Six-functor formalisms, and a modern approach to Fourier–Mukai transforms

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Categorical introduction

In usual category theory, the category Set plays a distinguished role:

- For two objects X, Y in a category C, the collection of maps $\mathrm{Hom}_C(X, Y)$ is a set. (Up to possible size issues.)

In an ∞-category, we have a notion of "morphism between morphisms" and now composition of maps is only unique up to isomorphism.

Correspondingly, in ∞-category theory, the category Set is replaced by the ∞-category Ani of anima. (Also called ∞-groupoids, spaces, ...)

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Correspondingly, in ∞-category theory, the category Set is replaced by the ∞-category Ani of anima. (Also called ∞-groupoids, spaces, ...)

How to think about anima?

- Description 1: Ani *is something like* $D^{\leq 0}(\mathrm{Set})$.
  We have functors $\pi_0 : \mathrm{Ani} \to \mathrm{Set}$ and $\pi_i : \mathrm{Ani} \to \mathrm{Grp}$ for $i > 0$, similar to cohomology groups. An anima X is a set iff $\pi_i(X) = 0$ for all $i > 0$.
  An anima X satisfying $\pi_i(X) = 0$ for $i > 1$ is equivalent to a groupoid; hence sheaves of anima generalize stacks.
- Description 2: Ani is to Set what Sch is to Var.
  Given a scheme X, its reduced subscheme X_{red} has the same topological space, we picture X as X_{red} *with some fuzz on top*.
  You can imagine an anima X in a similar way: as a set $\pi_0(X)$ with *some soul on top*.

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Every usual category is also an ∞-category: ∞-category theory generalizes usual category theory.

Basically always, "operations" in ∞-category generalize their usual counterparts. Simple example: given two categories C and D, you can do their product in ∞-category land and obtain the product category $C \times D$ you all know and love.

There's one key exception: if you localize a category in ∞-category land you may obtain a truly ∞-category. In particular, for a scheme X, there exists a usual category $D_{qc}(X)$ and an ∞-category that deserves the same notation.

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Algebraic geometers love to glue stuff, right?

From now on, $\mathrm{D}_{\mathrm{qc}}(X)$ will always denote this ∞-category. If needed, we'll denote the usual derived category by $\mathrm{hD}_{\mathrm{qc}}(X)$.

The ∞-category $\mathrm{D}_{\mathrm{qc}}(X)$ has a key advantage from $\mathrm{hD}_{\mathrm{qc}}(X)$: if $\{U_i\}$ is a Zariski (or even fpqc) cover of X, you can recover $\mathrm{D}_{\mathrm{qc}}(X)$ from the $\mathrm{D}_{\mathrm{qc}}(U_i)$.

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Correspondences and six-functor formalisms

The *yoga* of six operations

For a long time, six-functor formalisms had no rigorous definition; you knew one when you saw it. Here's what happens in all examples:

- You have a certain (possibly ∞-)category *of spaces* C.
- For every space X in C, there is a closed symmetric monoidal ∞-category $D(X)$ (i.e., we have $\otimes$ and $\underline{\mathrm{Hom}}$ functors, which are adjoint).
- For every morphism $f : X \to S$ in C, we have a symmetric monoidal functor $f^* : D(S) \to D(X)$ that admits a right adjoint f_*.
- There exists a collection E of morphisms in C such that, for every morphism $f : X \to S$ in E, we have adjoint functors

$$f_! : D(X) \rightleftarrows D(S) : f^!$$

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Base change and the projection formula

These functors should satisfy certain compatibilities. Notably:

- (Projection formula) For f in E, there exists a natural isomorphism
$$f_! - \otimes - \simeq f_!(- \otimes f^* -).$$
- (Base change) Given a cartesian diagram in C
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S, \end{tikzcd}$$
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How to define a six-functor formalism? Property vs data

Suppose we want to build a six-functor formalism from scratch: what data do we need to provide?

- The ∞-categories $D(X)$, with their symmetric monoidal structures. (Admitting a inner Hom is a property.)
- Functors $f^* : D(S) \to D(X)$, as well as their symmetric monoidal structures. (Admitting a right adjoint f_* is a property.)
- Functors $f_! : D(X) \to D(S)$. (Admitting a right adjoint $f^!$ is a property.)
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This is not enough!

- If $h = g \circ f$, we should provide natural isomorphisms $h^* \to f^*g^*$ and $h_! \to g_!f_!$.
- These isomorphisms should be compatible with the projection formula:

$$\begin{tikzcd}
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Not even this is enough!

