) \longrightarrow F_p^H M_{\widetilde{A}}^0 = F_{p+d}^{ord} M_{\widetilde{A}}^0 \\ P \mapsto P \cdot \partial^{(c_0,c)}\end{aligned}$$

where $\partial^{(c_0,c)} := \prod_{i=0}^n \partial_{\lambda_i}^{k_i}$ for any $\underline{k} = (k_0, \ldots, k_n)$ with $\widetilde{A} \cdot \underline{k} = (c_0, c)$. Since $\widetilde{A}$ is homogeneous we have $\sum k_i = c_0$. As a consequence, we have the following result.

Corollary 3.32. *The morphism*

$$\begin{array}{rcl}\phi : (M_{\widetilde{A}}^{-(c_0,c)}, F_{\bullet-c_0}^{ord}) & \longrightarrow & (M_{\widetilde{A}}^0, F_\bullet^{ord}) \\ P & \longmapsto & P \cdot \partial^{(c_0,c)}\end{array}$$

(where $\partial^{(c_0,c)}$ is as above) is strictly filtered.

4 Landau-Ginzburg models and non-commutative Hodge structures

In this final section we will give a first application of our main result. It is concerned with Hodge theoretic properties of differential systems occurring in toric mirror symmetry. More precisely, we will prove [RS12, Conjecture 6.13] showing that the so-called *reduced quantum $\mathcal{D}$-module* of a nef complete intersection inside a smooth projective toric variety underlies a (variation of) non-commutative Hodge structure(s). We will recall as briefly as possible the necessary notations and results of loc.cit. and then deduce this conjecture from our main Theorem 3.30.

Let X_Σ be smooth, projective and toric with $\dim_{\mathbb{C}}(X_\Sigma) = k$. Put $m := k + b_2(X_\Sigma)$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_l$ be globally generated line bundles on X_Σ (in particular, they are nef according to [Ful93, Section 3.4]) and assume that $-K_{X_\Sigma} - \sum_{i=1}^{l} c_1(\mathcal{L}_i)$ is nef. Put $\mathcal{E} := \oplus_{i=1}^{l}\mathcal{L}_i$, and let $\mathcal{E}^\vee$ the dual vector bundle. Its total space $\mathbb{V}(\mathcal{E}^\vee) := \mathbf{Spec}_{\mathcal{O}_{X_\Sigma}}(\mathrm{Sym}_{\mathcal{O}_{X_\Sigma}}(\mathcal{E}))$ is a quasi-projective toric variety with defining fan Σ'. The matrix $A \in M((k+l)\times(m+l), \mathbb{Z})$ whose columns are the primitive integral generators of the rays of Σ' then satisfies the conditions in Theorem 3.31. More precisely, we have $\mathbb{Z}\widetilde{A} = \mathbb{Z}^{d+1}$ and it follows from [RS12, Proposition 5.1] that the semi-group $\mathbb{N}\widetilde{A}$ is normal and that we have $int(\mathbb{N}\widetilde{A}) = \widetilde{c} + \mathbb{N}\widetilde{A}$, where $\widetilde{c} = \sum_{i=m+1}^{m+l} e_i = (l+1, \underline{0}, \underline{1})$, e_i being the i'th standard vector in $\mathbb{Z}^{1+m+l}$.

The strictly filtered duality morphism ϕ from Corollary 3.32 is more concretely given as

$$\begin{aligned}\phi : (\mathcal{M}_{\widetilde{A}}^{-(l+1,\underline{0},\underline{1})}, F^{ord}_{\bullet-l-1}) &\longrightarrow (\mathcal{M}^0_{\widetilde{A}}, F^{ord}_{\bullet}) \\ P &\longmapsto P\cdot\partial_{\lambda_0}\cdot\partial_{\lambda_{m+1}}\cdot\ldots\cdot\partial_{\lambda_{m+l}}\,.\end{aligned}$$

Proposition 4.1. *The image of ϕ underlies a pure Hodge module of weight $m+k+2l$, where the Hodge filtration is given by*

$$F^H_\bullet im(\phi) = im(\phi) \cap F^{ord}_{\bullet+k+l}\mathcal{M}^0_{\widetilde{A}}.$$

Proof. This is a consequence of [RS12, Theorem 2.16] and of Proposition 3.3. □

A main point in the paper [RS12] is to consider the partial localized Fourier transformations of the GKZ-systems $\mathcal{M}^{\beta}_{\widetilde{A}}$. We recall the main construction and refer to [RS12, Section 3.1] for details (in particular concerning the definition and properties of the Fourier-Laplace functor FL and its "localized" version FL^{loc}).

