Fourier-Mukai transformations

0.) Introductory

Let A be abelian variety /k. We have

$$\hat{A} := \mathrm{Pic}^0(A) := \{ L \to A \text{ line bdl.} \mid c_1(L) = 0 \} / \cong$$

$$A^\natural := \{ (L, \nabla) \mid L \in \hat{A}, \nabla : L \to L \otimes \Omega^1_A, \nabla^2 = 0 \} / \cong$$

and universal bundles $\mathcal{P} \to A \times A$, $\tilde{\mathcal{P}} \to A \times A^\natural$ giving rise to Fourier-Mukai-transformations;

$$FM_{\mathcal{P}} : D^b_{coh}(\mathcal{O}_A) \longrightarrow D^b_{coh}(\mathcal{O}_{\hat{A}})$$

$$F^\cdot \longmapsto q_*(\mathcal{P} \otimes p^* F^\cdot)$$

$$\begin{tikzcd} & A^\vee \times A \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & A^\vee \end{tikzcd}$$

$$FM_{\tilde{\mathcal{P}}} : D^b_{coh}(\mathcal{D}_A) \longrightarrow D^b_{coh}(\mathcal{O}_{A^\natural})$$

$$F^\cdot \longmapsto \tilde{q}_+(\tilde{\mathcal{P}} \otimes \tilde{p}^+ F^\cdot)$$

$$\begin{tikzcd} & A^\natural \times A \arrow[dl, "\tilde{p}"'] \arrow[dr, "\tilde{q}"] & \\ A & & A^\natural \end{tikzcd}$$

Here, we that $\exists\ \nabla : \tilde{\mathcal{P}} \to \tilde{\mathcal{P}} \otimes \Omega^1_{A^\natural \times A / A^\natural}$.

Aim: Show that $FM_{\mathcal{P}}$ & $FM_{\tilde{\mathcal{P}}}$ are equivalences.

1.) FM-transforms of $\mathcal{O}$-modules

preliminaries: Let X, Y schemes, $F \in D^b(\mathcal{O}_{X\times Y})$.

Write $X \xleftarrow{p_X} X\times Y \xrightarrow{p_Y} Y$ and define

$$S_{X\to Y, F} : D^b(\mathcal{O}_X) \longrightarrow D^b(\mathcal{O}_Y)$$

$$T \longmapsto p_{Y,*}\left(F \otimes p_X^{*} T\right)$$

Lemma: If $F \in D^b(\mathcal{O}_{X\times Y})$, $\mathcal{G} \in D^b(\mathcal{O}_{Y,Z}) \Rightarrow S_{Y\to Z,\mathcal{G}} \circ S_{X\to Y, F} \cong S_{X\to Z, \mathcal{H}}$

where $\mathcal{H} = p_{X,Z,*}\left(p^{*}_{X,Y} F \otimes p^{*}_{Y,Z}\, \mathcal{G}\right)$

$$\begin{tikzcd}
 & & \mathcal{H} \arrow[d] & & \\
 & & X\times Z \arrow[dddll, bend right, "p_X"'] \arrow[dddrr, bend left, "p_Z"] & & \\
 & & X\times Y\times Z \arrow[u, "p_{X,Z}"] \arrow[dl, "p_{X,Y}"] \arrow[dr, "p_{Y,Z}"'] & & \\
F \arrow[r] & X\times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & & Y\times Z \arrow[dl, "p_Y"] \arrow[dr, "p_Z"'] & \mathcal{G} \arrow[l] \\
X & & Y & & Z
\end{tikzcd}$$

Proof: $T\in D^b(\mathcal{O}_X) \rightsquigarrow S_{Y\to Z,\mathcal{G}}\left(S_{X\to Y,\mathcal{F}}(T)\right) =$

base change

$$p_{Z,*}\left[\mathcal{G}\otimes p_Y^* \, p_{Y*}\left(\mathcal{F}\otimes p_X^* T\right)\right] = p_{Z*}\left[\mathcal{G}\otimes p_{Y,Z,*}\, p_{X,Y}^*\left(\mathcal{F}\otimes p_X^* T\right)\right] =$$

proj. formula

$$= p_{Z,*}\left[p_{Y,Z,*}\left(p_{Y,Z}^*\mathcal{G}\otimes p_{X,Y}^*\mathcal{F}\otimes p_{X,Y}^*\, p_X^* T\right)\right]$$

