Applications for abelian varieties I: the structure theorem(s)

§1. Results

$A/\mathbb{C}$ ab var. $\quad A^\natural = \{(L,\nabla) \mid \text{line bundles w/ conn.}\}/\sim$

Def^n $\mathcal{M}$ left $\mathcal{D}_A$-module.

$$S^k_m(A,\mathcal{M}) = \{(L,\nabla)\in A^\natural \mid \dim \mathbb{H}^k(A, DR_A(\mathcal{M}\otimes(L,\nabla))) \geq m\} \subseteq A^\natural$$

Def^n a <u>linear subvariety</u> of $A^\natural$ is a subset of the form

$$(L,\nabla)\otimes \operatorname{im}(f^\natural : B^\natural \to A^\natural)$$

$f: A\to B$ surj, conn fibers, B ab var.

this is <u>arithmetic</u> if (L,∇) can be taken to be a torsion point

Theorem 1 (T2.2) $\mathcal{M}\in D^b_h(\mathcal{D}_A)$.

a) $S^k_m(A,\mathcal{M})$ is a finite union of linear subvars.

b) if $\mathcal{M}$ semisimple, regular, of geometric origin, the lin subvars can be taken to be arithmetic.

Def^n $\operatorname{Char} A$ = characters of $\pi_1(A,0)$. $\rho\in \operatorname{Char}(A) \rightsquigarrow \mathbb{C}_\rho$ (rk 1 local system.)

if $K\in D^b_c(\mathbb{C}_A)$,

$$S^k_m(A,K) = \{\rho\in\operatorname{Char}A \mid \dim \mathbb{H}^k(A, K\otimes_{\mathbb{C}}\mathbb{C}_\rho)\geq m\}.$$

Rmk $\Phi: A^\natural \to \operatorname{Char}A$; $(L,\nabla)\mapsto \ker\nabla$

analytic isomorphism.

<u>Method of proving T1 a)</u> <u>Simpson</u> $Z\subseteq A^\natural$, $Z, \Phi(Z)$ algebraic $\Rightarrow Z,\Phi(Z)$ are finite unions of linear subvars.

- show $S^k_m(\mathcal{M})\subseteq A^\natural$
 $S^k_m(DR(\mathcal{M}))\subseteq \operatorname{Char}A$ alg
- show $\Phi(S^k_m(\mathcal{M})) = S^k_m(DR(\mathcal{M}))$

recall $FM_A : D^b_{coh}(\mathcal{D}_A) \xrightarrow{\sim} D^b_{coh}(\mathcal{O}_{A^\natural})$

Theorem 2 (T3.2) $\mathcal{M}\in D^b_h(\mathcal{D}_A)$. Supp $FM_A(\mathcal{M})$ is a fin union of linear subvars. $\mathcal{M}$ geom origin $\Rightarrow$ arithmetic.

§2. <u>Cohomology of constructible complexes</u> (§13)

(prove $S^k_m(K)\subseteq \operatorname{Char}A$ algebraic.) use "key lemma"

Write $A = V/\Lambda$, V $\mathbb{C}$-vspace dim g.

$\Lambda\simeq \pi_1(A,0)$ lattice rk $2g$.

$\operatorname{Char}A = \operatorname{Spec}\mathbb{C}[\Lambda]$

Def^n M a $k[\Lambda]$-module. then $\Lambda \curvearrowright M \rightsquigarrow \mathcal{L}_M$ local system of $k[\Lambda]$-modules on A. <u>note</u> $\mathcal{L}_{\mathbb{C}[\Lambda]}\simeq \pi_!\mathbb{C}_V$, $\pi: V\to A$.

Lemma (13.3) $\mathcal{F}$ fine sheaf of $\mathbb{C}$-vspaces on A

($\mathcal{F}$ fine $\Leftrightarrow$ $\forall\{U_i\}$ $\exists f_i\in \operatorname{Hom}_{\mathbb{C}}(\mathcal{F},\mathcal{F})$
 $\cdot$ supp $f_i\subseteq U_i$, $\cdot$ $\{\operatorname{supp} f_i\}$ locally finite , $\cdot$ $\sum f_i = 1_{\mathcal{F}}$.)

Then $H^0(A,\mathcal{F}\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})$ is a flat $\mathbb{C}[\Lambda]$-module

and $H^0(A,\mathcal{F}\otimes\mathcal{L}_M)\simeq H^0(A,\mathcal{F}\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes_{\mathbb{C}[\Lambda]} M$.

Pf $F: M\mapsto H^0(A,\mathcal{F}\otimes\mathcal{L}_M)$ exact + preserves $\oplus$.

Eilenberg-Watts : $F\simeq H^0(A,\mathcal{F}\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes -$ ∎

Proposition (13.4) $K\in D^b_c(\mathbb{C}_A)$, M $\mathbb{C}[\Lambda]$-mod.

