Applications for abelian varieties II: Perverse coherent sheaves

Motivation: Riemann-Hilbert correspondence

$$
\begin{array}{cccc}
DR_X : & D^b_{reghol}(\mathcal{D}_X) & \xrightarrow{\sim} & D^b_c(\mathbb{C}_X) \\
& \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} \\
& \mathrm{Mod}_{reghol}(\mathcal{D}_X) & \xrightarrow{\sim} & \mathrm{Perv}(\mathbb{C}_X)
\end{array}
$$

complexes in degree 0

perverse sheaves

goal of today: Fourier-Mukai transform

$$
\begin{array}{cccc}
FM_A : & D^b_{coh}(\mathcal{D}_A) & \xrightarrow{\sim} & D^b_{coh}(\mathcal{O}_{A^\natural}) \\
& \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} \\
& \mathrm{Mod}_{hol}(\mathcal{D}_A) & \longrightarrow & {}^m\mathrm{Coh}(\mathcal{O}_{A^\natural})
\end{array}
$$

m-perverse coherent sheaves on $A^\natural$

more generally: Through FM_A, the standard t-structure on $D^b_{hol}(\mathcal{D}_A)$ corresponds to the m-perverse t-structure on $D^b_{coh}(\mathcal{O}_{A^\natural})$

§1 t-structures

Basic example to have in mind:

$D^b(\mathcal{A})$
$\quad$ └ abelian category

$$D^{\leq 0}(\mathcal{A}) := \{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \quad \forall i > 0 \}$$

$$D^{\geq 0}(\mathcal{A}) := \{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \quad \forall i < 0 \}$$

"standard t-structure" on a derived category

e.g. a derived category

Def.: Let $\mathcal{T}$ be a triangulated category.
A pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called a t-structure on $\mathcal{T}$ if

(1) $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$

$\quad\;\; \mathcal{T}^{\geq 0}[-1] \subseteq \mathcal{T}^{\geq 0}$

(2) $\mathrm{Hom}_{\mathcal{T}}(X[1], Y) = 0$

$\quad\;\; \forall X \in \mathcal{T}^{\leq 0},\ Y \in \mathcal{T}^{\geq 0}$

no neg. ext groups for $X, Y \in \mathcal{A}$

(3) $\forall X \in \mathcal{T}$ $\exists$ distinguished triangle

$$X_{<0} \longrightarrow X \longrightarrow X_{\geq 0} \xrightarrow{+1}$$

for some $X_{<0} \in \mathcal{T}^{\leq -1}$, $X_{\geq 0} \in \mathcal{T}^{\geq 0}$.

<u>Remarks</u>:

- One writes $\mathcal{T}^{\leq k} := \mathcal{T}^{\leq 0}[k]$ and $\mathcal{T}^{\geq k} := \mathcal{T}^{\geq 0}[k]$ for $k \in \mathbb{Z}$.
- The triangle in (3) is unique up to unique isomorphism.

  One has <u>truncation</u> functors

$$\tau_{\leq 0} : \mathcal{T} \longrightarrow \mathcal{T}^{\leq 0}$$

$$\tau_{\geq 0} : \mathcal{T} \longrightarrow \mathcal{T}^{\geq 0}$$

  which are right / left adjoints of the inclusions $\mathcal{T}^{\leq 0} \hookrightarrow \mathcal{T}$ / $\mathcal{T}^{\geq 0} \hookrightarrow \mathcal{T}$.

  Then $X_{<0} \cong \tau_{\leq -1} X$, $X_{\geq 0} \cong \tau^{\geq 0} X$.

§2. Perverse coherent sheaves [Kashiwara '04]

X smooth complex algebraic variety

For $x \in X$ we write $i_x : \{x\} \hookrightarrow X$ for the inclusion,

$\kappa(x)$ for its residue field, $\operatorname{codim}(x) := \dim \mathcal{O}_{X,x}$

$= \operatorname{codim}_X(\overline{\{x\}})$

<u>Def.</u>: [Sch, 17.1] A map $p : X \to \mathbb{Z}$ is called <u>supporting function</u> if $y \in \overline{\{x\}}$ implies $p(y) \geq p(x)$ for any $x, y \in X$.

If p is a supporting function, we define its dual by $\hat{p}(x) := \operatorname{codim}(x) - p(x)$.

(This is not a supporting function a priori.)

<u>Def.-Lemma</u>: Let p be a supporting function such that $\hat{p}$ is also a supporting function.

