

Fourier transform for D-modules on a
connected, commutative, alg group / \mathbb{C}

①

Motivation

let $J = \text{con, conn, alg gp} / \mathbb{C}$

$$\Omega_J := \{ \text{translation-invariant 1-forms on } J \} \cong T_0^* J$$

$J^{\natural} := \{ \text{line bdlrs w/ flat con } (L, \nabla) \text{ satisfying thm of the square} \}$

ie if
$$\begin{array}{ccc} J \times J & \xrightarrow{m} & J \\ \downarrow p_1 & & \downarrow p_2 \\ J & & J \end{array}$$

$$p_1^*(L, \nabla) \otimes p_2^*(L, \nabla) \cong m^*(L)$$

Fact we have exache sequence

$$0 \rightarrow V \times T \rightarrow J \rightarrow A \rightarrow 0$$

with A abelian variety

$$V \cong \mathbb{G}_a^k$$

$$T \cong \mathbb{G}_m^{\ell}$$

in previous talks, we computed $A^{\natural}, V^{\natural}, T^{\natural}$

today: construct J^{\natural} together with universal object

$$\begin{array}{ccc} (\mathcal{E}, \nabla) \text{ on } & J \times J^{\natural} & \\ & \downarrow p_1 & \downarrow p_2 \\ & J & J^{\natural} \end{array}$$

define Fourier as
$$F(\mathcal{M}) = Rq_* DR_{J \times J^{\natural} / J^{\natural}} p^* \mathcal{M} \otimes (\mathcal{E}, \nabla)$$

1) Preliminaries on extensions (Serre Alg gps & class fields) (2)

$k = \mathbb{C}$

A, B commutative alg gps / \mathbb{C}

$$\text{Ext}(A, B) = \left\{ \begin{array}{l} \text{extensions } 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0 \\ \text{w/ } X \text{ commutative} \end{array} \right\} / \sim$$

two extensions are equivalent if

$$\begin{array}{c} 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0 \\ \quad \quad \quad \uparrow \quad \downarrow \\ 0 \rightarrow B \rightarrow X' \rightarrow A \rightarrow 0 \end{array}$$

Rq: If $X \in \text{Ext}(A, B)$

X has unique alg gp struct compatible w $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$

$\text{Ext}(A, B)$ functorial in A and B :

if $\sigma \in \text{Hom}(A', A)$, $\sigma^* : \text{Ext}(A, B) \rightarrow \text{Ext}(A', B)$ pullback

$$X \mapsto X \times_A A'$$

$u \in \text{Hom}(B, B')$

$u_* : \text{Ext}(A, B) \rightarrow \text{Ext}(A, B')$ pushout

$$X \mapsto X' := X \oplus B' / B \leftarrow \text{antidiag}$$

Additive structure on $\text{Ext}(A, B) \ni X, X'$

$$\begin{array}{ccccccc} 0 & \rightarrow & A \oplus A & \rightarrow & X \oplus X' & \rightarrow & B \oplus B \rightarrow 0 \\ & & \uparrow \delta & & & & \downarrow \delta \\ & & A & & & & B \end{array}$$

$$X + X' := s_* \delta^*(X \oplus X')$$

2) Basic facts on extensions of com alg gps

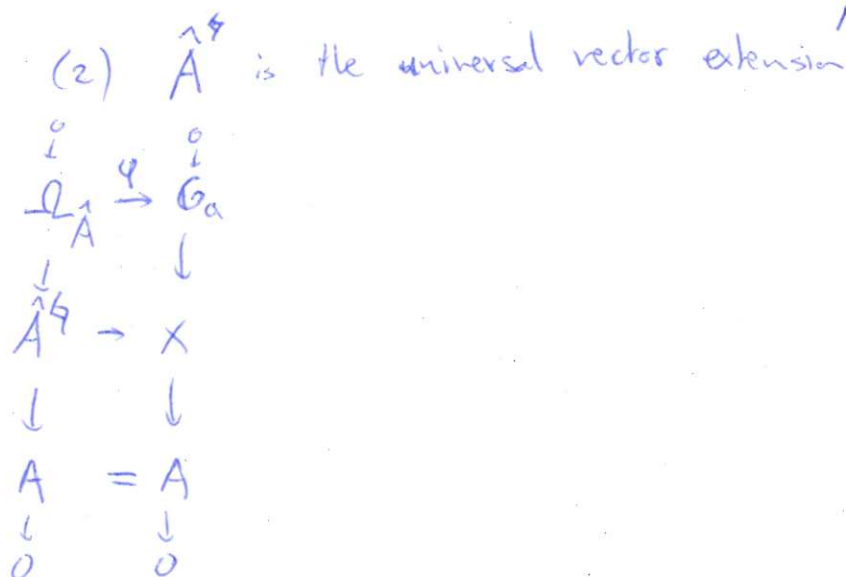
(3)

