Fourier transform for D-modules on a connected, commutative, alg group / $\mathbb{C}$

Motivation

let J = com, conn, alg gp / $\mathbb{C}$

$\Omega_J := \{$translation-invariant 1-forms on $J\} \simeq T_0^\vee J$

$J^\natural := \{$line bdles w/ flat con (L, ∇) satisfying thm of the square$\}$

ie if
$$\begin{tikzcd} J \times J \arrow[r, "m"] \arrow[d, "p_1"'] \arrow[rd, "p_2"] & J \\ J & J \end{tikzcd} \qquad p_1^*(L,\nabla) \otimes p_2^*(L,\nabla) \simeq m^*(L,\nabla)$$

Fact we have exact sequence

$$0 \to V \times T \to J \to A \to 0$$

with A abelian variety

$V \simeq \mathbb{G}_a^k$

$T \simeq \mathbb{G}_m^\ell$

in previous talks, we computed $A^\natural$, $V^\natural$, $T^\natural$

today: construct $J^\natural$ together with universal object

$$(\mathcal{P}, \nabla) \text{ on } \begin{tikzcd} & J \times J^\natural \arrow[dl, "p"'] \arrow[dr, "q"] & \\ J & & J^\natural \end{tikzcd}$$

define Fourier as $F(M) = Rq_* \, DR_{J \times J^\natural / J^\natural}\, p^* M \otimes (\mathcal{P}, \nabla)$

1) Preliminaries on extensions (Serre Alg gps & class fields)

$k = \mathbb{C}$

A, B commutative alg gps $/\mathbb{C}$

$$\mathrm{Ext}(A,B) = \left\{ \begin{array}{l} \text{extensions } 0 \to B \to X \to A \to 0 \\ \text{w/ } X \text{ commutative} \end{array} \right\} /\sim$$

Rq: If $X \in \mathrm{Ext}(A,B)$
X has unique alg gp structure compatible w $0 \to B \to X \to A \to 0$

two extensions are equivalent if

$$\begin{array}{ccccccccc} 0 & \to & B & \to & X & \to & A & \to & 0 \\ & & \| & & \exists \wr\!\downarrow \psi & & \| & & \\ 0 & \to & B & \to & X' & \to & A & \to & 0 \end{array}$$

$\mathrm{Ext}(A,B)$ functorial in A and B:

if $\sigma \in \mathrm{Hom}(A', A)$, $\quad \sigma^* : \mathrm{Ext}(A,B) \to \mathrm{Ext}(A',B)$ pullback

$$X \longmapsto X \times_A A'$$

$u \in \mathrm{Hom}(B, B')$ $\quad u_* : \mathrm{Ext}(A,B) \to \mathrm{Ext}(A,B')$ pushout

$$X \longmapsto X' := X \oplus B' / B \quad \leftarrow \text{antidiag}$$

Additive structure on $\mathrm{Ext}(A,B) \ni X, X'$

$$\begin{array}{ccccccccc} 0 \to & A \oplus A & \to & X \oplus X' & \to & B \oplus B & \to 0 \\ & \delta \uparrow & & & & \downarrow s & \\ & A & & & & B & \end{array}$$

$$X + X' := s_* \delta^* (X \oplus X')$$

2) Basic facts on extensions of com alg gps

- If $A, B \in \{\mathbb{G}_a, \mathbb{G}_m\}$ $\quad \mathrm{Ext}(A,B) = 0$ $\quad$ ⚠ If char $k>0$ $\mathrm{Ext}(\mathbb{G}_a, \mathbb{G}_a) \neq 0$

- If A abelian variety $\quad \mathrm{Ext}(A, \mathbb{G}_m) \simeq \hat{A} := \mathrm{Pic}^0(A)$ $\quad (1)$

$$\mathrm{Ext}(A, \mathbb{G}_a) \simeq T_0\hat{A} \simeq \Omega^*_{\hat{A}} \quad (2)$$

Isomorphism (1) given by $\mathcal{L} \in \mathrm{Pic}^0(A) \mapsto$ $\mathcal{L} \setminus \{\text{zero section}\} \xrightarrow{\mathbb{C}^\times} A$

(2) $\hat{A}^\natural$ is the universal vector extension

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega_{\hat{A}} \arrow[r, "\varphi"] \arrow[d] & \mathbb{G}_a \arrow[d] \\
\hat{A}^\natural \arrow[r] \arrow[d] & X \arrow[d] \\
A \arrow[r, equal] \arrow[d] & A \arrow[d] \\
0 & 0
\end{tikzcd}$$

