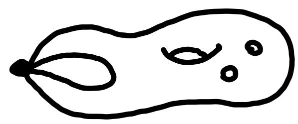
1 - Motives

Fix $k = \bar{k}$ of char $= 0$

1. Motivation: Hodge theory of curves (Deligne, Hodge III)



C nodal curve $/ k$

$\cap$

$\bar{C}$ compactificat^n w/ $S := \bar{C} \setminus S \subset Sm(\bar{C})$

Abel-Jacobi map:

smooth pt^s on $\bar{C}$

$$\begin{tikzcd} \mathbb{Z}^S \arrow[r, "u"] \arrow[dr, "\varphi"'] & \mathrm{Pic}(\bar{C}) \arrow[d, "\deg"] \\ & \mathbb{Z}^I \end{tikzcd}$$

$u : (n_i) \mapsto \mathcal{O}_{\bar{C}}(\sum_i n_i p_i)$

$I := \{\text{irred. cpt}^s \text{ of } \bar{C}\}$

$$\mathrm{Pic}^\circ(\bar{C}) := \ker(\deg)$$

$\rightsquigarrow$ Get a 1-motive

$$\Big[\underbrace{\ker(\varphi)}_{=:\, X} \xrightarrow{u} \underbrace{\mathrm{Pic}^\circ(\bar{C})}_{=:\, G} \Big] =: M(C)$$

Why "motive"?

Prop For $k = \mathbb{C}$,

the MHS on $H^1(C, \mathbb{Z}(1))$

can be functorially recovered from $M(C)$:

a) $\exists$ Cartesian square

$$\begin{tikzcd} H^1(C,\mathbb{Z}(1)) \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near end] & \mathrm{Lie}(G) \arrow[d, "\exp"] \\ X \arrow[r] & G \end{tikzcd}$$

b) $W_0 = H^1(C, \mathbb{Z}(1))$

$\cup$

$W_{-1} = \ker \beta = \ker(\exp) \simeq H_1(G, \mathbb{Z})$

$\cup$

$W_{-2} = H_1(T, \mathbb{Z})$, $T \subset G$ torus part

c) $F^0 = \ker \alpha_{\mathbb{C}} \subset F^{-1} = H^1(X, \mathbb{Z}(1))_{\mathbb{C}}$

Def A (Deligne) <u>1-motive</u> is a cplex

$$M = [X \xrightarrow{u} G] \quad (\text{in fppf sheaves on } Sch_k)$$

X: free abelian gp of finite rank

G: semiabelian variety

Lemma $\exists$ equiv. of cat.

$$(1\text{-motives}/\mathbb{C}) \xrightarrow{\sim} \text{torsionfree MHS of types } (p,q) \text{ w/ } p,q \in \{0,-1\} \text{ w/ polarizable } Gr^W_{-1}$$

Idea of pf.

- $M = [X \to G] \mapsto H_{\mathbb{Z}} := X \times_G \mathrm{Lie}(G)$

- $H_{\mathbb{Z}} \mapsto M = [X \to G]$ $\quad H_1(T,\mathbb{Z}) = Gr^W_{-2} H_{\mathbb{Z}}$

w/

$$\begin{array}{lcl} G & := & W_{-1}H_{\mathbb{Z}} \backslash W_{-1}H_{\mathbb{C}} / F^0 \cap W_{-1}H_{\mathbb{C}} \\ \downarrow & & \\ A & := & H_{\mathbb{Z}} \backslash Gr^W_{-1} H_{\mathbb{C}} / F^0 Gr^W_{-1} H_{\mathbb{C}} \end{array}$$

$\square$

$\underline{Q}$ Pick a MHS $H \longleftrightarrow M = [X \to G]$

then $H' := \mathrm{Hom}(H, \mathbb{Z}(1)) \longleftrightarrow M' = ?$

$\underline{\text{Deligne}}$: For $0 \to T \to G \to A \to 0$

we get $M' = [X' \to G']$

w/ $X' = \underline{\mathrm{Hom}}(T, \mathbb{G}_m)$

$$\begin{array}{ccccccccc} 0 & \to & T' & \to & G' & \to & A' & \to & 0 \\ & & \| & & & & \| & & \\ & & \underline{\mathrm{Hom}}(X, \mathbb{G}_m) & & & & \mathrm{Pic}^0(A) & & \end{array}$$

$\underline{\text{Laumon}}$: Why assume G semiabelian?