- The composition isomorphisms should be compatible with the base change isomorphisms. (I'll spare you the diagram…)
- The base change isomorphisms should be compatible with the projection formula.
- All of the above has to be compatible with the symmetric monoidal structures.
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Note that, since we are working with ∞-categories, commuting is not a property that some diagram may satisfy; it is extra data that has to be provided.

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Lurie's brilliant idea: correspondences

Suppose that C is a category admitting fiber products and that E is stable under composition and base change. We will construct a ∞-category $\mathrm{Span}(C, E)$ of correspondences.

The objects of $\mathrm{Span}(C, E)$ are the same as the objects of C. A morphism $X \to Y$ in $\mathrm{Span}(C, E)$ amounts to a diagram

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X & & Y, \end{tikzcd}$$

where $W \to Y$ lies in E.

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where $W \to Y$ lies in E.

To compose two morphisms $X \leftarrow W \rightarrow Y$ and $Y \leftarrow V \rightarrow Z$, we take a fiber product:

$$\begin{tikzcd}
 & & W \times_Y V \arrow[dl] \arrow[dr] & & \\
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Note that, since fiber products are only unique up to unique isomorphism, the composition is not uniquely defined. As a result, $\mathrm{Span}(\mathrm{C}, E)$ is only a ∞-category.

Correspondences II

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The modern definition of three-functor formalisms

The category $\mathrm{Span}(\mathrm{C}, E)$ has a symmetric monoidal structure, where the tensor product of two spaces X and Y is their cartesian product $X \times Y$. Similarly, the ∞-category Cat_∞ admits a cartesian symmetric monoidal structure.

Definition (Lurie, Liu–Zheng)

A three-functor formalism is a lax symmetric monoidal functor

$$\mathrm{D}\colon \mathrm{Span}(\mathrm{C}, E) \to \mathrm{Cat}_\infty.$$

Here, the adjective lax means that instead of having isomorphisms $\mathrm{D}(X) \times \mathrm{D}(Y) \simeq \mathrm{D}(X \times Y)$, we only have a natural transformation

$$\boxtimes\colon \mathrm{D}(-) \times \mathrm{D}(-) \to \mathrm{D}(- \times -).$$

The modern definition of three-functor formalisms

The category $\mathrm{Span}(\mathrm{C}, E)$ has a symmetric monoidal structure, where the tensor product of two spaces X and Y is their cartesian product $X \times Y$. Similarly, the ∞-category Cat_∞ admits a cartesian symmetric monoidal structure.

Definition (Lurie, Liu–Zheng)

A three-functor formalism is a lax symmetric monoidal functor

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Here, the adjective lax means that instead of having isomorphisms $\mathrm{D}(X) \times \mathrm{D}(Y) \simeq \mathrm{D}(X \times Y)$, we only have a natural transformation

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Let's unpack this definition

Given a three-functor formalism $\mathrm{D}\colon \mathrm{Span}(\mathrm{C}, E) \to \mathrm{Cat}_\infty$, we define:

- $f^*\colon \mathrm{D}(Y) \to \mathrm{D}(X)$ as the image of $Y \xleftarrow{f} X = X$ by D;
- $f_!\colon \mathrm{D}(X) \to \mathrm{D}(Y)$ as the image of $X = X \xrightarrow{f} Y$ by D;
- $\otimes$ as the composition

$$\mathrm{D}(X) \times \mathrm{D}(X) \xrightarrow{\boxtimes} \mathrm{D}(X \times X) \xrightarrow{\Delta_X^*} \mathrm{D}(X),$$

where $\Delta_X\colon X \to X \times X$ is the diagonal map.

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A six-functor formalism is a three-functor formalism in which the functors f^*, $f_!$ and $\otimes$ admit right adjoints.

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How this encodes the compatibilities

This elegant definition contains all the data previously described.

To give an example, consider a cartesian diagram as in the base change axiom and fit it into a composition of correspondences:

$$\begin{tikzcd}
 & & X' \arrow[dl, "f'"'] \arrow[dr, "g'"] & & \\
 & S' \arrow[dl, equal] \arrow[dr, "g"] & & X \arrow[dl, "f"'] \arrow[dr, equal] & \\
S' & & S & & X.
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On the other hand, we may write the large roof $S' \xleftarrow{f'} X' \xrightarrow{g'} X$ as the composition

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Since a functor $D\colon \mathrm{Span}(C, E) \to \mathrm{Cat}_\infty$ encodes all the data on a six-functor formalism... constructing it must be impossible, right?