$p_X\circ p_{X,Y} = p_X\circ p_{X,Z}$

$=$

$$p_{Z,*}\left[p_{Y,Z,*}\left(p_{Y,Z}^*\mathcal{G}\otimes p_{X,Y}^*\mathcal{F}\otimes p_{X,Z}^*\, p_X^* T\right)\right]$$

$p_Z\circ p_{Y,Z} = p_Z\circ p_{X,Z}$

$=$

$$p_{Z,*}\left[p_{X,Z,*}\left(p_{Y,Z}^*\mathcal{G}\otimes p_{X,Y}^*\mathcal{F}\otimes p_{X,Z}^*\, p_X^* T\right)\right] =$$

proj. form.

$=$

$$p_{Z,*}\left[\underbrace{p_{X,Z,*}\left(p_{X,Y}^*\mathcal{F}\otimes p_{Y,Z}^*\mathcal{G}\right)}_{\mathcal{H}}\otimes p_{X,Z,*}\, p_{X,Z}^*\, p_X^* T\right]$$

$$p_{Z,*}\left[\mathcal{H}\otimes p_X^* T\right] = S_{X\to Z,\mathcal{H}}(T) \qquad \square$$

Now let $X := A$ be abelian variety $/k$ of $\dim_k(A) = g$.

$\forall a \in A$, $t_a : A \to A$, $b \mapsto a+b$, then

$$\hat{A} := \mathrm{Pic}^0(A) \overset{!}{=} \{ L \in \mathrm{Pic}(A) \mid t_a^* L \simeq L \quad \forall a \in A \}$$

Facts: Let $L \in \hat{A}$, then:

1.) $\iota^* L \simeq L^\vee$, where $\iota : A \to A$, $a \mapsto -a$

2.) $n^* L \simeq L^{\otimes n}$, where $n : A \to A$, $a \mapsto n \cdot a$

3.) $m^* L \simeq p_1^* L \otimes p_2^* L$, where $m : A \times A \to A$, $(a,b) \mapsto a+b$

4.) $L \not\simeq \mathcal{O}_A \Rightarrow H^i(A, L) = 0 \quad \forall i \in \mathbb{Z}$

see, e.g. Huybrechts, FM, chap. 9.

Thm: $\hat{A}$ represents functor: $S \mapsto \mathrm{Pic}^0_A(S) :=$
$\{ L \in \mathrm{Pic}(A \times S) \mid L_s \in \mathrm{Pic}^0(A) \ \forall s \in S \text{ closed} \} / \sim$

Def.: Consider iso: $\phi : \mathrm{Hom}(-, \hat{A}) \to \mathrm{Pic}^0_A(-)$, and define $\mathcal{P} := \phi(\mathrm{id}_{\hat{A}}) \in \mathrm{Pic}^0_A(\hat{A}) \subset \mathrm{Pic}(A \times \hat{A})$ to be the Poincaré bundle on $A \times \hat{A}$. By definition $\forall S : \forall L \in \mathrm{Pic}^0_A(S) : \exists! f : S \to \hat{A} : L \simeq f^* \mathcal{P}$. Hence: $\{\text{pts. of } \hat{A}\} \simeq \mathrm{Pic}^0(A)$

Thm: we have nat. iso $A \to \hat{\hat{A}}$

Def.: (FM for $\mathcal{O}_A$-modules): A, $\hat{A}$ and $\mathcal{P}$ as above

$$FM_{\mathcal{P}} : D^b_{coh}(A) \longrightarrow D^b_{coh}(\hat{A}), \quad F \longmapsto q_{*}\left(\mathcal{P} \otimes_{\mathcal{O}_{A\times\hat{A}}} p^{*}F\right)$$

Similarly, using $\hat{\mathcal{P}} \in Pic(\hat{A}\times\hat{\hat{A}}) \cong Pic(\hat{A}\times A)$,

we have $FM_{\hat{\mathcal{P}}} : D^b_{coh}(\hat{A}) \longrightarrow D^b_{coh}(A)$

Theorem (Mukai): $\exists$ isomorphisms of functors

$$\iota^{*}[-g] \cong FM_{\hat{\mathcal{P}}} \circ FM_{\mathcal{P}} : D^b(A) \to D^b(A)$$

$$\hat{\iota}^{*}[-g] \cong FM_{\mathcal{P}} \circ FM_{\hat{\mathcal{P}}} : D^b(\hat{A}) \to D^b(\hat{A})$$