Idea of proof: For $B = \mathbb{G}_m$ or $B = \mathbb{G}_a$ $\qquad$ sheaf $U \mapsto \mathrm{Mor}(U, B)$

$$\mathrm{Ext}(A,B) \hookrightarrow \{\text{Principal } B\text{-bundles over } A\}/\sim \;\simeq H^1(A, \underline{B})$$

$\simeq$

If $B = \mathbb{G}_m$, $\underline{B} = \mathcal{O}_A^*$, $H^1(A, \mathcal{O}_A^*) \simeq \mathrm{Pic}(A)$

Fact: Bundle has a group structure iff $\mathcal{L} \in \mathrm{Pic}^0(A)$

If $B = \mathbb{G}_a$, $\underline{B} = \mathcal{O}_A$ $\quad H^1(A, \mathcal{O}_A) \simeq H^0(A, \Omega^1_A) \simeq \Omega_A \simeq \Omega^*_{\hat{A}}$

<u>Summary</u> If J is a connected commutating alg gp $/\mathbb{C}$, then

$$0 \to V \times T \to J \to A \to 0 \quad \text{w/ } V \simeq \mathbb{G}_a^k \quad T \simeq \mathbb{G}_m^\ell$$

3) The extension by a vector space

Suppose $0 \to V \to J \to A \to 0$
vec sp. a.v.

We have the following commutative diag

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_A \arrow[r] \arrow[d] & \Omega_J \arrow[r] \arrow[d] & \Omega_V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A^\natural \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & J^\natural \arrow[r] \arrow[d] & V^\natural \arrow[r] \arrow[d] & 0 \\
 & \hat{A} \arrow[r, equal] \arrow[d] & \hat{A} \arrow[d, "\leftarrow \text{surjectivity by commutativity of } 1"] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

forgetful map

$\mathrm{Pic}(J) \simeq \mathrm{Pic}(A) = \hat{A}$

middle row exact by four lemma:

$$\begin{tikzcd}
 & A \arrow[r, two heads] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, hook] \arrow[d] & B' \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C' \arrow[d] & \\
0 \arrow[r] & D \arrow[r, hook] & D' &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
 & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & D \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & E' &
\end{tikzcd}$$

exact columns

Moreover, by the universal prop. of univ extension, we have

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega_{\hat{A}} \arrow[r, dashed] \arrow[d] & V \arrow[d] \\
\hat{A}^\natural \arrow[r, dashed, "\exists!"] \arrow[d] & J \arrow[d] \\
A \arrow[r, equal] \arrow[d] & A \arrow[d] \\
0 & \,
\end{tikzcd}$$

taking duals, we get $\varphi^*: \Omega_J \to \Omega_{A^\natural} = H^1_{DR}(\hat{A})^*$
$= H^1_{DR}(A) =$
$\simeq \mathrm{Lie}(A^\natural)$

the restric° to $\Omega_A \subset \Omega_J \to H^1_{DR}(A)$
$\cup$ Hodge dec
Ω_A

This induces a canonical splitting

$$0 \to \mathrm{Lie}(A^\natural) \to \mathrm{Lie}(J^\natural) \to \mathrm{Lie}(V^\natural) \to 0$$

Recall:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \Omega_A \arrow[r] \arrow[d] & \Omega_J \arrow[r] \arrow[d] \arrow[l, dashed, bend left, "(1)"] & \Omega_V \\
 & \mathrm{Lie}\, A^\natural \arrow[r] \arrow[d] & \mathrm{Lie}\, J^\natural \arrow[d] \arrow[l, dashed, bend left, "(2)"] & \\
 & \mathrm{Lie} A \arrow[r, equal] \arrow[d] & \mathrm{Lie} A \arrow[d] \arrow[l, bend left, "\nu"] & \\
 & 0 & 0 &
\end{tikzcd}$$

In particular, the subring

$$\underset{\mathcal{D}_{J^\natural/A^\natural}}{\underset{\parallel}{G_{J^\natural} \otimes_k \mathrm{Sym}_k(V^\vee)}} = G_{J^\natural} \otimes_k \mathrm{Lie}(V^\natural) \subset G_{J^\natural} \otimes_k \mathrm{Sym}_k \mathrm{Lie}\, J^\natural = \mathcal{D}_{J^\natural}$$

is well defined

Universal object on $J \times J^\natural$?