It's often simple to construct a lax symmetric monoidal functor $C^{op} \to \mathrm{Cat}_\infty$ encoding the functors f^* and $\otimes$.

Suppose there exist classes of morphisms $J, P \subset E$ such that every map $f \in E$ factors as $f = p \circ j$, for some $j \in J$ and $p \in P$. Then, we may try to construct $p_!$ and $j_!$ separately.

- Since we expect that $p_! \simeq p_*$, the functor $p_!$ should be right adjoint to p^*.
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Did base change and the projection formula become *properties*?

In general, base change and the projection formula are relations between the functors $f_!$ and f^*, which need not be adjoint in any direction. When they are, the adjunction provides natural maps that can be checked to be isomorphisms.

For example, if $j_!$ is left adjoint to j^*, the unit $\eta\colon \mathrm{id} \to j^*j_!$ induces a map

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The construction theorem

This leads to the following definition.

Definition (Ayoub, Liu–Zheng, Mann)

A three-functor seed is a lax symmetric monoidal functor $\mathrm{D}\colon \mathrm{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ such that:

- For $j \in J$, the functor j^* admits a left adjoint $j_!$ satisfying base change and the projection formula.
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Theorem (Ayoub, Cisinski–Déglise, Liu–Zheng, Gaitsgory–Rozenblyum, Mann, Dauser–Kuijper, Cnossen–Lenz–Linskens)

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Suppose that C is endowed with a subcanonical Grothendieck topology and that D is a six-functor formalism such that

$$\mathrm{C}^{\mathrm{op}} \to \mathrm{Span}(\mathrm{C}, E) \xrightarrow{\mathrm{D}} \mathrm{Cat}_\infty$$

is a sheaf. In this case we say that D is sheafy.

Theorem (Liu–Zheng, Mann, Scholze, Heyer–Mann)

There exists a “large” collection of maps E' in $\mathrm{X} = \mathrm{Sh}(\mathrm{C}; \mathrm{Ani})$, containing E, such that every sheafy six-functor formalism on (C, E) extends uniquely to (X, E').

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Not the end of the talk, but time for a pause.

Questions?

The six-functor formalism of quasi-coherent sheaves

Proper base change for quasi-coherent sheaves

Consider the following cartesian diagram of affine schemes:

$$\begin{tikzcd} \operatorname{Spec} A \otimes_B B' \arrow[r, "g'"] \arrow[d, "f'"'] & \operatorname{Spec} A \arrow[d, "f"] \\ \operatorname{Spec} B' \arrow[r, "g"'] & \operatorname{Spec} B. \end{tikzcd}$$

In order to have a six-functor formalism of quasi-coherent sheaves, one must have a natural isomorphism

$$\mathrm{L}g^* \mathrm{R}f_! \simeq \mathrm{R}f'_! \mathrm{L}g'^{,*} : \mathrm{D}(A) \to \mathrm{D}(B').$$

There's no natural functor $\mathrm{R}f_!$ for quasi-coherent sheaves, but we could *define* $\mathrm{R}f_!$ to be $\mathrm{R}f_*$ and see what happens...

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Proper base change for quasi-coherent sheaves II

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\begin{tikzcd}
M \otimes^{\mathsf{L}}_A (A \otimes_B B') \arrow[ddd, mapsto] & & & M \arrow[lll, mapsto] \arrow[dddd, mapsto] \\
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It's now clear that the desired isomorphism

$$M \otimes^{\mathsf{L}}_A (A \otimes_B B') \simeq M \otimes^{\mathsf{L}}_B B'$$

holds if either A or B' is a flat B-algebra; then $A \otimes_B B' \simeq A \otimes^{\mathsf{L}}_B B'$.

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Derived geometry to the rescue!

In general, the solution is to *replace* the naive tensor product $A \otimes_B B'$ by $A \otimes^{\mathrm{L}}_B B'$. However, to *do* algebraic geometry with the latter, it best be some sort of ring. ("Classically", we can see $A \otimes^{\mathrm{L}}_B B'$ as an object of either $\mathrm{D}(A)$ or $\mathrm{D}(B')$. Either way, it's some sort of *module*; not a ring.)

This is precisely the problem that *derived algebraic geometry* solves: we study a notion of animated commutative rings, which generalize commutative rings in the same way that objects of $\mathrm{D}(\mathbb{Z})$ generalize abelian groups.

