Characterization of irreducible $\mathcal{D}$-modules on the torus

①

Let $(\mathbb{G}_m)^p = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, \dots, x_p^{\pm 1}]$ be the p-dim torus. Will write $x = (x_1, \dots, x_p)$

$\mathcal{D}_{\mathbb{G}_m^p} = \mathbb{C}[x, x^{-1}]\langle x\partial_x \rangle = \mathbb{C}[s]\langle \tau, \tau^{-1}\rangle$ for $s_i := -x_i\partial_{x_i}$, $\tau_i = x_i$.

The commutator relation is $\tau_i s_i = (s_i + 1)\tau_i$. ~~Analogously in higher d~~

M hol. $\mathcal{D}_{\mathbb{G}_m}$-module $\Rightarrow M(s) := \mathbb{C}(s) \otimes_{\mathbb{C}[s]} M$ fin. dim. over $\mathbb{C}(s)$ (Bernstein's thm 2)

$DR(M) := \Omega^\bullet_{(\mathbb{G}_m)^p} \otimes_{\mathcal{O}_{(\mathbb{G}_m)^p}} M[p]$ ~~(p dim of T)~~, $\chi(\mathbb{G}_m^p, M)$ = Euler char. of this complex.

Thm (Katz, $p=1$ case)

Let M be a hol. $\mathcal{D}$-module on $T := \mathbb{G}_m$.

1. $\chi(T, M) = \dim_{\mathbb{C}(s)} M(s)$
2. Assume $\dim_{\mathbb{C}(s)} M(s) = 1$. Then the following are equivalent:
   a) M is irreducible
   b) M has neither a submodule nor a quotient that is $\mathbb{C}[s]$-torsion
   c) $M \cong \mathcal{H}_{P,Q} := \underbrace{\mathbb{C}[s]\langle \tau, \tau^{-1}\rangle}_{\mathcal{D}_T} / \underbrace{\mathbb{C}[s]\langle \tau, \tau^{-1}\rangle}_{\mathcal{D}_T} \cdot (P(s)\tau + Q(s))$

      where P and Q have no common zero mod $\mathbb{Z}$.

Q is the Γ-function w.r.t. x which makes no sense. $\Gamma(x)$ is not a divisor

Proof: 2. a$\Leftrightarrow$b: a$\Rightarrow$b clear, a$\Leftarrow$b: Assume M not irreducible and M' submodule. Then either $\dim_{\mathbb{C}(s)} M'(s) = 0$ and M' is $\mathbb{C}[s]$ torsion or $\dim_{\mathbb{C}(s)} M'(s) = 1$ and M/M' is $\mathbb{C}[s]$ torsion.

c) $\Rightarrow$ a) Prop. by Katz

a) $\Rightarrow$ c): m generator of $M \Rightarrow \tau \cdot m = a(s) \cdot m$ for some $a(s) \in \mathbb{C}(s)$.

$h \in \mathbb{C}(s)\setminus\{0\}$: $m' = h(s) \cdot m$ different basis of $M(s) \Rightarrow a'(s) = \frac{h(s+1)}{h(s)} a(s)$

$\Rightarrow \exists$ basis m' s.t. $a'(s) = -\frac{Q(s)}{P(s)}$ s.t. P, Q have no common zero mod $\mathbb{Z}$.

$M' := \underbrace{\mathbb{C}[s]\langle \tau, \tau^{-1}\rangle}_{\mathcal{D}} \cdot m' \subseteq M(s) \Rightarrow \mathcal{H}_{P,Q} \longrightarrow M'$, $[1] \mapsto m'$ is surj. and inj. because $\mathcal{H}_{P,Q}$ irred.. Let $h(s) = \frac{p(s)}{q(s)}$.