Set

$${}^{p}D^{\leq 0}_{coh}(\mathcal{O}_X) := \{ F \in D^b_{coh}(\mathcal{O}_X) \mid Li_x^* F \in D^{\leq p(x)}_{coh}(\kappa(x)) \ \forall x \in X \}$$

$${}^{p}D^{\geq 0}_{coh}(\mathcal{O}_X) := \{ F \in D^b_{coh}(\mathcal{O}_X) \mid i_x^! F \in D^{\geq p(x)}_{coh}(\kappa(x)) \ \forall x \in X \}$$

These define a t-structure on $D^b_{coh}(\mathcal{O}_X)$.
(Kashiwara, Arinkin-Bezrukavnikov).

Its heart is denoted by

$${}^{p}Coh(\mathcal{O}_X) := {}^{p}D^{\leq 0}_{coh}(\mathcal{O}_X) \cap {}^{p}D^{\geq 0}_{coh}(\mathcal{O}_X)$$

and its objects are called <u>perverse coherent sheaves.</u>

If $p = 0$: standard t-structure.

Without the condition on p, one can a priori only define a t-structure on <u>quasi-coherent</u> $\mathcal{O}_X$-modules in this way.

<u>The problem is:</u> It can happen that the perverse cohomology of a coherent complex w.r.t. this t-structure is not coherent [Kas, Ex. 5.1].

Now, we concretely consider the following supporting function:

$$m(x) := \left\lfloor \tfrac{1}{2} \operatorname{codim}(x) \right\rfloor$$

i.e. $$\hat{m}(x) := \left\lceil \tfrac{1}{2} \operatorname{codim}(x) \right\rceil$$

(For $y \in \overline{\{x\}}$, $\operatorname{codim}(y) \geq \operatorname{codim}(x)$, therefore $m(y) \geq m(x)$ and $\hat{m}(y) \geq \hat{m}(x)$.)

Lemma:

$${}^{m}D^{\leq 0}_{coh}(\mathcal{O}_X) = \{ F \in D^b_{coh}(\mathcal{O}_X) \mid \operatorname{codim} \operatorname{supp} \mathcal{H}^i(F) \geq 2i \ \forall i \in \mathbb{Z} \}$$

$${}^{m}D^{\geq 0}_{coh}(\mathcal{O}_X) = \{ F \in D^b_{coh}(\mathcal{O}_X) \mid \operatorname{codim} \operatorname{supp} \mathcal{H}^i(DF) \geq 2i-1 \ \forall i \in \mathbb{Z} \}$$

where $DF := R\mathcal{H}om(F, \mathcal{O}_X)$.

(observation: not self-dual)

Idea of proof: $Li_x^* F \in D^{\leq \frac{1}{2} \operatorname{codim}(x)}$

means $x \in \operatorname{supp} H^k(F)$ if $k \leq \frac{1}{2} \operatorname{codim}(x)$

$\iff \operatorname{codim}(x) \geq 2k$

$\forall x \in \operatorname{supp} H^k(F)$

$\Longleftrightarrow$ codim supp $H^k(F) \geq 2k$

(any irreducible subset has a generic point, [Kas, Lemma 5.5])

Similarly:

$$k \leq \frac{1}{2}\operatorname{codim}(x) + 0{,}5$$

$${}^{\hat{m}}D^{\leq 0}_{coh}(\mathcal{O}_x) = \{F \in D^b_{coh}(\mathcal{O}_x) \mid \operatorname{codim}\operatorname{supp}\mathcal{H}^k(F) \geq 2k-1 \ \forall k \in \mathbb{Z}\}$$

and we see easily that

${}^{\hat{m}}D^{\leq 0}_{coh}(\mathcal{O}_x)$ is dual to ${}^{m}D^{\geq 0}_{coh}(\mathcal{O}_x)$

$\square$

§3 Fourier-Mukai transformation and t-structures

Theorem: Let $\mathcal{M} \in D^b_{hol}(\mathcal{D}_A)$.
[Sch, 18.1] Then $\mathcal{M} \in D^{\leq 0}_{hol}(\mathcal{D}_A) \iff FM_A(\mathcal{M}) \in {}^m D^{\leq 0}_{coh}(\mathcal{O}_{A^\natural})$
$\mathcal{M} \in D^{\geq 0}_{hol}(\mathcal{D}_A) \iff FM_A(\mathcal{M}) \in {}^m D^{\geq 0}_{coh}(\mathcal{O}_{A^\natural})$.