$$\downarrow$$

- We have universal 1-form $\tilde{\omega}$ on $J \times V^\natural$ obtained by composing

$$\omega_V : V^\natural \to \Omega_V \hookrightarrow \Omega_J$$

using splitting constructed above

- $\tilde{\mathcal{P}}$ Poincaré bundle obtained by pulling back $\mathcal{P}$ via

$$J \times J^\natural \to A \times \hat{A}$$

$\exists$ cover $U \subset \hat{A}$ st $J^\natural|_U \cong A^\natural|_U \times V^\natural$

define $\tilde{\nabla} : \tilde{\mathcal{P}} \to \Omega_{J \times J^\natural / A^\natural} \otimes \tilde{\mathcal{P}}$ locally as $\nabla_A \boxtimes \tilde{\omega}$

Then show they glue together using universal property

4) Extension by T

Situation: $\quad 0 \to T \to J \to A \to 0$

let $\quad Y := \mathrm{Hom}_{k\text{-gp}}(J, \mathbb{G}_m) \hookrightarrow X := \mathrm{Hom}(T, \mathbb{G}_m)$

$$\hat{J} := \mathrm{Ext}^1(J, \mathbb{G}_m)$$

long exact sequence of Ext: $\quad 0 = \mathrm{Hom}(A, \mathbb{G}_m) \to Y \to \cancel{X} \to \underset{\mathrm{Ext}(A,\mathbb{G}_m) =}{\hat{A}} \to \hat{J} \to \underset{\mathrm{Ext}(T,\mathbb{G}_m) =}{0}$

We have:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \arrow[dddll, bend left, red] & \\
0 \arrow[r] & \Omega_A \arrow[r] \arrow[d] & \Omega_J \arrow[r] \arrow[d] & \Omega_T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^\natural \arrow[r] \arrow[d] & J^\natural \arrow[r] \arrow[d] & T^\natural \arrow[r] \arrow[d] & 0 \\
 & \hat{A} \arrow[r] \arrow[d] & \hat{J} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Let $\quad K := \Omega_J \oplus A^\natural / \Omega_A \qquad$ (pushout of $A^\natural$ by $\Omega_A \to \Omega_J$)

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X/Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_J \arrow[r] & K \arrow[r] & \hat{A} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "x"'] & J \arrow[r] \arrow[d] & A \arrow[d, equal] & \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] & L_x \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Construction of $X \to K$: $\quad x \in X$ induces by pushout

$\mathcal{L}_x \in \hat{A}$ corresponding line bdle

choose $\quad \omega \in \Omega_{L_x} \quad$ st $\quad \omega|_{\mathbb{G}_m} = \frac{dx}{x} \qquad$ (well def up to Ω_A)

then $\exists!$ connection ∇ on $\mathcal{L}_x$ st $\quad (\mathcal{L}_x, \nabla)|_{L_x} \cong (G_{L_x}, d + \omega)$

then $\quad X \to K$

$\quad x \mapsto (\mathcal{L}_x, \nabla), \omega|_J \in A^\natural \oplus \Omega_J / \Omega_A$

Definition of the universal object

$\hat{\mathcal{P}}$ on $J \times K$

$\tilde{\mathcal{P}}$ pullback from $J \times \hat{J}$ of Poincaré bdle

- connection $\tilde{\nabla}$ defined locally (over $\hat{A}$) by

$U \subset \hat{A}$ st $\quad K_{|U} \cong (\Omega_A \times U \oplus \Omega_J)/_{\Omega_A}$

on $J \times \Omega_J$ we have $\tilde{\omega}$ universal 1-form

on $J \times A^\natural_{|U}$ we pull back $(\mathcal{P}_A, \nabla_A)$ via $J \times A^\natural \to A \times A^\natural$

then $(\tilde{\mathcal{P}}, \tilde{\nabla}) := \left((\mathcal{P}_A, \nabla_A)_{|J \times A^\natural_{|U}} \boxtimes (\mathcal{O}_{J \times \Omega_J}, \tilde{\omega}) \right) / \Omega_A$

Ω_A-invariant, descends to quotient

- We have an action on $\tilde{\mathcal{P}}$ compatible with $X \Rightarrow K$

5) Extension by $V\times T$

Now suppose

$$0 \to V\times T \to J \times A \to 0 \qquad (1)$$

by pushout we get two extensions

$$0 \to V \to J_V \to A \to 0$$
$$0 \to T \to J_T \to A \to 0$$

We obtain (1) by pullback via diagonal $A \hookrightarrow A\times A$

$$\begin{tikzcd} 0 \arrow[r] & V\times T \arrow[r] & J_V \times J_T \arrow[r] & A\times A \arrow[r] & 0 \\ 0 \arrow[r] & V\times T \arrow[r] \arrow[u, equal] & J \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