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Derived geometry to the rescue II

Our previous observation shows that the assignment $A \mapsto \mathrm{D}(A)$ satisfies base change. Proving the projection formula is just as easy; we obtain a six-functor seed on

$$\mathcal{C} = \mathrm{AniRing}^{\mathrm{op}},\; P = E = \{\text{all}\},\; J = \{\text{isomorphisms}\}.$$

The construction theorem + the extension theorem then give:

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There exists a six-functor formalism D_{qc} on derived stacks[1].

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Commutative group stacks and 1-motives

Abelian group objects

Definition

Denote by Lat the full subcategory of Ab whose objects are free abelian groups of finite rank. An abelian group object G in an ∞-category C is a functor

$$G \colon \mathrm{Lat}^{\mathrm{op}} \to \mathrm{C}$$

that commutes with finite products. We denote by Ab(C) the ∞-category of such functors.

We think as $G(\mathbb{Z})$ as being the underlying object of G in C. The addition in G is induced by the addition in $\mathbb{Z}$:

$$G(\mathbb{Z}) \times G(\mathbb{Z}) \simeq G(\mathbb{Z} \times \mathbb{Z}) \xrightarrow{\mathrm{G(add)}} G(\mathbb{Z}).$$

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The Dold–Kan correspondence

This definition generalizes every example you might imagine:

$$\mathrm{Ab}(\mathrm{Set}) \simeq \mathrm{Ab}, \quad \mathrm{Ab}(\mathrm{Sch}) \simeq \{\text{commutative group schemes}\}, \cdots .$$

Theorem (Dold–Kan, Quillen, Lurie)

We have an equivalence of ∞-categories

$$\mathrm{Ab}(\mathrm{Ani}) \simeq \mathrm{D}^{\leq 0}(\mathbb{Z}).$$

Under this equivalence, π_i corresponds to H^{-i}.

In particular, this restricts to $\mathrm{Ab}(\mathrm{Set}) \simeq \mathrm{Ab}$ and

$$\begin{aligned} \{\text{two-term complexes in } \mathrm{D}(\mathbb{Z})\} &\xrightarrow{\sim} \mathrm{Ab}(\mathrm{Grpd}) \\ [H \to G] &\longmapsto [G/H] \end{aligned}$$

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“1-motives = commutative group stacks”

Definition

We define a stack as an fppf sheaf of groupoids on Ring, and we denote their ∞-category as Stk. A commutative group stack is an object of Ab(Stk).

Recall that a derived stack is an fppf sheaf of anima on AniRing; hence Stk is naturally a full subcategory of dStk.

Corollary from Dold–Kan (first proved by Deligne)

Abusing notation, denote by Ab(Ring) the category of fppf abelian sheaves on Ring. We have an equivalence:

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The classifying stack of line bundles

Ab(Stk) has a distinguished object: the classifying stack $\mathrm{B}\mathbb{G}_m$.
(Denoting by $(-)^\circ$ the inverse equivalence, this is $[\mathbb{G}_m \to 0]^\circ$.)

For a stack $\mathcal{X}$, a map of stacks $\mathcal{X} \to \mathrm{B}\mathbb{G}_m$ amounts to a line bundle on $\mathcal{X}$. Similarly, for a commutative group stack $\mathcal{G}$ with group law $m\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, we have:

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This discussion leads to the following definition:

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We define the stacky Cartier dual $\mathcal{G}^\vee$ of a commutative group stack $\mathcal{G}$ as $\underline{\mathrm{Hom}}(\mathcal{G}, \mathrm{B}\mathbb{G}_m)$.

This is a commutative group stack whose associated two-term complex is

$$(\mathcal{G}^\vee)^\circ \simeq \tau_{\leq 0}\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{G}^\circ, \mathbb{G}_m[1]).$$

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You may recall this functor from Laumon's 85' preprint.

Théorème 7.3. Le foncteur

$$D(-) = \tau_{\leq 0}\mathrm{R}\underline{\mathrm{Hom}}(-, \mathbb{G}_m[1])$$

induit une dualité parfaite sur la catégorie M_k , qui sera dite de Cartier.

de Rham spaces to the rescue!

This $\mathcal{G}^\vee$ is a moduli space of "multiplicative" line bundles on $\mathcal{G}$. How do we go from line bundles to line bundles with connection?