(The same holds with $\leq k$ / $\geq k$ since FM_A commutes with shifts.)

In particular, $\mathcal{M} \in Mod_{hol}(\mathcal{D}_A) \iff FM_A(\mathcal{M}) \in {}^m Coh(\mathcal{O}_{A^\natural})$

and

$$\operatorname{codim} S^k_1(A, \mathcal{M}) \geq |2k|$$

for any $\mathcal{M} \in Mod_{hol}(\mathcal{D}_A)$.

Note:

This is <u>not</u> saying

$$FM_A(D^{\leq 0}_{hol}(\mathcal{D}_A)) = {}^m D^{\leq 0}_{coh}(\mathcal{O}_{A^\natural})$$

and $FM_A(D^{\geq 0}_{hol}(\mathcal{D}_A)) = {}^m D^{\geq 0}_{coh}(\mathcal{O}_{A^\natural})$

Def.: Let $\lambda \in \mathbb{C}$. A $\underline{\lambda\text{-connection}}$ on a
[Sch, 10.1]
locally free $\mathcal{O}_A$-module $\mathcal{E}$ is a $\mathbb{C}$-linear morphism

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_A \underset{\mathcal{O}_A}{\otimes} \mathcal{E}$$

such that $\forall f \in \mathcal{O}_A$, $s \in \mathcal{E}$ (local sections)

$$\nabla(f \cdot s) = f \cdot \nabla s + df \otimes \lambda s.$$

It is called integrable if $\nabla \circ \nabla = 0$.

Denote by $\underline{E(A)}$ the moduli space of line bundles with λ-connection for some $\lambda \in \mathbb{C}$ on A.

One naturally has morphisms

$$\pi : E(A) \longrightarrow \hat{A}, \quad (L, \nabla) \longmapsto L$$

$$\lambda : E(A) \longrightarrow \mathbb{C}, \quad (L, \nabla) \longmapsto \lambda$$

λ-connection for some $\lambda \in \mathbb{C}$

and $\tilde{P} := (id \times \pi)^* P$

Poincaré bundle on $A \times \hat{A}$ (similar to Christian's talk)

with a canonical generalized relative connection

$$\tilde{\nabla} : \tilde{P} \longrightarrow \Omega^1_{A \times E(A)/E(A)} \otimes \tilde{P}.$$

Prop.: (*) $\mathcal{M} \in \mathrm{Mod}_{hol}(\mathcal{D}_A)$, $i > 0$
[Sch, 18.2] Then $\mathrm{supp}\, \mathcal{H}^i FM_A(\mathcal{M})$ is a proper subset of $A^\natural$.

Preparation for the proof:
Consider a good filtration $F_\bullet \mathcal{M}$ and the Rees construction

$$R_F \mathcal{M} := \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M}$$

over $$\mathcal{R}_A := \bigoplus_{k \geq 0} \underbrace{F_k \mathcal{D}_A}_{\substack{\text{order} \\ \text{filtration}}} \otimes z^k \subseteq \mathcal{D}_A[z]$$

Then one can define the extended Fourier-Mukai transform

$$\widetilde{FM}_A(R_F \mathcal{M}) :=$$

$$R p_{2*} DR_{A \times E(A) / E(A)} \big((\mathrm{id} \times \lambda)^* \underbrace{R_F \mathcal{M}}_{\mathcal{R}_{A \times \mathbb{C}/\mathbb{C}}\text{-module}} \otimes_{\mathcal{O}_{A \times E(A)}} \widetilde{P} \big)$$

$p_2 : A \times E(A) \to E(A)$

$$\in D^b_{qcoh}(\mathcal{O}_{E(A)})$$

($\widetilde{FM}_A$ actually gives an equivalence)

Prop.: (i) $\widetilde{FM}_A(R_F\mathcal{M})\big|_{\lambda=1} \cong FM_A(\mathcal{M})$

[Sch, 12.1]

same for any $\lambda \neq 0$

note: $\lambda^{-1}(1) = A^{\natural}$

(line bundles with connections)

$$\begin{array}{c} A \times H^0(A,\Omega_A^1) = \lambda^{-1}(0) \\ \uparrow p_{13} \\ A \times \hat{A} \times H^0(A,\Omega_A^1) \\ \downarrow p_{23} \\ \hat{A} \times H^0(A,\Omega_A^1) \end{array}$$