Let (as done already in section in 3.1) Λ be the affine space $\mathbb{C}^{m+l}$ with coordinates $\lambda_1, \ldots, \lambda_{m+l}$ (so that $V = \mathbb{C}_{\lambda_0}\times\Lambda$) and put $\widehat{V} := \mathbb{C}_z \times \Lambda$. Let $\widehat{\mathcal{M}}^{(\beta_0,\beta)}_A$ be the $\mathcal{D}_{\widehat{V}}$-module $\mathcal{D}_{\widehat{V}}[z^{-1}]/\mathcal{I}$, where $\mathcal{I}$ is the left ideal generated by the operators $\widehat{\Box}_{\underline{l}}$ (for all $l \in \mathbb{L}_A$), $\widehat{E}_j - \beta_j z$ (for $j = 1, \ldots, k+l$) and $\widehat{E} - \beta_0 z$, which are defined by

$$\begin{aligned}\widehat{\Box}_{\underline{l}} &:= \prod_{i:l_i<0}(z\cdot\partial_{\lambda_i})^{-l_i} - \prod_{i:l_i>0}(z\cdot\partial_{\lambda_i})^{l_i}\,, \\ \widehat{E} &:= z^2\partial_z + \textstyle\sum_{i=1}^{m+l} z\lambda_i\partial_{\lambda_i}\,, \\ \widehat{E}_j &:= \textstyle\sum_{i=1}^{m+l} a_{ji}\, z\lambda_i\partial_{\lambda_i}\,.\end{aligned}$$

We denote the corresponding $\mathcal{D}_{\widehat{V}}$-module by $\widehat{\mathcal{M}}^{(\beta_0,\beta)}_A$. Then we have (see [RS12, Lemma 3.2])

$$\mathrm{FL}^{loc}_\Lambda\left(\mathcal{M}^{(\beta_0,\beta)}_{\widetilde{A}}\right) = \widehat{\mathcal{M}}^{(\beta_0+1,\beta)}_A.$$

Consider the filtration on $\mathcal{D}_{\widehat{V}}$ for which z has degree -1, ∂_z has degree 2 and $\deg(\lambda_i) = 0$, $\deg(\partial_{\lambda_i}) = 1$. Write $\mathrm{MF}^z(\mathcal{D}_{\widehat{V}})$ for the category of well-filtered $\mathcal{D}_{\widehat{V}}$-modules (that is, $\mathcal{D}_{\widehat{V}}$-modules equipped with a filtration compatible with the filtration on $\mathcal{D}_{\widehat{V}}$ just described and such that the corresponding Rees module is coherent over the corresponding Rees ring). Denote by $G_\bullet$ the induced filtrations on the module $\widehat{\mathcal{M}}^{(\beta_0,\beta)}_A$, which are $\mathcal{R}_{\mathbb{C}_z\times\Lambda}$-modules. We have

$$G_0\widehat{\mathcal{M}}^{(\beta_0,\beta)}_A = \mathcal{R}_{\mathbb{C}_z\times\Lambda}/\mathcal{R}_{\mathbb{C}_z\times\Lambda}(\widehat{\Box}_{\underline{l}})_{l\in\mathbb{L}_A} + \mathcal{R}_{\mathbb{C}_z\times\Lambda}\widehat{E} + \mathcal{R}_{\mathbb{C}_z\times\Lambda}(\widehat{E}_j)_{k=1,\ldots,k+l}$$