$p: A\to pt$

$$\mathbb{R}p_*(K\otimes_{\mathbb{C}}\mathcal{L}_M)\simeq \mathbb{R}p_*(K\otimes_{\mathbb{C}}\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes^{\mathbb{L}}_{\mathbb{C}[\Lambda]} M$$

Pf we can choose a bdd complex of fine sheaves $\mathcal{F}^\bullet$ which is qiso to K :

$K\simeq DR_A(\mathcal{M})$. $\quad K \overset{\text{Poincaré lemma}}{\simeq} [\mathcal{A}^0_A\otimes\mathcal{M}\to\mathcal{A}^1_A\otimes\mathcal{M}\to\dots\to\mathcal{A}^{2g}_A\otimes\mathcal{M}]$

it is easy to see that any $\mathcal{A}^0_A$-module is a fine sheaf.

$$\begin{aligned}\mathbb{R}p_*(K\otimes\mathcal{L}_M) &\simeq [H^0(A,\mathcal{F}^\bullet\otimes_{\mathbb{C}}\mathcal{L}_M)]\\ &\overset{\text{lemma}}{\simeq} [H^0(A,\mathcal{F}^\bullet\otimes_{\mathbb{C}}\mathcal{L}_{\mathbb{C}[\Lambda]})]\otimes^{\mathbb{L}}_{\mathbb{C}[\Lambda]} M\\ &\simeq \mathbb{R}p_*(K\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes^{\mathbb{L}}_{\mathbb{C}[\Lambda]} M.\ ∎\end{aligned}$$

Corollary $\rho\in\operatorname{Char}A$.

$$\mathbb{R}p_*(K\otimes\mathbb{C}_\rho)\simeq \mathbb{R}p_*(K\otimes\mathbb{C}[\Lambda])\otimes^{\mathbb{L}}_{\mathbb{C}[\Lambda]}\frac{\mathbb{C}[\Lambda]}{\mathfrak{m}_\rho}$$

Pf $\mathbb{C}_\rho = \mathcal{L}_{\mathbb{C}[\Lambda]/\mathfrak{m}_\rho}$. ∎

<u>Key lemma</u> (13.7) E perfect complex on scheme X.

$$\{x\in X \mid \dim H^k(E\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{O}_{X,x}/\mathfrak{m}_x)\geq m\}$$

is the set of closed pts of a closed subscheme.

Proof local. $X = \operatorname{Spec} R$. $P\subseteq R$ prime.

set of primes for which $\dim H^k(E_P)\geq m$ is equal to

$$\bigcup_{i\geq 0}\{P \mid rk(\ker d_k)\geq m+i\}\cap\{P \mid rk(\ker d_{k-1})\geq rk(E_{k-1})-i\}$$

this is defined by vanishing of minors of d_k, d_{k-1}. ∎

Theorem (13.6) $K\in D^b_c(\mathbb{C}_A)$, $S^k_m(A,K)\subseteq\operatorname{Char}A$ is algebraic.

Proof

$$S^k_m(A,K) = \{\rho\in\operatorname{Char}A \mid \dim H^k(\mathbb{R}p_*(K\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes^{\mathbb{L}}_{\mathbb{C}[\Lambda]}\tfrac{\mathbb{C}[\Lambda]}{\mathfrak{m}_\rho})\geq m\}$$

apply key lemma $\quad X = \operatorname{Spec}\mathbb{C}[\Lambda]$, $E = \mathbb{R}p_*(K\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})$. ∎

§3. <u>Comparison theorems</u>

$\Phi: A^\natural\to\operatorname{Char}A$.

theorem (14.1) $\mathcal{M}\in D^b_h(\mathcal{D}_A)$

$$\Phi(S^k_m(A,\mathcal{M})) = S^k_m(A, DR_A(\mathcal{M}))$$

Proof given (L,∇). $\quad \ker\nabla\otimes_{\mathbb{C}}\mathcal{O}_A\simeq L$

$$(\Omega^{g+i}_A\otimes_{\mathcal{O}_A}\mathcal{M}^j)\otimes_{\mathbb{C}}\ker\nabla \xrightarrow{\sim} \Omega^{g+i}_A\otimes_{\mathcal{O}_A}\mathcal{M}^j\otimes_{\mathcal{O}_A} L$$

compatible with differentials.

$$DR_A(\mathcal{M})\otimes_{\mathbb{C}}\ker\nabla\xrightarrow{\sim} DR_A(\mathcal{M}\otimes_{\mathcal{O}_A}L)$$

$\dim\mathbb{H}^k(DR(\mathcal{M})\otimes\ker\nabla)\geq m \Leftrightarrow \dim\mathbb{H}^k(DR(\mathcal{M}\otimes L))\geq m$ ∎

theorem (14.2) $\mathcal{M}\in D^b_h(\mathcal{D}_A)$

$$\mathbb{R}\Phi_* FM_A(\mathcal{M})\simeq \mathbb{R}p_*(DR_A(\mathcal{M})\otimes_{\mathbb{C}}\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes_{\mathbb{C}[\Lambda]}\mathbb{C}\{\Lambda\}$$

§4. <u>Structure theorem(s)</u> (§15,16)

theorem [Simpson] Z cl subset of $A^\natural$. If $Z,\Phi(Z)$ algebraic, then Z (and $\Phi(Z)$) are finite unions of linear subvars.