(ii) $\widetilde{FM}_A(R_F\mathcal{M})\big|_{\lambda=0} \cong$

$$R p_{23*}\left(p_{12}^* P \otimes p_1^*\Omega_A^g \otimes p_{13}^*(\mathrm{id}\times\iota)^* \mathrm{gr}^F\mathcal{M}\right)$$

(see [Sch, (10.8)] for notation)

Proof of Prop. ⊛

<u>Idea:</u> "being a proper subset" is an open condition with respect to λ, so if

$$\mathrm{supp}\,\mathcal{H}^i\left(\widetilde{FM}_A(R_F\mathcal{M})\big|_{\lambda=0}\right) \subsetneq \hat{A}\times H^0(A,\Omega_A^1)$$

then the claim follows.

$$\dim \mathrm{supp}\, \mathrm{gr}^F\mathcal{M} = \dim A = g$$

($\mathcal{M}$ holonomic)

$$\Rightarrow \dim \mathrm{supp}\; p_{13}^*(\mathrm{id}\times\iota)^*\mathrm{gr}\,\mathcal{M} = 2g$$

$$\Rightarrow \dim \mathrm{supp}\left(p_{12}^*P \otimes \ldots\ldots \mathrm{gr}^F\mathcal{M}\right) = 2g$$

This dimension is equal to that of $\hat{A} \times H^0(A, \Omega^1_A)$, the target of p_{23}.

$\Rightarrow$ The fibers $p_{23}^{-1}((L, \Theta)) \cap \operatorname{supp}(\dots)$ are generically 0-dimensional

$\Rightarrow$ higher cohomologies vanish generically $\square$

Proof of Theorem:

$$FM_A\left(D^{\le 0}_{hol}(\mathcal{D}_A)\right) \subseteq {}^m D^{\le 0}_{coh}(\mathcal{O}_X)$$

clear if $\dim A = 0$

now for $\dim A > 0$:

Let Z be an irred. component of

$\operatorname{Supp} \mathcal{H}^\ell FM_A(\mathcal{M})$

$\overset{\text{Henry}}{\Longrightarrow}$ $Z = \underset{\in A^\natural}{(L,\nabla)} \otimes \operatorname{im}(f^\natural : B^\natural \to A^\natural)$

(linear subvariety)

$f: A \to B$ surj.

since (Prop.) $\operatorname{codim} Z > 0$

$\Rightarrow \dim B < \dim A$

$\Rightarrow \operatorname{codim} Z = 2r := 2(\dim A - \dim B)$

$= \dim A^\natural - \dim B^\natural$

Now, by Christian's talk

$$FM_A(\mathcal{M})\big|_Z = FM_B\Big(\underbrace{f_+(\mathcal{M}\otimes(L,\nabla))}_{\in D^{\le r}_{hol}(\mathcal{D}_B)}\Big)$$

$\overset{\text{ind. hyp.}}{\in} {}^m D^{\le r}_{coh}(\mathcal{O}_Z)$.

[Hotta]

i.e. $FM_A(\mathcal{M})|_Z\,[r] \in {}^{m}D^{\leq 0}_{coh}(\mathcal{O}_Z)$

$\Rightarrow$ $\operatorname{codim}_Z \operatorname{supp} \mathcal{H}^{\ell} FM_B(f_+(\mathcal{M}\otimes(L,\nabla))) \geq 2\ell - 2r$

but $\operatorname{supp}(\ldots) = Z \Rightarrow \operatorname{codim} \ldots = 0$

$$\Rightarrow \ell \leq r$$

$\Rightarrow$ $\operatorname{codim} Z \geq 2\ell$

<u>To prove $\Leftarrow$</u> (i.e. if $FM_A(\mathcal{M}) \in {}^{m}D^{\leq 0}_{coh}(\mathcal{O}_{A^\natural})$,

then $\mathcal{M} \in D^{\leq 0}_{hol}(\mathcal{D}_A)$)

consider the inverse FM-transform:

$$\mathcal{M} \cong \langle -1_A\rangle^* R p_{1*}\left(P^\natural \otimes_{A\times\hat{A}} p_2^* \pi_* FM_A(\mathcal{M})\right)[g]$$

as complexes of $\mathcal{O}_A$-modules, $\pi : A^\natural \to A$
fibre dim. g

If $\operatorname{codim} \operatorname{supp} \mathcal{H}^{\ell} FM_A(\mathcal{M}) \geq 2\ell$

then $\operatorname{codim} \operatorname{supp} \mathcal{H}^{\ell} \pi_* FM_A(\mathcal{M}) \geq \ell$