Try $0 \to V \times T \to G \to A \to 0 \ldots$

$\lightning$ $\underline{\mathrm{Hom}}(V, \mathbb{G}_m)$ is not represented

by an algebraic gp $\lightning$

2. Cartier duality & formal gps

Ex $\mathfrak{X} := \mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m)$

$\Rightarrow$ For any comm. k-algebra R,

$$\mathfrak{X}(R) = \mathrm{Hom}_{gp/R}(\mathbb{G}_{a,R}, \mathbb{G}_{m,R})$$

$$= \mathrm{Hom}_{\mathrm{Hopf\,alg}/R}(k[t,t^{-1}], k[x])$$

$$= \{ p(x) \in R[x]^* : p(x+y) = p(x)p(y) \;\&\; p(0) = 1 \}$$

$$= \{ p(x) = \exp(ax) \mid a \in \mathrm{Nil}(R) \}$$

$$\simeq \mathrm{Nil}(R)$$

$$= \varinjlim_n \underbrace{\left(\mathrm{Spec}\, k[x]/(x^n)\right)}_{=:\, \mathfrak{X}_n}(R)$$

$$= \widehat{\mathbb{G}}_a(R)$$

<u>Def</u> A <u>formal scheme</u> / k is a functor

$$\mathfrak{X} = \varinjlim_n X_n : R\text{-alg} \to \text{Sets}$$

with all $X_n \in \text{Sch}_k$.

A <u>formal gp</u> / k is a comm. gp object $\mathfrak{X}$ in formal schemes sth

- each X_n is finite / k
- $\mathfrak{X}(k)$ is a fingen. ab. gp
- $\text{Lie}(\mathfrak{X})$ has dim $< \infty$.

<u>Fact</u> Any formal gp $\mathfrak{X}$ decomposes as

$$\mathfrak{X} = \mathfrak{X}^{\circ} \times \mathfrak{X}^{ét} \quad \text{w/} \quad \mathfrak{X}^{et} \simeq \mathbb{Z}^r \times \text{torsion}$$

$$\mathfrak{X}^{\circ} \simeq \hat{\mathbb{G}}_a^s$$

In fact:

$$(\text{connected formal gps}) \xrightarrow{\sim} (\text{fin.dim. v'spaces} / k)$$

$$\mathfrak{X} \longmapsto \text{Lie}(\mathfrak{X})$$

(e.g. $\hat{\mathbb{G}}_a = \mathrm{Spf}\, k[\![x]\!] \xrightarrow[\exp]{\sim} \hat{\mathbb{G}}_m = \mathrm{Spf}\, k[\![t, t^{-1}]\!]$)

Thm (Cartier duality) $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$

gives an involutive equivalence

$$\begin{pmatrix} \text{comm. affine} \\ \text{gps} / k \end{pmatrix} \xleftrightarrow{\sim} \begin{pmatrix} \text{formal} \\ \text{gps} / k \end{pmatrix}$$

$$(\text{connected}) \longleftrightarrow (\text{torsion-free})$$

[e.g. Demazure, Lectures on p-divisible gps]

Concretely:

$$\mathbb{G}_m^r \longleftrightarrow \mathbb{Z}^r$$

$$\mathbb{G}_a^s \longleftrightarrow \hat{\mathbb{G}}_a^s$$

or better:

$$T \longleftrightarrow X^*(T) := \underline{\mathrm{Hom}}(T, \mathbb{G}_m)$$

$$V \longleftrightarrow \underbrace{\hat{W}}_{\text{completion at } 0} \text{ for } W := V^*$$

3. Laumon 1-motives

Def A (<u>Laumon / generalized) 1-motive</u> is a

cplex $M = [\mathfrak{X} \to G]$

$\mathfrak{X}$: torsion free formal gp

G: conn. comm. alg gp

(of fppf sheaves of ab gps on Sch_k)

Prop $\exists$ natural involutive duality functor

$$M = [\mathfrak{X} \xrightarrow{u} G] \longmapsto M' = [\mathfrak{X}' \xrightarrow{u'} G']$$

where for $0 \to L \to G \to A \to 0$:

$$\mathfrak{X}' = \underline{Hom}(L, \mathbb{G}_m)$$

$$0 \to L' \to G' \to A' \to 0$$

$$L' = \underline{Hom}(\mathfrak{X}, \mathbb{G}_m), \qquad A' = \mathrm{Pic}^0(A)$$

Pf.