and $G_k\widehat{\mathcal{M}}_A^{(\beta_0,\beta)} = z^k \cdot G_0\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$ In general, the modules $\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$ and their filtration steps may be quite complicated. However, we have considered in [RS12] their restriction to a specific Zariski open subset $\Lambda^\circ \subset \left(\Lambda\setminus\bigcup_{i=1}^{m+l}\{w_i=0\}\right) \subset \Lambda$ (called W° in [RS12, Remark 3.8]), which contains the critical locus of the family of Laurent polynomials associated to the matrix A (but excludes certain singularities at infinity of this family). Denote by ${}^\circ\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$ the restriction $(\widehat{\mathcal{M}}_A^{(\beta_0,\beta)})_{|\mathbb{C}_z\times\Lambda^\circ}$ together with the induced filtration $G_\bullet{}^\circ\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$. Then $G_k{}^\circ\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$ is $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-locally free for all k. Moreover, the multiplication by z is invertible on $\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$, filtered with respect to $G_\bullet$ (shifting the filtration by one) and so is its inverse. Hence, we have a strict morphism

$$\cdot z : ({}^\circ\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}, G_\bullet) \longrightarrow ({}^\circ\widehat{\mathcal{M}}_A^{(\beta_0-1,\beta)}, G_{\bullet+1}).$$

We also need a slightly modified version of the Fourier-Laplace transformed GKZ-systems. More precisely, define the modules ${}^\circ\widehat{\mathcal{N}}_A^\beta$ as the cyclic quotients of $\mathcal{D}_{\mathbb{C}_z\times\Lambda^\circ}[z^{-1}]$ by the left ideal generated by $\widetilde{\Box}_{\underline{l}}$ for $\underline{l}\in\mathbb{L}_A$ and $\widehat{E}_j - z\beta_j$ for $j=0,\ldots,k+c$, where

$$\begin{aligned}\widetilde{\Box}_{\underline{l}} \quad := \quad & \prod_{i\in\{1,\ldots,m\}:\, l_i>0} \lambda_i^{l_i}(z\cdot\partial_i)^{l_i} \prod_{i\in\{m+1,\ldots,m+l\}:\, l_i>0}\ \prod_{\nu=1}^{l_i}(\lambda_i(z\cdot\partial_i)-z\cdot\nu) \\ & - \prod_{i=1}^{m+l}\lambda_i^{l_i}\cdot \prod_{i\in\{1,\ldots,m\}:\, l_i<0}\lambda_i^{-l_i}(z\cdot\partial_i)^{-l_i} \prod_{i\in\{m+1,\ldots,m+l\}:\, l_i<0}\ \prod_{\nu=1}^{-l_i}(\lambda_i(z\cdot\partial_i)-z\cdot\nu).\end{aligned}$$

Consider the *invertible* morphism

$$\Psi : {}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})} \longrightarrow {}^\circ\widehat{\mathcal{M}}_A^{-(2l,\underline{0},\underline{1})} \tag{43}$$

given by right multiplication with $z^l\cdot\prod_{i=m+1}^{m+l}\lambda_i$ (recall that $\lambda_i\neq 0$ on Λ°). We define $\widetilde{\phi}$ to be the composition $\widetilde{\phi} := \widehat{\phi}\circ\Psi$, where $\widehat{\phi}$ is the morphism

$$\widehat{\phi} : {}^\circ\widehat{\mathcal{M}}_A^{-(2l,\underline{0},\underline{1})} \longrightarrow {}^\circ\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})},$$

given by right multiplication with $\partial_{\lambda_{m+1}}\cdot\ldots\cdot\partial_{\lambda_{m+l}}$. In concrete terms, we have:

$$\begin{aligned}\widetilde{\phi} : {}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})} \quad &\longrightarrow \quad {}^\circ\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})}, \\ x \quad &\longmapsto \quad \widehat{\phi}(x\cdot z^l\cdot\lambda_{m+1}\cdot\ldots\cdot\lambda_{m+l}) = x\cdot(z\lambda_{m+1}\partial_{m+1})\cdot\ldots\cdot(z\lambda_{m+l}\partial_{m+l}).\end{aligned}$$

We have an induced filtration $G_\bullet{}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})}$ which satisfies

$$G_0{}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})} = \mathcal{R}_{\mathbb{C}_z\times\Lambda^\circ}/\mathcal{R}_{\mathbb{C}_z\times\Lambda^\circ}(\widetilde{\Box}_{\underline{l}})_{\underline{l}\in\mathbb{L}_A} + \mathcal{R}_{\mathbb{C}_z\times\Lambda^\circ}(\widehat{E}_j - z\beta_j)_{j=0,\ldots,m+l}$$

and $G_k{}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})} = z^k\cdot G_0{}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})}$
In order to obtain the lattices $G_\bullet$ we need to extend the functor $\mathrm{FL}_\Lambda^{loc}$ to the category of filtered $\mathcal{D}$-modules.