$\Rightarrow \mathcal{H}_{P,Q} \cong \underbrace{\mathbb{C}[s]\langle \tau, \tau^{-1}\rangle}_{\mathcal{D}} \cdot m' \overset{\text{irred.}}{=} \underbrace{\mathbb{C}[s]\langle \tau, \tau^{-1}\rangle}_{\mathcal{D}} (q(s) m') = \underbrace{\mathbb{C}[s]\langle \tau, \tau^{-1}\rangle}_{\mathcal{D}} (p(s) m) \overset{\text{irred}}{=} \mathbb{C}[s]\langle \tau, \tau^{-1}\rangle = M$

1. $DR(M)$: $0 \to \overset{-1}{M} \xrightarrow{x\partial_x} \overset{0}{M} \to 0$. Let $i: \{0\} \hookrightarrow \operatorname{Spec} \mathbb{C}[s]$. Then $Li^*M = 0 \to M \xrightarrow{s} M \to 0$. Let $N \subset M$ fin. gen. $\mathbb{C}[s]$-module s.th $\mathbb{C}(s) \otimes_{\mathbb{C}[s]} N = M(s)$. Then $\chi(Li^* N) = \underbrace{~~\chi(\dots)~~}_{R \otimes \mathbb{C}[s]} = \underbrace{~~\dots~~}_{R \otimes \mathbb{C}(s)}$

$= \dim N(s) = \dim M(s)$. for $R \to N$ free resolution

To show: $\chi(L_i^* N) = \chi(L_i^* M)$. Let $M_0 = N$ and for $k \geq 0$ $M_k = \sum_{|j| \leq k} \tau^j N$. Since $M_1 \otimes_{\mathbb{C}[s]} \mathbb{C}(s) = M_0 \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$, there is $b(s) \in \mathbb{C}[s]$ s.t.

$b(s) \cdot M_1 / M_0 = \{0\}$. $\Rightarrow b(s+k)\, b(s-k) \cdot M_{k+1}/M_k = \{0\}$.

For $k \gg 0$ $s \nmid b(s+k)\, b(s-k) \Rightarrow s : M_{k+1}/M_k \to M_{k+1}/M_k$ is bijective.

$\Rightarrow 0 \to M \xrightarrow{s} M \to 0$ is quasi-isomorphic to $0 \to M_k \xrightarrow{s} M_k \to 0$.

~~Tak~~ Now take M_k as the new N $\square$.

Now the general case for higher dimensions:

Let $L : \mathbb{C}^p \to \mathbb{C}$ given by $(s_1, \dots, s_p) \mapsto (\sum l_i s_i,\ l_i \in \mathbb{Z})$ and

$i_L : G_m \to (G_m)^p$, $x \mapsto (x^{l_i})_i$. ~~Then Mellin~~ Let M be a hol. $\mathcal{D}_{G_m}$-module.

The Mellin-transform of $i_{L+} M$ is

$L^* M = \mathbb{C}[s] \otimes_{\mathbb{C}[\sigma]} M$ for $\sigma = L(s)$ with the action of τ_i being

$\tau_i (1 \otimes m) = 1 \otimes \tau^{l_i} m$. In particular $L^* M$ is holonomic. $\square$

Def: Let $\mathcal{H}_{P,Q,L} = L^* \mathcal{H}_{P,Q}$

Thm (Sabbah, Loeser)

Let M be a holonomic $\mathcal{D}$-module on $(G_m)^p$.

1. $\chi((G_m)^p, M) = \dim_{\mathbb{C}(s)} M(s)$
2. Let $\dim_{\mathbb{C}(s)} M(s) = 1$. Then the following are equivalent:

a) M is irreducible

b) M has neither a submodule nor a quotient that is $\mathbb{C}[s]$-torsion.

c) M is isomorphic to the unique irreducible holonomic $\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ submodule of a tensor product over fin. many $\mathcal{H}_{P,Q,L}$, ~~RQ~~ for each factor P, Q with no common zero mod $\mathbb{Z}$.

Rmk: Let M' be a tensor product as in c). Then it is holonomic, has finite length and thus an irred. submodule. Let M_1, M_2 be irred. submodules. Then $M_1 \cap M_2 = \{0\}$, thus $\exists\ M_1 \hookrightarrow M'/M_2$, since $M'(s) = M_2(s)$ the latter is torsion, so $M_1 = \{0\}$.

Prf: $a \Leftrightarrow b$ as in the $p=1$ case, $c) \Rightarrow a)$ trivial.