let $K := J_V^\natural \times K_T / A^\natural$ ← diagonal embedding

def previously

We have $V^\vee \to \mathrm{Lie}(J_V^\natural) \to \mathrm{Lie}(K)$

$$X_T \to K_T \longrightarrow K$$

let $\mathcal{D}_{K/(A^\natural, \Omega_T)} := \mathcal{O}_K \otimes_k \mathrm{Sym}_k(V^\vee)$

$$J^\natural = \text{"} K / X_T \text{"}$$

<u>Def</u> A q.c. $\mathcal{D}_{J^\natural/A^\natural, T^\natural}$-module is q.c. $\mathcal{D}_{K/A^\natural, \Omega_T}$-module

together with action of X_T compatible with $X_T \to K$

We have on $J\times K$ a universal object $\tilde{\mathcal{P}}$ with

- relative connection $\tilde{\nabla}$
- action of X_T

$$\begin{array}{ccc} & J \times K & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \\ J & & K \end{array}$$

$$F: \quad D^b_{qc}(\mathcal{D}_J) \longrightarrow D^b_{qc}(\mathcal{D}_{J^\natural}/A^\natural, T^\natural)$$

$$M \longmapsto Rq_* \, DR_{J \times K / K}(p^* M \otimes \tilde{\mathcal{S}})[g]$$

$$F': \quad D^b_{qc}(\mathcal{D}_{J^\natural}/A^\natural, T^\natural) \to D^b_{qc}(\mathcal{D}_J)$$

$$M \longmapsto R p_*^{\times_+} \, DR_{J \times K / J \times (A^\natural, \Omega_+)} \; q^* M \otimes \tilde{\mathcal{S}}$$

Representable functors (Alg geom I Görtz-Wedhorn Chap 8)

Let $\mathscr{C}$ be the category of schemes $/S$

$\hat{\mathscr{C}} :=$ category of functors $F : \mathscr{C}^{op} \to set$

By Yonedas lemma we have an embedding

$$\mathscr{C} \hookrightarrow \hat{\mathscr{C}}$$
$$X \mapsto Hom_S(-, X)$$

Recall: Yoneda's Lemma

The map

$$Hom_{\hat{\mathscr{C}}}(h_X, F) \to F(X)$$
$$\alpha \longmapsto \alpha(X)(Id_X)$$

is bijective

Def A functor $F : \mathscr{C}^{op} \to Sets$ is representable if $\exists X \in \mathscr{C}$ and isomorphism $\xi : h_X \xrightarrow{\sim} F$

Def (Open subfunctor) An open subfunctor F' of F is morphism $\eta : F' \to F$ st $\forall X \in \mathscr{C}$, $F' \times_F h_X$ is representable (by say X') and

$$X' \simeq F' \times_F h_X \xrightarrow{pr_X} X \quad \text{is open immersion}$$

the fiber product $F(U) \times_{H(U)} G(U)$ is the set-theoretic fiber product $\forall F, G, H \in \hat{\mathscr{C}}$

Def (Open cover)

$(F_i)_{i\in I}$ is a (Zariski) open cover of F if

$F_i \hookrightarrow F$ open subfunctor $\forall i$

$\forall X \in \mathcal{C}$ $\left(X_i := F_i \times_F X \right)_{i\in I}$ is a (Zariski) open cover of X

Thm Let $F \in \hat{\mathcal{C}}$ functor st

(a) F is a sheaf in Zariski topology

(b) $\exists (F_i)_{i\in I}$ open cover of F with F_i representable

Then F is representable

Def (Sheaf in Zariski top)

$F \in \hat{\mathcal{C}}$ such that $\forall X \in \mathcal{C}$, $\forall (U_i) \to X$ cover

given $\xi_i \in F(U_i)$ with $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$

$\Rightarrow \exists! \, \xi \in F(X)$ with $\xi|_{U_i} = \xi_i$

Principal G-bundles

<u>Def</u> (G fibration) $\pi : E \to X$ with $G \circlearrowright E$ st π is G-equivariant

principal G-bundle if $\exists\, U_i$ covering of X (in étale topology)

st $E|_{U_i} \simeq G \times U_i$

Let $\underline{G} :=$ Sheaf $U \to Mor(U, G)$

Then $\{$principal G-bundles $/X\}/$isomorphism $\simeq H^1_{et}(X, \underline{G})$

$\left\{\begin{matrix}\text{principal G-bundles} \\ \text{trivial in Zariski top}\end{matrix}\right\}/\sim \;\; \simeq H^1_{Zar}(X, \underline{G})$

Fact (Grothendieck) If $G = Gl_n$, Sl_n or Sp_{2n}, then

$$H^1_{et}(X, \underline{G}) \simeq H^1_{Zar}(X, \underline{G})$$