• If $A, B \in \{G_a, G_m\}$ $\text{Ext}(A, B) = 0$ ⚠ \neq char > 0 $\text{Ext}(G_a, G_a) \neq 0$

• If A abelian variety $\text{Ext}(A, G_m) \cong \hat{A} := \text{Pic}^0(A)$ (1)

$\text{Ext}(A, G_a) \cong T_0 \hat{A} \cong \Omega_A^*$ (2)

Isomorphism (1) given by $\mathcal{L} \in \text{Pic}^0(A) \mapsto X \setminus \{\text{zero section}\}$



Idea of proof: For $B = G_m$ or $B = G_a$ sheaf $U \mapsto \text{Mor}(U, B)$

$$\text{Ext}(A, B) \longleftrightarrow \{ \text{Principal } B\text{-bundles over } A \} / \sim \cong H^1(A, \underline{B})$$

If $B = G_m$, $\underline{B} = G_A^*$, $H^1(A, G_A^*) \cong \text{Pic}(A)$

Fact: Bundle has a group structure iff $\mathcal{L} \in \text{Pic}^0(A)$

If $B = G_a$, $\underline{B} = G_A$ $H^1(A, G_A) \cong H^0(A, \Omega_A^*) \cong \Omega_A \cong \Omega_{\hat{A}}^*$

Summary If J is a connected commutative alg gp / \mathbb{C} , then

$$0 \rightarrow V \times T \rightarrow J \rightarrow A \rightarrow 0 \quad \text{w/ } V \cong G_a^{rk} \quad T \cong G_m^d$$

3) The extension by a vector space

(4)

Suppose $0 \rightarrow V \rightarrow J \rightarrow A \rightarrow 0$
 resp. a.v.

We have the following commutative diag

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \rightarrow & \Omega_A & \rightarrow & \Omega_J & \rightarrow & \Omega_V \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & A^{\wedge 2} & \rightarrow & J^{\wedge 2} & \rightarrow & V^{\wedge 2} \rightarrow 0 \\
 & & \downarrow \textcircled{1} & & \downarrow & & \downarrow \\
 & & \hat{A} & = & \hat{A} & & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

forgetful map

$\text{Pic}(J) \cong \text{Pic}(A) = \hat{A}$

\downarrow ← surjectivity
 \circ by commutativity of $\textcircled{1}$

middle row exact by four lemma:

$$\begin{array}{ccc}
 A \twoheadrightarrow A' \rightarrow 0 & & B \rightarrow B' \rightarrow 0 \\
 \downarrow & \downarrow & \downarrow \\
 0 \rightarrow B \hookrightarrow B' & & C \rightarrow C' \rightarrow 0 \\
 \downarrow & \downarrow & \downarrow \\
 0 \rightarrow C \rightarrow C' & & D \rightarrow D' \rightarrow 0 \\
 \downarrow & \downarrow & \downarrow \\
 0 \rightarrow D \hookrightarrow D' & & 0 \rightarrow E \rightarrow E'
 \end{array}$$

exact columns

Moreover, by the universal prop. of univ. extension, we have

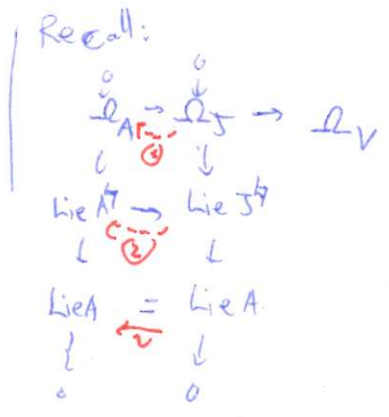
$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \Omega_A & \dashrightarrow & V \\
 \downarrow & & \downarrow \\
 A^{\wedge 2} & \xrightarrow{\cong} & J^{\wedge 2} \\
 \downarrow & & \downarrow \\
 A & = & A \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

taking duals, we get $\varphi^*: \Omega_J \rightarrow \Omega_A^{\wedge 2} = H_{DR}^1(\hat{A})^*$
 $= H_{DR}^1(A) =$
 $\cong \text{Lie}(A^{\wedge 2})$

The restricⁿ to $\Omega_A \subset \Omega_J \rightarrow H_{DR}^1(A)$
 $\cong \bigcup_{S \subset A} \text{Hodge dec}$

This induces a canonical splitting

$$0 \rightarrow \text{Lie}(A^{\wedge k}) \rightarrow \text{Lie}(J^k) \rightarrow \text{Lie}(V^{\wedge k}) \rightarrow 0$$



In particular, the splitting

$$G_{J^k} \otimes_{\mathbb{R}} \text{Sym}_k(V^{\vee}) = G_{J^k} \otimes_{\mathbb{R}} \text{Lie}(V^{\wedge k}) \subset G_{J^k} \otimes_{\mathbb{R}} \text{Sym}_k \text{Lie } J^k = \mathcal{D}_{J^k}$$

!!
 $\mathcal{D}_{J^k}/A^{\wedge k}$ is well defined

Universal object on $J \times J^k$?