Definition 4.2. *Let $(\mathcal{M},F_\bullet)\in\mathrm{MF}(\mathcal{D}_V)=\mathrm{MF}(\mathcal{D}_{\mathbb{C}_{\lambda_0}\times\Lambda})$. Define $\mathcal{M}[\partial_{\lambda_0}^{-1}] := \mathcal{D}_V[\partial_{\lambda_0}^{-1}]\otimes_{\mathcal{D}_V}\mathcal{M}$ and consider the natural localization morphism $\widehat{\mathrm{loc}} : \mathcal{M}\to\mathcal{M}[\partial_{\lambda_0}^{-1}]$. We define the saturation of $F_\bullet$ to be*

$$F_k\mathcal{M}[\partial_{\lambda_0}^{-1}] := \sum_{j\geq 0}\partial_{\lambda_0}^{-j}\widehat{\mathrm{loc}}\,(F_{k+j}\mathcal{M})\ . \tag{44}$$

and we denote by $G_\bullet\widehat{\mathcal{M}}$ the filtration induced from $F_k\mathcal{M}[\partial_{\lambda_0}^{-1}]$ on $\widehat{\mathcal{M}} := \mathrm{FL}_\Lambda^{loc}(M)\in M_h(\mathcal{D}_{\widehat{V}}) = M_h(\mathcal{D}_{\mathbb{C}_z\times\Lambda})$. Notice that for $(\mathcal{M},F_\bullet) = (\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}, F_\bullet^{ord})$, the two definitions of $G_\bullet$ coincide: As we have

$$F_k^{ord}\mathcal{M}_A^{(\beta_0-1,\beta)}[\partial_{\lambda_0}^{-1}] = im\left(\partial_{\lambda_0}^k\mathbb{C}[\lambda_0,\lambda_1,\ldots,\lambda_{m+l}]\langle\partial_{\lambda_0}^{-1},\partial_{\lambda_0}^{-1}\partial_{\lambda_1},\ldots,\partial_{\lambda_0}^{-1}\partial_{\lambda_{m+l}}\rangle\right) \text{ in } \mathcal{M}_{\widetilde{A}}^{(\beta_0-1,\beta)}[\partial_{\lambda_0}^{-1}],$$

the filtration induced by $F_k^{ord}\mathcal{M}_{\widetilde{A}}^{(\beta_0-1,\beta)}[\partial_{\lambda_0}^{-1}]$ on $\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$ is precisely $G_k\widehat{\mathcal{M}}_A^{(\beta_0,\beta)}$.
We denote by $(\mathrm{FL}_\Lambda^{loc},\mathrm{Sat})$ the induced functor from the category $\mathrm{MF}(\mathcal{D}_V)$ to the category $\mathrm{MF}^z(\mathcal{D}_{\widehat{\Lambda}})$ which sends $(\mathcal{M},F_\bullet)$ to $(\widehat{\mathcal{M}},G_\bullet)$.

From the above duality considerations, we deduce the following result.

Proposition 4.3. *The morphism*

$$\widetilde{\phi} : {}^{\circ}\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})} \longrightarrow {}^{\circ}\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})}$$

is strict with respect to the filtration $G_\bullet$, in particular, we have

$$\widetilde{\phi}\left(G_0{}^{\circ}\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})}\right) = G_0{}^{\circ}\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})} \cap \mathrm{im}(\widetilde{\phi})$$

Moreover, the object $\left(im(\widetilde{\phi}), G_\bullet\right)$ is obtained via the functor $(\mathrm{FL}_\Lambda^{loc}, \mathrm{Sat})$ *from* $\left(im(\phi), F_\bullet^H = F_{\bullet+k+l}^{ord}\right)$*, which underlies a pure Hodge module of weight $m+k+2l$ by Proposition 4.1.*