Proposition (15.2) $\mathcal{M}\in D^b_{coh}(\mathcal{D}_A)$. Then $S^k_m(A,\mathcal{M})$ is algebraic.

Proof given (L,∇)

$$\begin{array}{ccccc}\{(L,\nabla)\} & \xleftarrow{p} & A\times\{(L,\nabla)\} & \simeq & A\\ \downarrow i_{(L,\nabla)} & & \downarrow i & \nearrow p_1 & \\ A^\natural & \xleftarrow{p_2} & A\times A^\natural & & \end{array}$$

$$\begin{aligned}\mathbb{L}i^*_{(L,\nabla)} FM_A(\mathcal{M}) &\simeq \mathbb{R}p_{2,*}\,\mathbb{L}i^* DR_{A\times A^\natural/A^\natural}(p_1^*\mathcal{M}\otimes P^\natural)\\ &\simeq \mathbb{R}p_{2,*} DR_A(\mathcal{M}\otimes\mathbb{L}i^*P^\natural)\\ &\simeq \mathbb{R}p_{2,*} DR_A(\mathcal{M}\otimes(L,\nabla))\end{aligned}$$

$FM_A(\mathcal{M}) \overset{\text{qiso}}{\simeq} \mathcal{E}^\bullet$ ← loc free

$$S^k_m(A,\mathcal{M}) = \{(L,\nabla)\in A^\natural \mid \dim H^k(i^*_{(L,\nabla)}(\mathcal{E}^\bullet))\geq m\}$$

by key lemma, ∎

Thm 1.a) follows immediately.

<u>Proof Thm 2</u> Supp $FM_A(\mathcal{M})$ is algebraic.

$$\begin{aligned}\Phi(\operatorname{Supp} FM_A(\mathcal{M})) &\simeq \operatorname{Supp}\mathbb{R}\Phi_* FM_A(\mathcal{M})\\ &\simeq \operatorname{Supp}(\mathbb{R}p_*(DR(\mathcal{M})\otimes\mathcal{L}_{\mathbb{C}[\Lambda]})\otimes_{\mathbb{C}[\Lambda]}\mathbb{C}\{\Lambda\})\end{aligned}$$

is algebraic. ∎

Def^n $\mathcal{M}\in D^b_{h,ss}(\mathcal{D}_A)$ is of <u>geometric origin</u> if it can be obtained from $\mathcal{O}_-$ after applications of $H^if_+, H^if^+, H^if_!, H^if^!, \psi_g, \phi_{g,1}, \mathbb{D}, \boxtimes, \otimes, R\mathcal{H}om$, subquotients.

<u>Fact</u> any ss rh $\mathcal{D}$-mod of geom origin is a direct summand of rh $\mathcal{D}$-mods of geom origin w/ $\mathbb{Q}$-structure.

Now, given $\sigma\in\operatorname{Aut}(\mathbb{C}/\mathbb{Q})$. $A^\sigma, L^\sigma, \nabla^\sigma$.

$c_\sigma: A^\natural\to(A^\sigma)^\natural$ iso of ab grps

Def^n $Z\subseteq_{cl} A^\natural$ <u>absolute closed</u> if

$\Phi(c_\sigma(Z))\subseteq\operatorname{Char}(A^\sigma)$ closed + $\overline{\mathbb{Q}}$-str $\forall\sigma$.

<u>thm</u> [Simpson] Z abs cl $\Rightarrow$ fin union of arithmetic linear subvars.

<u>Lemma</u> $\mathcal{M}\in D^b_{rh}(\mathcal{D}_A)$ w/ $\overline{\mathbb{Q}}$-structure.

Then $S^k_m(DR(\mathcal{M}))\subseteq\operatorname{Char}A$ has $\overline{\mathbb{Q}}$-structure.

<u>Proof of T1.b)</u>

<u>case 1</u> <u>$\mathcal{M}$ has $\overline{\mathbb{Q}}$-structure</u> $\quad c_\sigma(S^k_m(\mathcal{M})) = S^k_m(\mathcal{M}^\sigma)$

so by lemma, $c_\sigma(S^k_m(\mathcal{M}^\sigma))$ closed + defined over $\overline{\mathbb{Q}}$.

Simpson $\Rightarrow$ $S^k_m(\mathcal{M})$ fin union arith lin subvars

<u>case 2</u> <u>general $\mathcal{M}$</u> $\quad\mathcal{M}$ summand of ss rh $\mathcal{D}$-mods w/ $\mathbb{Q}$-structure of geom origin.

$\mathcal{M}$ = direct sum of simple subquotients of rh $\mathcal{D}$-mods w/ $\mathbb{Q}$-str.

it's easy to see (L16.6) that such simple subquots have $\overline{\mathbb{Q}}$-structure. $\mathcal{M}$ has $\overline{\mathbb{Q}}$-structure. ∎