(recall $\iota : A \to A$, $a \mapsto -a$, similarly for $\hat{\iota}$)

Proof: we apply Lemma for $X = A$, $Y = \hat{A}$, $Z = \hat{\hat{A}} \cong A$

$\rightsquigarrow FM_{\hat{\mathcal{P}}} \circ FM_{\mathcal{P}} = S_{\hat{A}\to A, \hat{\mathcal{P}}} \circ S_{A\to\hat{A}, \mathcal{P}} \cong S_{A\to A, \mathcal{K}}$, where

$\mathcal{K} = p_{1,2,*}\left(p_{1,3}^{*}\mathcal{P} \otimes p_{2,3}^{*}\mathcal{P}\right)$ where p_{ij} proj. from $A\times A\times\hat{A}$

Claim: $p_{1,3}^{*}\mathcal{P} \otimes p_{2,3}^{*}\mathcal{P} \cong (m\times id_{\hat{A}})^{*}\mathcal{P}$ [follows from Facts, 3.]

$\rightsquigarrow \mathcal{K} \cong p_{1,2,*}(m\times id_{\hat{A}})^{*}\mathcal{P} \overset{\text{base change}}{\cong} m^{*}p_{1,*}\mathcal{P}$

$$\begin{tikzcd}
A\times A\times\hat{A} \arrow[r, "p_{12}"] \arrow[d, "m\times id_{\hat{A}}"'] & A\times A \arrow[d, "m"] \\
A\times\hat{A} \arrow[r, "p_1"'] & A
\end{tikzcd}$$

Facts 4.) $\Rightarrow R p_{1,*} \mathcal{P} = R^g p_{1,*} \mathcal{P} = k_{\{0\}}$
+ Mumford

Hence $\mathcal{K} = m^* k_{\{0\}}[-g] \simeq \mathcal{O}_{\Gamma_\iota}[-g]$, where

$\Gamma_\iota = \mathrm{Graph}(\iota : A \to A)$

exercise: $f : X \to Y$, $F = \mathcal{O}_{\Gamma_f} \Rightarrow S_{X \to Y, F} = f_*$, $S_{Y \to X, F} = f^*$

hence $FM_{\hat{\mathcal{P}}} \circ FM_{\mathcal{P}} = S_{A \to A, \mathcal{K}} = \iota^*[-g]$ □

Construction of $A^\natural$: Put $\mathrm{Pic}^\natural(A) := \{(L, \nabla) \mid L \in \mathrm{Pic}^0(A),$
$\nabla : L \to L \otimes \Omega^1_A, \nabla^2 = 0\}/\cong$: group with $\otimes$-product. We have

$$\begin{tikzcd}
 & & & \text{torsor under } H^0(A,\Omega^1_A) \arrow[d] & \\
0 \arrow[r] & H^0(A,\Omega^1_A) \arrow[r, "i"] & \mathrm{Pic}^\natural(A) \arrow[r, "\pi"] & \mathrm{Pic}^0(A) \arrow[r] & 0 \\
 & & (L,\nabla) \arrow[r, mapsto] & L & ? \arrow[u] \\
 & \omega \arrow[r, mapsto] & (\mathcal{O}_A, d+\omega) & &
\end{tikzcd}$$

again: $S \mapsto \mathrm{Pic}^\natural_A(S) := \{L \in \mathrm{Pic}^0(A \times S) \mid \nabla : L \to L \otimes \Omega^1_{A\times S/S}, \nabla^2 = 0\}$

is functor, represented by a group scheme $A^\natural$ (smooth, qproj. of dim $2g$)

as before, the identity in $\mathrm{Hom}(A^{\natural}, A^{\natural}) = \mathrm{Pic}_{A}^{\natural}(A^{\natural})$ defines $\widetilde{P} \in \mathrm{Pic}^{\circ}(A \times A^{\natural})$ with $\nabla : \widetilde{P} \to \widetilde{P} \otimes \Omega^{1}_{A \times A^{\natural}/A^{\natural}}$.