Proof. The morphism Ψ is invertible, filtered (shifting the filtration by $-l$) and its inverse is also filtered. Hence it is strict. Therefore the strictness of $\widetilde{\phi}$ follows from the strictness of $z\widehat{\phi}$. We will deduce it from the strictness property of the morphism ϕ in Corollary 3.32.
Notice that the morphism $\widehat{\phi}$ is obtained from ϕ by linear extension in $\partial_{\lambda_0}^{-1}$. Recall that the morphism

$$\phi : (\mathcal{M}_{\widetilde{A}}^{-(l+1,\underline{0},\underline{1})}, F_\bullet^{ord}) \longrightarrow (\mathcal{M}_{\widetilde{A}}^{\underline{0}}, F_{\bullet+l+1}^{ord})$$

was strict, hence equation (44) yields the strictness of

$$\widehat{\phi} : (\widehat{\mathcal{M}}_A^{-(2l,\underline{0},\underline{1})}, G_\bullet) \longrightarrow (\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})}, G_{\bullet+l})$$

Finally, as already noticed above, this yields the strictness of

$$\widetilde{\phi} = \widehat{\phi} \circ \Psi : (\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})}, G_\bullet) \longrightarrow (\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})}, G_\bullet).$$

□

The next corollary is now a direct consequence of [Sab08, Corollary 3.15].

Corollary 4.4. *The free $\mathcal{O}_{\mathbb{C}_z\times\Lambda^\circ}$-module $G_0\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})} \cap im(\widetilde{\phi})$ underlies a variation of pure polarized non-commutative Hodge structures on Λ° (see [Sab11] for a detailed discussion of this notion).*

The main result in [RS12] concerns a mirror statement for several quantum $\mathcal{D}$-modules which are associated to the toric variety X_Σ and the split vector bundle $\mathcal{E}$. In particular, one can consider the reduced quantum $\mathcal{D}$-module $\overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E})$ which is a vector bundle on $\mathbb{C}_z \times H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*$, where $B_\varepsilon^* := \{q \in (\mathbb{C}^*)^{b_2(X_\Sigma)}, |\, 0 < |q| < \varepsilon\}$ together with a flat connection

$$\nabla : \overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E}) \to \overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E}) \otimes_{\mathcal{O}_{\mathbb{C}_z\times H^0(X_\Sigma,\mathbb{C})\times B_\varepsilon^*}} z^{-1}\Omega^1_{\mathbb{C}_z\times H^0(X_\Sigma,\mathbb{C})\times B_\varepsilon^*}\left(\log(\{0\}\times H^0(X_\Sigma,\mathbb{C})\times B_\varepsilon^*)\right).$$

We refer to [MM11] for a detailed discussion of the definition of $\overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E})$, a short version can be found in [RS12, Section 4.1]. Notice that in loc.cit., $\overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E})$ is defined on some larger set, but in mirror type statements only its restriction to $H^0(X_\Sigma, \mathbb{C}) \times \mathbb{C}_z \times B_\varepsilon^*$ is considered. In the sequel, we will need to consider a Zariski open subset of $\mathcal{KM}^\circ \subset (\mathbb{C}^*)^{b_2(X_\Sigma)}$ which contains B_ε^*. We recall the main result from [MM11], which gives a GKZ-type description of $\overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E})$. We present it in a slightly different form, taking into account [RS12, Proposition 6.9]. Let $\mathcal{R}_{\mathbb{C}_z\times\mathcal{KM}^0}$ be the sheaf of Rees rings on $\mathbb{C}_z \times \mathcal{KM}^\circ$, and $R_{\mathbb{C}_z\times\mathcal{KM}^0}$ its module of global sections. If we write $q_1, \ldots, q_r$ for the coordinates on $(\mathbb{C}^*)^r$ (with $r := b_2(X_\Sigma)$, then $R_{\mathbb{C}_z\times\mathcal{KM}^0})$ is generated by $zq_i\partial_{q_i}$ and $z^2\partial_z$ over $\mathcal{O}_{\mathbb{C}_z\times\mathcal{KM}^\circ}$.