Theorem (Simpson, '96)

Let X be a smooth algebraic variety. There exists a stack X_{dR} such that

$$\left\{ \begin{array}{c} \text{Line bundles} \\ \text{on } X_{\mathrm{dR}} \end{array} \right\} = \left\{ \begin{array}{c} \text{Line bundles with} \\ \text{flat connection on } X \end{array} \right\}.$$

For a commutative algebraic group G, one can prove that

$$G_{\mathrm{dR}} \simeq [G/\widehat{G}],$$

where $\widehat{G}$ is the formal completion of G at the identity. (I.e., G_{dR} is the stack associated to the 1-motive $[\widehat{G} \to G]$.)

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Computing the stacky Cartier dual

The map $\widehat{G} \to G$ is injective, and so G_{dR} is a usual abelian sheaf. Also, it satisfies $\underline{\mathrm{Hom}}(G_{\mathrm{dR}}, \mathbb{G}_m) = 0$; simplifying the computation of $G_{\mathrm{dR}}^{\vee}$.

Easy lemma

Let $\mathscr{G}$ be an abelian sheaf.

- If $\underline{\mathrm{Hom}}(\mathscr{G}, \mathbb{G}_m) = 0$, then $\mathscr{G}^{\vee} \simeq \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)$.
- If $\underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m) = 0$, then $\mathscr{G}^{\vee} \simeq \mathrm{B}\underline{\mathrm{Hom}}(\mathscr{G}, \mathbb{G}_m)$.

Examples

- For an abelian variety A, we have that $A^{\vee}$ is the dual abelian variety and $A_{\mathrm{dR}}^{\vee} \simeq A^{\natural}$.
- For a torus T with character group X, we have $T^{\vee} \simeq \mathrm{B}X$ and $T_{\mathrm{dR}}^{\vee} \simeq \Omega_T / X$.

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The unipotent case

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Theorem (R.–Rosengarten, Gabber)

Let R be a $\mathbb{Q}$-algebra. Then there is a functorial isomorphism:

$$\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(R) \simeq \mathrm{Pic}(\mathbb{G}_{a,R_{\mathrm{red}}})_{\mathrm{mult}},$$

where the right-hand side denotes the group of multiplicative line bundles on $\mathbb{G}_{a,R_{\mathrm{red}}}$.

These groups vanish if and only if $\operatorname{Spec} R_{\mathrm{red}}$ is seminormal. In particular, it does not vanish for $R = \mathbb{Q}[x, y]/(y^2 - x^3)$.

The Laumon dual

As a result, if G has a unipotent subgroup, the stacky Cartier dual $G^\vee_{\mathrm{dR}}$ is *mysterious*...

In my paper "A Moduli Space of Character Sheaves", I give a simpler construction of the dual of $[\widehat{G} \to G]$ defined by Thomas last time.

Denoting it by $G^\triangleright_{\mathrm{dR}}$, I construct a natural map

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Moral of the story

This operation $\mathcal{G} \mapsto \mathcal{G}^\vee$ exists in extreme generality, and gives *the right results* whenever there's no $\mathbb{G}_a$ in sight. However, it can be difficult to compute.

On the other hand, there is a different operation $\mathcal{G} \mapsto \mathcal{G}^\triangleright$ that only exists when $\mathcal{G}$ is the stack associated to a 1-motive in the sense of Laumon. However, it is easy to compute and always gives *the right result.*

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The Fourier–Mukai transform

Main definitions

Let $\mathcal{G}$ be a *flat*[2] commutative group stack. The evaluation map

$$\mathcal{G} \times \mathcal{G}^\vee \to \mathrm{B}\mathbb{G}_m$$

induces a Poincaré bundle $\mathcal{P}$ on $\mathcal{G} \times \mathcal{G}^\vee$.

This gives rise to the Fourier–Mukai transform

$$\begin{aligned} \mathrm{FT}_{\mathcal{G}} \colon \mathrm{D}_{\mathrm{qc}}(\mathcal{G}) &\to \mathrm{D}_{\mathrm{qc}}(\mathcal{G}^\vee) \\ M &\mapsto \mathrm{pr}_{2,!}(\mathcal{P} \otimes \mathrm{pr}_1^* M), \end{aligned}$$

where pr_i are the projections from $\mathcal{G} \times \mathcal{G}^\vee$.