⓪ From $M : [\mathcal{X} \xrightarrow{u} G]$ we get $\bar{u} : \mathcal{X} \to A$.

① $\mathrm{Hom}(\mathcal{X}, A) \simeq \mathrm{Ext}^1(A', L')$

Via $\mathcal{X} \simeq \mathbb{Z}^r \times \hat{W}$ & $L' \simeq \mathbb{G}_m^r \times V$ $(V := W^*)$

one reduces to:

a) $\mathrm{Hom}(\mathbb{Z}, A) = A \simeq \mathrm{Pic}^\circ(A') \simeq \mathrm{Ext}^1(A', \mathbb{G}_m)$

b) $\mathrm{Hom}(\hat{W}, A) \xrightarrow[\mathrm{Lie}(-)]{\sim} \mathrm{Hom}(W, \mathrm{Lie}(A))$

$$\xrightarrow[\text{dual}]{\sim} \mathrm{Hom}(\underbrace{\mathrm{Lie}(A)^*}_{\Omega_A}, \underbrace{W^*}_{V})$$

$$\xrightarrow{\sim} \mathrm{Ext}^1(A', V)$$

$\varphi \mapsto \varphi_*(A^\natural)$

for the universal vector extension

$0 \to \Omega_A \to A^\natural \to A' \to 0$

② By the above,

- $\bar{u} : \mathfrak{X} \to A$ defines an ext^n $0 \to L' \to G' \to A' \to 0$
- $0 \to L \to G \to A \to 0$ defines a morph $\bar{u}' : \mathfrak{X}' \to A'$

<u>Claim:</u> $\exists$ bijection between

a) lifts $u : \mathfrak{X} \to G$ of $\bar{u}$

b) lifts $u' : \mathfrak{X}' \to G'$ of $\bar{u}'$

Indeed these correspond to splittings:

a)
$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X} \arrow[r] \arrow[d, "\bar{u}"] \arrow[l, dashed, bend right, "u"'] & 0 \\
0 \arrow[r] & L \arrow[r] & G \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Cartier dual

b)
$$\begin{tikzcd}
0 \arrow[r] & L' \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X}' \arrow[r] \arrow[d, "\bar{u}'"] \arrow[l, dashed, bend right, "u'"'] & 0 \\
0 \arrow[r] & L' \arrow[r] & G' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

& Cartier duality sends splittings to splittings. $\square$

$\underline{Ex}$ $M = [\hat{G} \xrightarrow{u} G]$:

formal completion at $1 \in G$

a) $G = V \rightsquigarrow M' = [\hat{W} \to W]$

same shape with $W = V^*$

b) $G = T \rightsquigarrow M' = [X^*(T) \to \Omega_T]$

$$\chi \longmapsto \frac{d\chi}{\chi}$$

c) $G = A \rightsquigarrow M' = [0 \to A^\natural]$

4. Sheaves on 1-motives

<u>Idea</u>: Think of $M = [\mathfrak{X} \to G]$ as $G/\mathfrak{X}$

Sheaves on M $\rightsquigarrow$ equivar. sheaves on G wrt the gp action

$$\begin{tikzcd} G \hat{\times} \mathfrak{X} \arrow[r, "\rho"] \arrow[dr, "\mathrm{id}\times u"'] & G \\ & G\times G \arrow[u, "m"'] \end{tikzcd}$$

<u>Def</u> A qcoh sheaf on $M = [\mathfrak{X} \xrightarrow{u} G]$

is a qcoh sheaf $\mathcal{E}$ on G

together w/ an iso

$\varphi : pr_1^* \mathcal{E} \xrightarrow{\sim} \rho^* \mathcal{E}$ on $G \hat{\times} \mathfrak{X}$

satisfying the cocycle condition

[$\varphi|_{G\times e} = \mathrm{id}$ &

$(\rho \hat{\times} \mathrm{id})^*(\varphi) \circ pr_{12}^*(\varphi) = (\mathrm{id} \hat{\times} m)^*(\varphi)$

on $G \hat{\times} \mathfrak{X} \hat{\times} \mathfrak{X}$]

Prop For $M = [\hat{G} \to G]$:

$$\mathrm{QCoh}(M) \simeq \mathrm{Mod}_{qc}(\mathcal{D}_G).$$

Pf. $\hat{G} \simeq \hat{\mathbb{G}}_a = \mathrm{Spf}\, k[\![x_1, .., x_n]\!]$

w/ $m^*(x_i) = x_i \otimes 1 + 1 \otimes x_i$.