Theorem 4.5. *For any $\mathcal{L} \in \mathrm{Pic}(X_\Sigma)$, write $\widehat{\mathcal{L}} \in R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ for the associated "quantized operator" as defined in [MM11, Notation 4.2.] or [RS12, Theorem 6.7]. Define the left ideal J of $R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ by*

$$J := R_{\mathbb{C}_z \times \mathcal{KM}^\circ}(Q_{\underline{l}})_{\underline{l} \in \mathbb{L}_{A'}} + R_{\mathbb{C}_z \times \mathcal{KM}^\circ} \cdot \widehat{E}\,,$$

where

$$\begin{aligned} Q_{\underline{l}} &:= \prod_{i \in \{1,\ldots,m\}: l_i > 0} \prod_{\nu=0}^{l_i - 1} \left(\widehat{\mathcal{D}}_i - \nu z\right) \prod_{j \in \{1,\ldots,c\}: l_{m+j} > 0} \prod_{\nu=1}^{l_{m+l}} \left(\widehat{\mathcal{L}}_j + \nu z\right) \\ &- \underline{q}^{\underline{l}} \cdot \prod_{i \in \{1,\ldots,m\}: l_i < 0} \prod_{\nu=0}^{-l_i - 1} \left(\widehat{\mathcal{D}}_i - \nu z\right) \prod_{j \in \{1,\ldots,c\}: l_{m+j} < 0} \prod_{\nu=1}^{-l_{m+l}} \left(\widehat{\mathcal{L}}_j + \nu z\right), \\ \widehat{E} &:= z^2 \partial_z - \widehat{K}_{\mathbb{V}(\mathcal{E}^\vee)}\,. \end{aligned}$$

Here we write $\mathcal{D}_i \in \mathrm{Pic}(X_\Sigma)$ for a line bundle associated to the torus invariant divisor D_i, where $i = 1, \ldots, m$. Let $K \subset R_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ be the ideal

$$K := \left\{ P \in R_{\mathbb{C}_z \times \mathcal{KM}^\circ} \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} : \prod_{i=0}^{k} \prod_{j=1}^{c} (\widehat{\mathcal{L}} + p + i) P \in J \right\}$$

and $\mathcal{K}$ the associated sheaf of ideals in $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$.
Suppose as above that the bundle $-K_{X_\Sigma} - \sum_{j=1}^{l} \mathcal{L}_j$ is nef, and moreover that each individual bundle $\mathcal{L}_j$ is ample. Then there is a map $\mathrm{Mir} : B_\varepsilon^* \to H^0(X_\Sigma, \mathbb{C}) \times B_\varepsilon^*$ such that we have an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$-modules

$$(\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}/\mathcal{K})_{|\mathbb{C}_z \times B_\varepsilon^*} \xrightarrow{\cong} (\mathrm{id}_{\mathbb{C}_z} \times \mathrm{Mir})^* \,\overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E}).$$

In order to relate the quantum $\mathcal{D}$-module $\overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E})$ with our results on GKZ-systems, we will use the restriction map $\overline{\rho} : \mathcal{KM}^\circ \hookrightarrow \Lambda$ as constructed in [RS12] (discussion before Definition 6.3. in loc.cit.). Then it follows from the results of loc.cit., Proposition 6.10, that we have an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$-modules

$$\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}/\mathcal{K} \cong (\mathrm{id}_{\mathbb{C}_z} \times \overline{\rho})^* \left(\widetilde{\phi}\left(G_0{}^\circ\widehat{\mathcal{N}}_A^{(0,\underline{0},\underline{0})}\right)\right)$$

Now we can deduce from Corollary 4.4 the main result of this section.