Remark:- If $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_{A})$, $\tilde{p} : A \times A^{\natural} \to A$,

then $\mathcal{N} := \widetilde{P} \underset{\mathcal{O}_{A \times A^{\natural}}}{\otimes} \tilde{p}^{*}\mathcal{M}$ has relative connection

$\nabla : \mathcal{N} \to \mathcal{N} \otimes \Omega^{1}_{A \times A^{\natural}/A^{\natural}}$ (Leibniz rule)

- If $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathcal{O}_{A \times A^{\natural}})$ with $\nabla : \mathcal{N} \to \mathcal{N} \otimes \Omega^{1}_{A \times A^{\natural}/A^{\natural}}$

then we have (relative) de Rham complex:

$$DR^{\circ}(\mathcal{N}) := \left[0 \to \mathcal{N} \to \mathcal{N} \otimes \Omega^{1}_{A \times A^{\natural}/A^{\natural}} \to \cdots \to \mathcal{N} \otimes \Omega^{g}_{A \times A^{\natural}/A^{\natural}} \to 0 \right][g]$$

which is $\tilde{q}^{-1}\mathcal{O}_{A^{\natural}}$-linear, $\tilde{q} : A \times A^{\natural} \to A^{\natural}$

- If $R \in \mathrm{Mod}(\mathcal{O}_{A^{\natural}})$, then $\widetilde{P} \underset{\mathcal{O}_{A \times A^{\natural}}}{\otimes} \tilde{q}^{*} R$

has relative (to $A^{\natural}$) connection coming only from $\widetilde{P}$

Def.: (FM for $\mathcal{D}_A$-modules)

$$FM_{\widetilde{\mathcal{P}}} : D^b_{qc}(\mathcal{D}_A) \longrightarrow D^b_{qc}(\mathcal{O}_{A^\natural})$$

$$F^\cdot \longmapsto \widetilde{q}_+\left(\widetilde{\mathcal{P}} \otimes_{\mathcal{O}_{A\times A^\natural}} \widetilde{p}^* F^\cdot\right)$$

where $\widetilde{q}_+(-) := \widetilde{q}_*\left(DR^\cdot(-)\right)$

and $FM^\natural_{\widetilde{\mathcal{P}}} : D^b_{qc}(\mathcal{O}_{A^\natural}) \longrightarrow D^b_{qc}(\mathcal{D}_A)$

$$R^\cdot \longmapsto \widetilde{p}_*\left(\widetilde{\mathcal{P}} \otimes_{\mathcal{O}_{A\times A^\natural}} \widetilde{p}^* R^\cdot\right)$$

Prop.: Recall $FM_{\mathcal{P}} : D^b(\mathcal{O}_A) \to D^b(\mathcal{O}_{\hat{A}})$. Then we have for $G^\cdot \in D^b_{qcoh}(\mathcal{O}_A)$: $FM_{\widetilde{\mathcal{P}}}(\mathcal{D}_A \otimes_{\mathcal{O}_A} G^\cdot) \simeq \pi^* FM_{\mathcal{P}}(G^\cdot) \otimes \alpha^*_{A^\natural}(\omega_{A|\{0\}})$

Idea of proof: - $\exists$ iso $\underbrace{\widetilde{\mathcal{P}} \otimes \widetilde{p}^*(\mathcal{D}_A \otimes G^\cdot)}_{\text{left } \mathcal{D}\text{-mod. via Leibniz}} \simeq \underbrace{\mathcal{D}_{A\times A^\natural/A^\natural} \otimes (\underbrace{\widetilde{\mathcal{P}} \otimes \widetilde{p}^* G^\cdot}_{\text{no } \mathcal{D}\text{-mod}})}_{\text{"trivial" left } \mathcal{D}\text{-mod}}$

$\simeq$

$$\widetilde{\mathcal{P}} \otimes \mathcal{D}_{A\times A^\natural/A^\natural} \otimes p^* G^\cdot$$

$$P_\cdot\left[e \otimes (1 \otimes f)\right] \longmapsfrom P \otimes (e \otimes f)$$

Then $FM_{\tilde{\mathcal{P}}}(\mathcal{D}_A \otimes G) \simeq \tilde{q}_* DR^{\cdot}_{A\times A^\natural/A^\natural}\left(\mathcal{D}_{A\times A^\natural/A^\natural} \otimes (\tilde{\mathcal{P}} \otimes \tilde{p}^* G)\right)$

$$\underset{\uparrow}{\simeq} \tilde{q}_*\left[\Omega^g_{A\times A^\natural/A^\natural} \otimes \tilde{\mathcal{P}} \otimes \tilde{p}^* G\right] \underset{\uparrow}{\simeq} FM_{\mathcal{P}}(G) \otimes a^{\natural *}\omega_{A|\{0\}}$$