$\varphi : pr_1^* \mathcal{E} \xrightarrow{\sim} g^* \mathcal{E}$

amounts to a map $\psi : \mathcal{E} \to \mathcal{E}[\![x_1, .., x_n]\!]$

$:= \mathcal{E} \hat{\otimes} k[\![x_1, .., x_n]\!]$

sth

① $\mathcal{O}$-linearity:

$\psi(f \cdot s) = m^*(f) \cdot \psi(s)$ for $f \in \mathcal{O}_G$

② Cocycle condition:

- $\psi \equiv \mathrm{id}$ modulo $(x_1, \ldots, x_n)$
- $(\psi \otimes \mathrm{id}) \circ \psi = (\mathrm{id} \hat{\otimes} m^*) \circ \psi$

Concretely:

Write $\psi(s) = \sum_\alpha D_\alpha(s) \otimes x^\alpha$

① $D_\alpha(fs) = \sum_{\beta+\gamma=\alpha} \frac{1}{\beta!} \partial^\beta(f) \cdot D_\gamma(s)$

$m^*(f) = \sum_\beta \frac{1}{\beta!} \partial^\beta(f) \otimes x^\beta$ in $\mathcal{O}_G[\![x_1,..,x_n]\!]$

$\Rightarrow$ Leibniz rule for $\nabla_i := D_{e_i}$:

$$\nabla_i(fs) = \partial_i(f) \cdot s + f \cdot \nabla_i(s)$$

② $(\psi \times id)\psi(s) = \sum_{\beta,\gamma} D_\beta(D_\gamma(s)) \otimes x^\beta \otimes x^\gamma$

$$(id \otimes m^*)\psi(s) = \sum_\alpha D_\alpha(s) \otimes \underbrace{m^*(x^\alpha)}_{= \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} x^\alpha \otimes x^\beta}$$

$$\Rightarrow D_\beta D_\gamma = \binom{\beta+\gamma}{\beta} \cdot D_{\beta+\gamma}$$

Unique solution (char $k = 0$):

$$D_\alpha = \frac{1}{\alpha!} \cdot \nabla_1^{\alpha_1} \cdots \nabla_n^{\alpha_n}$$

Note: $\beta + \gamma = \gamma + \beta$

$\Rightarrow D_\beta D_\gamma = D_\gamma D_\beta$

$\Rightarrow \nabla$ gives a flat connection

(as $[\partial_i, \partial_j] = 0$ since $\mathrm{Lie}(G)$ is abelian) $\square$

<u>Thm</u> (Laumon) For any 1-motive M,

$\exists$ FM trafo

$$\mathcal{F} : D^b_{qc}(M) \xrightarrow{\sim} D^b_{qc}(M')$$

(involutive up to shift and $(-\mathrm{id})^!$

& intertwining f_* w/ $f^!$ up to shift)

Ex Take $M = [\hat{G} \to G]$:

a) $G = V$:

$$\mathcal{F} : D^b_{qc}(\mathcal{D}_V) \xrightarrow{\sim} D^b_{qc}(\mathcal{D}_W), \quad W = V^*$$

b) $G = T$:

$$\mathcal{F} : D^b_{qc}(\mathcal{D}_T) \xrightarrow{\sim} D^b_{qc}(\Delta_T)$$

via

$QCoh(\Delta_T) =$ qcoh sheaves on $Lie(T)^*$ equivariant wrt $X^*(T)$ acting by translations

c) $G = A$:

$$\mathcal{F} : D^b_{qc}(\mathcal{D}_A) \xrightarrow{\sim} D^b_{qc}(\mathcal{O}_{A^\natural}).$$