- We have universal 1-form $\tilde{\omega}$ on $J \times V^{\wedge k}$ obtained by composing

$$\omega_V: V^{\wedge k} \rightarrow \Omega_V \xrightarrow{\quad} \Omega_J$$

↑
using splitting constructed above

- $\tilde{\mathcal{P}}$ Poincaré bundle obtained by pulling back \mathcal{P} via

$$\begin{aligned}
 J \times J^k &\rightarrow A \times \hat{A} \\
 \text{J covers } U \subset \hat{A} &\text{ st } J^k|_U \cong A^{\wedge k}|_U \times V^{\wedge k}
 \end{aligned}$$

define $\tilde{\mathcal{D}}: \tilde{\mathcal{P}} \rightarrow \mathcal{D}_{J \times J^k}/A^{\wedge k} \otimes \hat{\mathcal{P}}$ locally as $\nabla_A \boxtimes \tilde{\omega}$

Then show they glue together using universal property

4) Extension by T

(6)

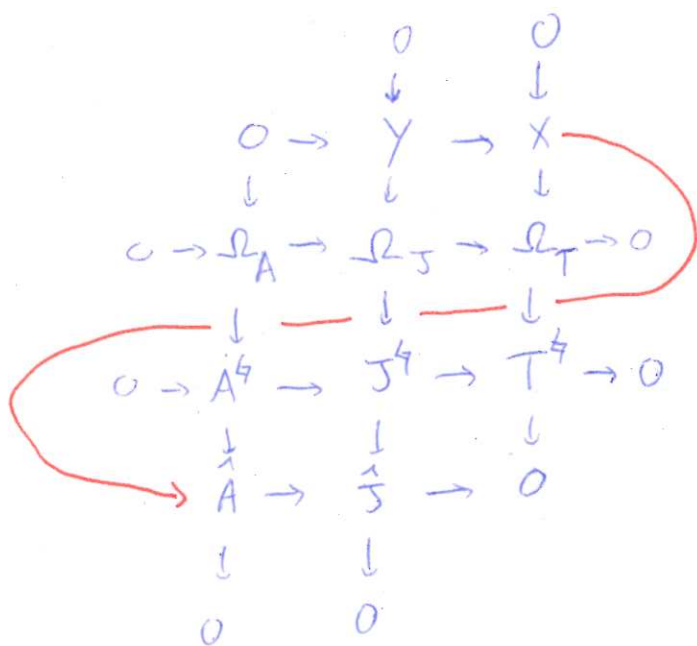
Situation: $0 \rightarrow T \rightarrow J \rightarrow A \rightarrow 0$

let $Y := \text{Hom}_{k\text{-gr}}(J, G_m) \hookrightarrow X := \text{Hom}(T, G_m)$

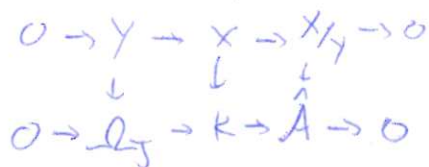
$\hat{J} := \text{Ext}^1(J, G_m)$

long exact sequence of Ext: $0 = \text{Hom}(A, G_m) \rightarrow Y \rightarrow X \rightarrow \hat{A} \rightarrow \hat{J} \rightarrow 0$

We have:



Let $K := \Omega_J \oplus A^{\wedge 2} / \Omega_A$ (pushout of $A^{\wedge 2}$ by $\Omega_A \rightarrow \Omega_J$)



Construction of $X \rightarrow K$: $X \in X$ induces by pushout $L_X \subset \hat{A}$ corresponding line bundle

choose $w \in \Omega_{L_X}$ st $w|_{G_m} = \frac{dx}{x}$ (well def up to Ω_A)

then $\exists!$ connection ∇ on L_X st $(L_X, \nabla)|_{L_X} \cong (G_{L_X}, d+w)$

then $X \rightarrow K$
 $\ast \mapsto (L_X, \nabla), w|_T \in A^{\wedge 2} \oplus \Omega_J / \Omega_A$

Definition of the universal object

\tilde{P} on $J \times K$

\tilde{P} pullback from $J \times \hat{J}$ of Poincaré bundle

- connection $\tilde{\nabla}$ defined locally (over \hat{A}) by

UCA st
$$K|_U \cong (\mathcal{P}_A \times U \oplus \Omega_J) / \Omega_A$$

on $J \times \mathcal{P}_J$ we have $\tilde{\omega}$ universal 2-form

on $J \times A^{\hat{A}}|_U$ we pullback $(\mathcal{P}_A, \nabla_A)$ via $J \times A^{\hat{A}} \rightarrow A \times A^{\hat{A}}$