$a) \Rightarrow c)$ Choose a basis m of $M(s)$. Let $\tau_i \cdot m = \varphi_i \cdot m$ for $\varphi_i \in \mathbb{C}(s)$

Then $\tau_i \tau_j \cdot m = \tau_i\, \varphi_j(s) \cdot m = \varphi_j(s + 1_i) \cdot \cancel{\varphi} \tau_i \cdot m = \varphi_j(s+1_i)\, \varphi_i(s) \cdot m$

On the other hand $\tau_j \tau_i m = \tau_i \tau_j m = \varphi_i(s+1_j)\,\varphi_j(s)\cdot m$.

Thus we obtain the integrability condition:

$$\frac{\varphi_i(s+1_j)}{\varphi_i(s)} = \frac{\varphi_j(s+1_i)}{\varphi_j(s)}.$$

Lemma: ~~There is another basis m' with $\tau_i m' = \varphi_i m'$~~, up to base change s.t.

$$\varphi_i(s) = c_i \cdot \prod_{L \in \mathcal{L}} \prod_{\alpha \in \mathbb{Q}/\mathbb{Z}} \prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha + \lambda)^{n_i(L,\alpha,\lambda)}, \quad \mathcal{L} \text{ integral linear form}$$

Rmk: Let $m' = h \cdot m$. Then $\varphi_i' m' = \varphi_i'(s) h(s) m = \cancel{\tau_i h(s) m = h(s+1_i)\tau_i m = h(s+1_i)\varphi_i m}$

$$\Longrightarrow \varphi_i'(s) = \frac{h(s+1_i)}{h(s)} \varphi_i(s).$$

Prf: $p \geq 2$ Let $P(s)$ be an irreducible factor of some φ_i. Let $\prod_{\sigma \in \mathbb{Z}^p} P(s+\sigma)^{n_i(P,\sigma)}$ be all integer translates of P appearing in φ_i for any i.

If P is not invariant under any translation by some $\sigma \in \mathbb{Z}^p$, this expression is unique. From the integrability condition (counting the exponent of $P(s+\sigma)$)

$$n_i(P, \sigma - 1_j) - n_i(P, \sigma) = n_j(P, \sigma - 1_i) - n_j(P, \sigma) =: n_{ij}(\sigma)$$

Let $m(\sigma) = -\sum_{k \geq 0} n_i(P, \sigma - k1_i)$. Then $m(\sigma - 1_i) - m(\sigma) = n_i(P,\sigma)$ and

$$-n_i(P, \sigma - k1_i) = \sum_{l \geq 0} n_{ij}(\sigma - k1_i - l1_j) \Longrightarrow m(\sigma) = \sum_{l \geq 0}\sum_{k \geq 0} n_{ij}(\sigma - k1_i - l1_j)$$

doesn't depend on i. m has finite support, because n_i has and

$$-\sum_{k \in \mathbb{Z}} n_i(P, \sigma - k1_i) = -\sum_{k \in \mathbb{Z}} n_i(P, \sigma - l1_j - k1_i) = 0 \text{ for } l \gg 0.$$

NEED TWO coords for this!!!

$\Longrightarrow$ For $h(s) = \prod_{\sigma \in \mathbb{Z}^p} P(s+\sigma)^{m(\sigma)}$ we have $\frac{h(s+1_i)}{h(s)} = \prod_{\sigma \in \mathbb{Z}^p} P(s+\sigma)^{n_i(P,\sigma)}$

So after base change, the factor P vanishes.

If P is stable under translation by some sublattice $\Gamma \subset \mathbb{Z}^p$, it descends through $\pi: \mathbb{C}^p \to \mathbb{C}^p / \mathbb{C}\otimes\Gamma \cong \mathbb{C}^q$ and we can repeat the argument. Let $q = 1$, then $\pi: \mathbb{C}^p \to \mathbb{C}$ is given by $L(s) = \lambda_1 s_1 + \dots + \lambda_p s_p$, $\lambda_i \in \mathbb{Z}$ coprime, s.t. $L(\Gamma) = 0$. P reduced to $t - \alpha$, t coordinate of $\mathbb{C}$. The pullback along π is $L(s) - \alpha$ which implies the lemma. $\square$

Lemma: $M(s)$ is isomorphic to a tensor product of fin. many $\mathcal{H}_{P,Q,L}(s)$, P, Q no common zeros mod $\mathbb{Z}$

Observation: Let $L(s) = \lambda_1 s_1 + \dots + \lambda_p s_p$. Then

analogous for $Q = s^{\lambda} - \alpha$. for λ_i

$$L^* \mathcal{H}_{1,t-\alpha} = \cancel{\ldots} / \tau_i + (L(s)-\alpha)(L(s)-\alpha+1)\cdots(L(s)-\alpha+\lambda_i-$$

Prf of the lemma: tensor product of $L^*\mathcal{H}_{p,Q,L}$ "=" multiplying (4)

the corresponding φ_i. Fix a $L \in \mathcal{L}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$.