$DR^{\cdot}(\mathcal{M}) \simeq \Omega^{top} \overset{\mathbb{L}}{\underset{\mathcal{D}}{\otimes}} \mathcal{M}$

base change & $\tilde{\mathcal{P}} = (id \times \pi)^* \mathcal{P}$

$$\begin{tikzcd}
 & & A \times A^\natural \arrow[ddll, bend right, "\tilde{p}"'] \arrow[dl, "id\times\pi"] \arrow[dr, bend left, "\tilde{q}"] & \\
 & A \times \hat{A} \arrow[dl, "p"'] \arrow[dr, "q"] & & A^\natural \arrow[dl, "\pi"] \\
A & & \hat{A} &
\end{tikzcd}$$

Corollary: $FM_{\mathcal{P}} : D^b_{coh}(\mathcal{D}_A) \to D^b_{coh}(\mathcal{O}_{A^\natural})$

Pf: every $\mathcal{M} \in Mod_{coh}(\mathcal{D}_A)$ has resolution by induced $\mathcal{D}_A$-modules $\mathcal{D}_A \otimes M_i$, with $M_i \in Mod_{coh}(\mathcal{O}_A)$. Then use proposition and $FM_{\mathcal{P}} : D^b_{coh}(\mathcal{O}_A) \to D^b_{coh}(\mathcal{O}_{\hat{A}})$.

Theorem: We have:

1.) $FM_{\tilde{\mathcal{P}}^{\natural}} \circ FM_{\tilde{\mathcal{P}}} \simeq \iota^*[-g] : D^b_{qc}(\mathcal{D}_A) \to D^b_{qc}(\mathcal{D}_A)$

2.) $FM_{\tilde{\mathcal{P}}} \circ FM_{\tilde{\mathcal{P}}^{\natural}} \simeq \iota^*[-g] : D^b_{qc}(\mathcal{O}_{A^\natural}) \to D^b(\mathcal{O}_{A^\natural})$

main ingredient of proof

Lemma: We have

1.) $$H^i q_* \tilde{\mathcal{P}} = \begin{cases} 0 & \forall i \neq g \\ k_{\{0\}} & i = g \end{cases}$$

and

2.) $$R^i p_* \tilde{\mathcal{P}} = \begin{cases} 0 & \forall i \neq g \\ H^0_{\{0\}} \mathcal{O}_A & i = g \end{cases}$$

Laumon claims that this lemma implies the theorem by base change & projection formula.

I guess one can apply the previous constructions (e.g. lemma on p. 2) here, using base change + proj. formula for $\mathcal{D}$-modules.

Proof of lemma: 1.) To show: $[q_+ \tilde{\mathcal{P}}]_{|\{x\}} = 0 \quad \forall x \neq 0$

$\underset{\text{base change}}{\Longleftrightarrow}$ $\forall (\mathcal{L}, \nabla) \in A^\natural$, $(\mathcal{L}, \nabla) \neq (\mathcal{O}_A, d)$, we have

$$H^{i+g}_{dR}(\mathcal{L}, \nabla) = H^i DR(\mathcal{L}, \nabla) = 0$$

If $(\mathcal{L}, \nabla) \neq (\mathcal{O}_A, d)$, then $\ker \nabla$ is non-trivial local system of $rk = 1$ on A $\Rightarrow$ $H^0(A, \ker \nabla) = \{0\}$, hence $H^0(\mathcal{L}, \nabla) = 0$ (first quadrant spectral sequence)

proof by contradiction: let $n > 0$ be minimal with $H^{n_0}(\mathcal{L}, \nabla) \neq 0$

from $\mathcal{L} \in \hat{A}$ one deduces $m^*(\mathcal{L}, \nabla) \simeq p_1^*(\mathcal{L}, \nabla) \otimes p_2^*(\mathcal{L}, \nabla)$

$$\begin{tikzcd} & A \times A \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\ A & A & A \end{tikzcd}$$

$\underset{}{\overset{\text{Künneth}}{\Longrightarrow}}$ $H^{n_0}_{dR}(m^* \mathcal{L}, m^* \nabla) \simeq \bigoplus_{i+j=n_0} H^i_{dR}(\mathcal{L}, \nabla) \otimes H^j(\mathcal{L}, \nabla)$