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Convolution

Denoting by $m\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ the group law, we have a natural *convolution* functor

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- * -\colon \mathrm{D}_{\mathrm{qc}}(\mathcal{G}) \times \mathrm{D}_{\mathrm{qc}}(\mathcal{G}) &\to \mathrm{D}_{\mathrm{qc}}(\mathcal{G}) \\
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The following result becomes *abstract non-sense* using the formalism of correspondences:

Proposition

The operation $*$ defines a symmetric monoidal structure on $\mathrm{D}_{\mathrm{qc}}(\mathcal{G})$ such that

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The Laumon transform

Let $\mathcal{G}$ be the stack associated to a Laumon 1-motive. Denoting by ℓ the natural map $\mathcal{G}^{\triangleright} \to \mathcal{G}^{\vee}$, we define the Laumon transform $\mathsf{LT}_{\mathcal{G}}$ as

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This is still a symmetric monoidal functor, since ℓ^* is. It agrees with the expected Fourier transforms in all examples that I know.

Laumon claims that $\mathsf{LT}_{\mathcal{G}}$ is an equivalence of categories. This sounds very reasonable, as it holds in every example, but he did not prove it.

I *believe* that I know how to do it, but I honestly did not have the time to write it carefully in time for this talk.

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The Laumon transform

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$$\mathsf{D}_{\mathrm{qc}}(\mathcal{G}) \xrightarrow{\mathsf{FT}_{\mathcal{G}}} \mathsf{D}_{\mathrm{qc}}(\mathcal{G}^{\vee}) \xrightarrow{\ell^*} \mathsf{D}_{\mathrm{qc}}(\mathcal{G}^{\triangleright}).$$

This is still a symmetric monoidal functor, since ℓ^* is. It agrees with the expected Fourier transforms in all examples that I know.

Laumon claims that $\mathsf{LT}_{\mathcal{G}}$ is an equivalence of categories. This sounds very reasonable, as it holds in every example, but he did not prove it.

I *believe* that I know how to do it, but I honestly did not have the time to write it carefully in time for this talk.

The moduli space of character sheaves

Character sheaves

Henceforth, let k be a field of characteristic zero and G be a commutative connected algebraic group over k.

Definition

Let S be a k-scheme. A character sheaf on G relative to S is a line bundle $\mathscr{L}$ on G_S equipped with a flat connection ∇ relative to S satisfying

$$m_S^*(\mathscr{L},\nabla) \simeq (\mathscr{L},\nabla) \boxtimes (\mathscr{L},\nabla).$$

In the absolute case $S = \operatorname{Spec} k$, we call such an object a character sheaf on G.

We denote by $\mathrm{H}^1_m(G_{\mathrm{dR}} \times S, \mathbb{G}_m)$ the group of isomorphism classes of character sheaves on G relative to S.

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Is $G^{\triangleright}_{\mathrm{dR}}$ a moduli space of character sheaves?

The Laumon dual $G^{\triangleright}_{\mathrm{dR}}$ has the nice property that $\mathrm{D}_{\mathrm{qc}}(G^{\triangleright}_{\mathrm{dR}})$ *is what it should be* (the target of the Fourier–Mukai transform).

(For $G = \mathbb{G}_a$, we have $G^{\triangleright}_{\mathrm{dR}} \simeq \mathbb{G}_{a,\mathrm{dR}}$ and so $\mathrm{D}_{\mathrm{qc}}(G^{\triangleright}_{\mathrm{dR}}) \simeq \mathrm{D}_{\mathrm{qc}}(\mathcal{D}_{\mathbb{G}_a})$.)

Moreover, it does parametrize character sheaves: for a seminormal k-scheme S, we have a functorial isomorphism

$$G^{\triangleright}_{\mathrm{dR}}(S) \simeq \mathrm{H}^1_m(G_{\mathrm{dR}} \times S, \mathbb{G}_m).$$

It is *not*, however, representable by a scheme or algebraic space. (Recall that $\mathbb{G}_{a,\mathrm{dR}} \simeq \mathbb{G}_a/\widehat{\mathbb{G}}_a$.)

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The moduli space of character sheaves

By somewhat surgically taken away the formal groups inside of $G_{\mathrm{dR}}^{\triangleright}$, one obtains a sheaf that *is* representable.

Theorem (R., '26)

There exists a commutative connected group algebraic space $G^\flat$ satisfying $\dim G \leq \dim G^\flat \leq 2 \dim G$ and a natural isomorphism

$$G^\flat(S) \simeq \mathrm{H}^1_m(G_{\mathrm{dR}} \times S, \mathbb{G}_m)$$

for every seminormal k-scheme S.

Questions?