Let $n_i(\lambda)$ be the exponent of $L(s) - \alpha + \lambda$ in ~~pth.~~ φ_i.

~~Example: For $L^*\mathcal{H}_{1,s,\ldots}$ we have $n_i(\lambda) = \mathbb{1}_{[0,\lambda_i-1]}$ for $\lambda_i > 0$ and $n_i(\lambda) = \mathbb{1}_{[\lambda_i+1,0]}$ for $\lambda_i < 0$ and $n_i(\lambda) = 0$ for $\lambda_i = 0$.~~

From the integrability condition: $n_i(\lambda - \lambda_j) - n_i(\lambda) = n_j(\lambda - \lambda_i) - n_j(\lambda) = n_{ij}(\lambda)$.

~~Analogously to the above~~ set: $m(\lambda) := \begin{cases} -\sum_{k \geq 0} n_i(\lambda - k\lambda_i) & \text{for } \lambda_i > 0 \\ \sum_{k \geq 1} n_i(\lambda + k\lambda_i) & \text{for } \lambda_i < 0 \\ 0 & \text{for } \lambda_i = 0. \end{cases}$

Again, we can write this as an expression in n_{ij}, so it doesn't depend on i. $m(\lambda)$ is constant for $\lambda \gg 0$, BUT not necessarily zero.

If $m(\lambda) = 0$ for $\lambda \gg 0$, by the same argument as above, from $m(\lambda - \lambda_i) - m(\lambda) = n_i(\lambda)$ we obtain

$$\prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha + \lambda)^{n(\lambda)} = \prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha + \lambda)^{m(\lambda - \lambda_i) - m(\lambda)} = \frac{\prod_{\lambda \in \mathbb{Z}} (L(s+1) - \alpha + \lambda)^{m(\lambda)}}{\prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha - \lambda)^{m(\lambda)}} = \frac{h(s+1)}{h(s)}$$

for $h \in \mathbb{C}(s)$. If $m(\lambda) \neq 0$ for $\lambda \gg 0$, wlog $m(\lambda) > 0$, divide φ_i by $(L(s) - \alpha)(L(s) - \alpha + 1) \ldots (L(s) - \alpha + \lambda_i - 1)$ to reduce m by one. Induction shows, that φ_i only consists of such factors up to equivalence. $\square$

Prf of $a \Rightarrow c$: Let M' be a tensor product of $\mathcal{H}_{p,Q,L}$ s.t. $M(s) = M'(s)$. Let M_1 be the unique irreducible submodule of M' and m_1 a generator of M_1, m generator of M. As in the $p=1$ case: for $m_1 = \frac{p(s)}{q(s)} m$ we have

$$M_1 = \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle \cdot m_1 = \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle (q(s) m_1) = \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle (p(s) m) = \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle \cdot m = M$$

$\square$

Sketch of 7: $N \subset M$ fin. gen. $\mathbb{C}[s]$-module s.t. $N(s) = M(s)$ and $M = \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle \cdot N$. $\exists\, b$ s.t. for $\mathbb{C}[s]_b = \mathbb{C}[s, (b(s+\sigma))^{-1}]_{\sigma \in \mathbb{Z}^p}$ holds $\mathbb{C}[s]_b \otimes_{\mathbb{C}[s]} N = \mathbb{C}[s]_b \otimes_{\mathbb{C}[s]} M$ (just invert what needs to be inverted.)

After integral linear change of coords $s_p + b(s+\sigma)$. Let $i_p : \{s_p = 0\} \hookrightarrow \operatorname{Spec} \mathbb{C}[s]$.

Use induction.