Consider

$$\begin{tikzcd} A\times A \arrow[r, "m"] \arrow[rr, bend right, "a_{A\times A}"'] & A \arrow[r, "a_A"] & \{pt\} \end{tikzcd}$$

, then

$$H^{n_0}_{dR}(m^\circ\mathcal{L}, m^\circ\nabla) \simeq H^{n_0-2g}\, a_{A\times A,+}\left(m^\circ(\mathcal{L},\nabla)\right) =$$

$$H^{n_0-2g}\, a_{A,+}\, \underbrace{m_+\left(m^\circ(\mathcal{L},\nabla)\right)}_{\parallel} =$$

$\longleftarrow$ proj. formula

$$H^{n_0-2g}\, a_{A,+}\left[\underbrace{m_+\mathcal{O}_{A\times A}}_{\simeq\, \Omega^{\cdot}_A[g]\otimes_{\mathbb{C}}\mathcal{O}_A} \otimes (\mathcal{L},\nabla)\right]$$

$$= H^{n_0-g}\, a_{A,+}\left[\left(\Omega^{\cdot}_A \otimes_{\mathbb{C}} \mathcal{O}_A\right) \otimes (\mathcal{L},\nabla)\right]$$

Künneth

$$\overset{\downarrow}{=} \bigoplus_{i+j=n_0} H^i(A,\mathbb{C}) \otimes H^j(\mathcal{L},\nabla).$$ Hence

$$\bigoplus_{i+j=n_0} H^i_{dR}(\mathcal{L},\nabla) \otimes H^j_{dR}(\mathcal{L},\nabla) \simeq \bigoplus_{i+j=n_0} H^i(A,\mathbb{C}) \otimes H^j_{dR}(\mathcal{L},\nabla)$$

By assumption $H^k_{dR}(\mathcal{L},\nabla)=0 \;\forall k\in\{0,\dots,n_0-1\} \Rightarrow \mathrm{LHS}=0$

but $H^{n_0}_{dR}(\mathcal{L},\nabla)\neq 0$ & $H^0(A,\mathbb{C})\neq 0 \Rightarrow \mathrm{RHS}\neq 0$ ↯

left to show: $H^i q_+ \hat{\mathcal{P}}_{|\{0\}} = \begin{cases} 0 & i \neq g \\ k & i = g \end{cases}$

and 2.) . . . . . $\square$

Some further properties: iso of fcts: $D^b_{coh}(\mathcal{D}_A) \to D^b_{coh}(\mathcal{O}_{A^\natural})$

1.) $(\mathcal{L},\nabla) \in A^\natural \rightsquigarrow t_{(\mathcal{L},\nabla)} : A^\natural \to A^\natural$, $(\mathcal{L}',\nabla') \mapsto$

$(\mathcal{L}\otimes\mathcal{L}', \nabla\otimes id + id\otimes\nabla')$. Then $\exists$ iso

$$FM_{\hat{\mathcal{P}}}(-\otimes(\mathcal{L},\nabla)) \simeq Lt^*_{(\mathcal{L},\nabla)}\left(FM_{\hat{\mathcal{P}}}(-)\right)$$

2.) $\exists$ iso: $FM_{\hat{\mathcal{P}}} \circ \mathbb{D} \simeq \iota_{A^\natural} RHom_{\mathcal{O}_{A^\natural}}(FM_{\hat{\mathcal{P}}}(-), \mathcal{O}_{A^\natural})$

3.) $f : A \to B$ morphism $\rightsquigarrow$ pull-back $f^\natural : B^\natural \to A^\natural$

Then $\exists$ iso's:

$$Lf^{\natural,*} \circ FM_{\hat{\mathcal{P}}_A} \simeq FM_{\hat{\mathcal{P}}_B} \circ f_+ : D^b_{coh}(\mathcal{D}_A) \to D^b_{coh}(\mathcal{O}_{B^\natural})$$

$$Rf^\natural_* \circ FM_{\hat{\mathcal{P}}_B} \simeq FM_{\hat{\mathcal{P}}_A} \circ f^+ : D^b_{coh}(\mathcal{D}_B) \to D^b_{coh}(\mathcal{O}_{A^\natural})$$