Then $(\tilde{P}, \tilde{\nabla}) := (\mathcal{P}_A, \nabla_A) /_{J \times A^{\hat{A}}|_U} \boxtimes (\mathcal{G}_{J \times \Omega_J}, \tilde{\omega}) / \Omega_A$

\uparrow
 Ω_A -invariant, descends to quotient

- We have an action on \tilde{P} compatible with $X \rightarrow K$

$J \times K$
 $P \leftarrow \cup$
 $J \quad K$

$F: D_{qc}^b(\mathcal{D}_J) \rightarrow D_{qc}^b(\mathcal{D}_{J^h} / A^h, T^h)$
 $M \longmapsto Rq_* DR_{J \times K / K} (p^* M \otimes \tilde{\mathcal{S}}) \in \mathcal{D}_J$

$F': D_{qc}^b(\mathcal{D}_{J^h} / A^h, T^h) \rightarrow D_{qc}^b(\mathcal{D}_J)$
 $M \longmapsto R_{P^*}^x DR_{J \times K} / J \times (A^h, T^h) \quad q^* M \otimes \tilde{\mathcal{S}}$

Representable functors

(Alg gem I Götz-Wedhorn)
Chap 8

Let \mathcal{C} be the category of schemes / S

$\hat{\mathcal{C}} :=$ category of functors $F: \mathcal{C}^{op} \rightarrow \text{Set}$

By Yoneda's lemma we have an embedding

$$\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$$

$$X \mapsto \text{Hom}_S(-, X)$$

Recall: Yoneda's Lemma

The map

$$\text{Hom}_{\hat{\mathcal{C}}}(\text{h}_X, F) \rightarrow F(X)$$

$$\alpha \mapsto \alpha(X)(\text{Id}_X)$$

is bijective

Def A functor $F: \mathcal{C}^{op} \rightarrow \text{Sets}$ is representable if $\exists X \in \mathcal{C}$ and isomorphism $\xi: \text{h}_X \xrightarrow{\sim} F$

Def (Open subfunctor) An open subfunctor F' of F is

morphism $\eta: F' \rightarrow F$

(by say X')

st $\forall X \in \mathcal{C}$, $F'_X \times_F \text{h}_X$ is representable and

$X' \simeq F'_X \times_F \text{h}_X \xrightarrow{\text{pr}_X} X$ is open immersion

the fiber product

$F(U) \times_{H(U)} G(U)$ is the

set-theoretic fiber product

$\forall F, G, H \in \hat{\mathcal{C}}$

Def (Open cover)

$(F_i)_{i \in I}$ is a (Zariski) open cover of F if

$F_i \hookrightarrow F$ open subfunctors $\forall i$

$\forall X \in \mathcal{C} \quad (X_i := F_i \times_F X)_{i \in I}$ is a (Zariski) open cover of X

Thm Let $F \in \hat{\mathcal{C}}$ functor st

(a) F is a sheaf in Zariski topology

(b) $\exists (F_i)_{i \in I}$ open cover of F with F_i representable

Then F is representable

Def (Sheaf in Zariski top)

$F \in \hat{\mathcal{C}}$ such that $\forall X \in \mathcal{C}, \forall (U_i) \rightarrow X$ cover

given $\mathcal{F}_i \in F(U_i)$ with $\mathcal{F}_i|_{U_{ij}} = \mathcal{F}_j|_{U_{ij}}$

$\Rightarrow \exists! \mathcal{F} \in F(X)$ with $\mathcal{F}|_{U_i} = \mathcal{F}_i$

Principal G -bundles

(G fibration)

Def $\pi: E \rightarrow X$ with $G \subset E$ st π is G -equivariant
principal G -bundle if $\exists U_i$ covering of X (in étale topology)
st $E|_{U_i} \cong G \times U_i$

Let $\underline{G} := \text{Sheaf } U \rightarrow \text{Mor}(U, G)$

Then $\{\text{principal } G\text{-bundles}/X\} / \text{isomorphism} \cong H_{\text{ét}}^1(X, \underline{G})$

$\left. \begin{array}{l} \{\text{principal } G\text{-bundles} \\ \text{trivial in Zariski top} \} \right\} / \sim \cong H_{\text{zar}}^1(X, \underline{G})$

Fact. (Grothendieck) If $G = GL_n, SL_n$ or Sp_{2n} , then

$$H_{\text{ét}}^1(X, \underline{G}) \cong H_{\text{zar}}^1(X, \underline{G})$$