Theorem 4.6. *Consider the above situation of a k-dimensional toric variety X_Σ, globally generated line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_l$ such that $-K_{X_\Sigma} - \mathcal{E}$ is nef, where $\mathcal{E} = \oplus_{j=1}^{l} \mathcal{L}_j$, $\mathcal{L}_j$ being ample for $j = 1, \ldots, l$. Then the smooth $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$-module $(\mathrm{id}_{\mathbb{C}_z} \times \mathrm{Mir})^* \overline{\mathrm{QDM}}(X_\Sigma, \mathcal{E})$ (i.e., the vector bundle over $\mathbb{C}_z \times \mathcal{KM}^\circ$ together with its connection operator ∇) underlies a variation of pure polarized non-commutative Hodge structures.*

Proof. The strictness of $\widetilde{\phi}$ as shown in Proposition 4.3 shows that $G_0\widehat{\mathcal{M}}_A^{(-l,\underline{0},\underline{0})} \cap im(\widetilde{\phi}) = \widetilde{\phi}\left(G_0\mathcal{N}^{(0,\underline{0},\underline{0})}\right)$, hence, by Corollary 4.4, $\widetilde{\phi}\left(G_0\mathcal{N}^{(0,\underline{0},\underline{0})}\right)$ underlies a variation of pure polarized non-commutative Hodge structures on Λ^0. Hence the assertion follows from the mirror statement of Theorem 4.5. □

References

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290.

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[Bry86] Jean-Luc Brylinski, *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*, Astérisque (1986), no. 140-141, 3–134, 251, Géométrie et analyse microlocales.

[DE03] Andrea D'Agnolo and Michael Eastwood, *Radon and Fourier transforms for $\mathcal{D}$-modules*, Adv. Math. **180** (2003), no. 2, 452–485.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[GKZ90] Israel M. Gel′fand, Mikhail M. Kapranov, and Andrei V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), no. 2, 255–271.

[GKZ94] ______, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR 0163910 (29 #1209)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *$\mathcal{D}$-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi.

[MM04] Philippe Maisonobe and Zoghman Mebkhout, *Le théorème de comparaison pour les cycles évanescents*, Éléments de la théorie des systèmes différentiels géométriques (Philippe Maisonobe and Luis Narváez Macarro, eds.), Sémin. Congr., vol. 8, Soc. Math. France, Paris, 2004, Papers from the CIMPA Summer School held in Séville, September 2–13, 1996, pp. 311–389.

[MM11] Etienne Mann and Thierry Mignon, *Quantum $\mathcal{D}$-modules for toric nef complete intersections*, preprint arXiv:1112.1552, 2011.

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941 (electronic).

[Moc15] Takuro Mochizuki, *Twistor property of GKZ-hypergeometric systems*, preprint arXiv:1501.04146, 2015.

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[Rei14] Thomas Reichelt, *Laurent Polynomials, GKZ-hypergeometric Systems and Mixed Hodge Modules*, Compositio Mathematica **(150)** (2014), 911–941.

[RS12] Thomas Reichelt and Christian Sevenheck, *Non-affine Landau-Ginzburg models and intersection cohomology*, 58 pages, preprint arxiv:1210.6527, 2012.

[RS15] ______, *Logarithmic Frobenius manifolds, hypergeometric systems and quantum $\mathcal{D}$-modules*, Journal of Algebraic Geometry **24** (2015), no. 2, 201–281.

[Sab05] Claude Sabbah, *Polarizable twistor $\mathcal{D}$-modules*, Astérisque (2005), no. 300, vi+208.

[Sab08] ______, *Fourier-Laplace transform of a variation of polarized complex Hodge structure.*, J. Reine Angew. Math. **621** (2008), 123–158.

[Sab11] ______, *Non-commutative Hodge structures*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 7, 2681–2717. MR 3112504

[Sai88] Morihiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989).

[Sai90] ______, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.

[Sai93] ______, *On b-function, spectrum and rational singularity*, Math. Ann. **295** (1993), no. 1, 51–74. MR 1198841 (95b:32051)

[Sai94] ______, *On the theory of mixed Hodge modules*, Selected papers on number theory, algebraic geometry, and differential geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 160, Amer. Math. Soc., Providence, RI, 1994, pp. 47–61.

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000. MR 1734566 (2001i:13036)

[SW09] Mathias Schulze and Uli Walther, *Hypergeometric $\mathcal{D}$-modules and twisted Gauß-Manin systems*, J. Algebra **322** (2009), no. 9, 3392–3409.

[Wal07] Uli Walther, *Duality and monodromy reducibility of A-hypergeometric systems*, Math. Ann. **338** (2007), no. 1, 55–74.

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