$\operatorname{codim} \operatorname{supp} \mathcal{H}^{\ell}(\mathcal{M}[-g]) \geq \ell$

$\forall i$: $\operatorname{codim} \operatorname{supp} \underbrace{\mathcal{H}^i \mathcal{M}}_{= \mathcal{H}^{i+g}\mathcal{M}[-g]} \geq i+g$ $\Rightarrow \mathcal{H}^i\mathcal{M} = 0 \;\; \forall i > 0$

$\square$

Proof of "in particular" in Theorem

$$\mathcal{M} \in \mathrm{Mod}_{hol}(\mathcal{D}_A) \implies FM_A(\mathcal{M}) \in {}^m\mathrm{Coh}(\mathcal{O}_{A^\natural})$$

i.e. $\quad \operatorname{codim} \operatorname{supp} \mathcal{H}^k FM_A(\mathcal{M}) \geq 2k$

$\quad \operatorname{codim} \operatorname{supp} \mathcal{H}^k \mathbb{D} FM_A(\mathcal{M}) \geq 2k-1 \quad \forall k \in \mathbb{Z}$

Recall (Henry)

$$i^*_{(L,\nabla)} \mathcal{H}^k FM_A(\mathcal{M}) \simeq R^k\Gamma(A, DR_A(\mathcal{M} \otimes_{\mathcal{O}_A} (L,\nabla)))$$

i.e.

$$\operatorname{supp} \mathcal{H}^k(FM_A(\mathcal{M})) \supseteq \underbrace{S^k_1(A,\mathcal{M})}_{= \{(L,\nabla) \in A^\natural \mid \dim R^k\Gamma \ldots \geq 1\}}$$

since $(L,\nabla) \in \operatorname{supp} \mathcal{H}^k(FM_A(\mathcal{M}))$

$\iff i^{-1}_{(L,\nabla)} \mathcal{H}^k(FM_A(\mathcal{M})) \neq 0$

$\Leftarrow i^*_{(L,\nabla)} \mathcal{H}^k FM_A(\mathcal{M}) \neq 0$

$\iff \dim R^k\Gamma(A, DR_A(\mathcal{M} \otimes_{\mathcal{O}_A} (L,\nabla))) \geq 1.$

Similarly, $\mathbb{D}\mathcal{M} \in \mathrm{Mod}_{hol}(\mathcal{D}_A)$

$\implies \operatorname{codim} \operatorname{supp} \mathcal{H}^k \mathbb{D} FM_A(\mathcal{M}) \geq 2k$

$\implies \operatorname{codim} S^k_1(A, \mathbb{D}\mathcal{M}) \geq 2k$

$\implies \operatorname{codim} S^k_1(A, \mathcal{M}) \leq -2k \qquad \square$

Extra: A conjecture [Sch, §6, §19]

What is the image

$$FM_A(D^b_{hol}(\mathcal{D}_A)) \subseteq D^b_{coh}(\mathcal{O}_{A^\natural})$$

or $$FM_A(Mod_{hol}(\mathcal{D}_A)) \subseteq {}^m Coh(\mathcal{O}_{A^\natural}) \;?$$

idea: "hyperkähler constructible complexes" / "hyperkähler perverse sheaves"
($A^\natural$ is a hyperkähler manifold, but not compact)
(tentative characterization of those objects in [Sch, Conj. 6.1])

"constructibility" is usually defined in terms (standard cohomology) of cohomology sheaves, which leads us to the following theorem:

Theorem: [Sch, 19.1] If $\mathcal{F} \in FM_A(D^b_{hol}(\mathcal{D}_A))$, then $\tau_{\le n}\mathcal{F}, \tau_{\ge n}\mathcal{F} \in FM_A(D^b_{hol}(\mathcal{D}_A))$.

Corollary: $\mathcal{F} \in D^b_{coh}(\mathcal{O}_{A^\natural})$

Then $\mathcal{F} \in FM_A(D^b_{hol}(\mathcal{D}_A))$

$\Longleftrightarrow \mathcal{H}^k\mathcal{F} \in FM_A(D^b_{hol}(\mathcal{D}_A)) \quad \forall k \in \mathbb{Z}$