Talk 2: Characterization of irreducible D-modules on the torus.

Let $\mathbb{G}_m^p = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, \dots, x_p^{\pm 1}]$.

$D_{\mathbb{G}_m^p} = \mathbb{C}[x, x^{-1}]\langle x\partial_x \rangle = \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ for $s_i = -x_i \partial_{x_i}$, $\tau_i = x_i$

$\tau_i s_i = (s_i + 1)\tau_i$

Let M be a hol. $D_{\mathbb{G}_m^p}$-module, $M(s) : \mathbb{C}(s) \otimes_{\mathbb{C}[s]} M$ is a fin. dim $\mathbb{C}(s)$ vector space.

$DR(M) := \Omega^\bullet_{\mathbb{G}_m^p} \otimes_{\mathcal{O}_{\mathbb{G}_m}} M[p]$, $\chi(\mathbb{G}_m^p, M)$ is the Euler char of $DR(M)$

Thm ($p = 1$ case, Katz)

Let M be a holonomic $D_{\mathbb{G}_m}$-module

1. $\chi(\mathbb{G}_m, M) = \dim_{\mathbb{C}(s)} M(s)$
2. Assume $\dim_{\mathbb{C}(s)} M(s) = 1$. TFAE.
   a) M is irreducible
   b) M has neither a submodule nor a quotient that is $\mathbb{C}[s]$-torsion.
   c) $M \cong \mathcal{H}_{P,Q} := \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle / P(s)\tau + Q(s)$, $P, Q \in \mathbb{C}[s]$ with no common zeros mod $\mathbb{Z}$.

Proof 2 a⇒b clear, b⇒a: Assume M not irreducible $M' \subset M$ submodule.

Either $\dim_{\mathbb{C}(s)} M'(s) = 0$, then M' is $\mathbb{C}[s]$-torsion,

or $\dim_{\mathbb{C}(s)} M'(s) = 1$, M/M' is $\mathbb{C}[s]$-torsion

c) ⇒ a) Thm (Katz)

a) ⇒ c) Let m be a generator of M as $D_{\mathbb{G}_m}$-module. Let

$$\tau \cdot m = a(s) \cdot m, \quad a(s) \in \mathbb{C}(s).$$

$m' = h(s) \cdot m$ other basis of $M(s)$.

$\tau m' = a'(s) \cdot m'$

$$a'(s) m' = \tau m' = \tau h(s) \cdot m = h(s+1) \tau m = h(s+1) a(s) m = \frac{h(s+1)}{h(s)} a(s) m'$$

$$\Rightarrow a'(s) = \frac{h(s+1)}{h(s)} a(s)$$

⇒ Can write $a'(s) = -\frac{Q(s)}{P(s)}$ s.t. P, Q have no common zeros mod $\mathbb{Z}$.

$M' := \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle m' \subset M(s)$ (,,D'')

$h(s) = \frac{p(s)}{q(s)}$ $\quad \mathcal{H}_{P,Q} \to M'$, $[1] \mapsto m'$ is an isomorphism

$h(s) = \frac{1}{q(s)}$ $\mathcal{H}_{P,Q} \to M'$, $[1] \to m'$ is an isomorphism

$$\mathcal{H}_{P,Q} \cong D \cdot m' = D \cdot q(s) \cdot m' = D \cdot p(s) \cdot m = D \cdot m = M.$$

1. $DR(M)$: $0 \to M \xrightarrow{\cdot x\partial} M \to 0$, Let $i: \{0\} \to \mathrm{Spec}\, \mathbb{C}[s]$. Then

$\mathbb{L} i^* M$: $0 \to M \xrightarrow{\cdot s} M \to 0$. Let N be a fin. gen. $\mathbb{C}[s]$-submodule of M s.t. $N(s) = M(s)$.

Then $\chi(\mathbb{L} i^* N) = \dim_{\mathbb{C}(s)} N(s) = \underline{\dim_{\mathbb{C}(s)}(M(s))}$

To show $\chi(\mathbb{L} i^* N) = \chi(\mathbb{L} i^* M)$

Let $M_0 = N$, for $k > 0$ $M_k = \sum_{|j| \le k} \tau^j N$.

$M(s) \supset M_1(s) = M_0(s)$.

$\Rightarrow \exists\, b(s) \in \mathbb{C}[s]$ $b(s) \cdot M_1/M_0 = \{0\}$

$b(s+k)\, b(s-k) \cdot M_{k+1}/M_k = \{0\}$

For $k \gg 0$ st $b(s+k)\, b(s-k)$

$\Rightarrow s: M_{k+1}/M_k \to M_{k+1}/M_k$ is bijective.

$0 \to M \xrightarrow{\cdot s} M \to 0$ is quasi-isomorphic to

$0 \to M_k \xrightarrow{\cdot s} M_k \to 0$ for some k.

Choose M_k as new N. $\square$

General case: Let $L: \mathbb{C}^p \to \mathbb{C}$, $(s_1, \dots, s_p) \mapsto (\lambda_1 s_1 + \dots + \lambda_p s_p)$, $\lambda_i \in \mathbb{Z}$.

and $i_L: \mathbb{G}_m \to \mathbb{G}_m^p$, $x \mapsto (x^{\lambda_i})_i$.

Let M be holonomic $D_{\mathbb{G}_m}$-module.

The Mellin-transform of $i_{L+} M$ is

$L^* M = \mathbb{C}[s] \otimes_{\mathbb{C}[\sigma]} M$, for $\sigma = L(s)$ and

$\tau_i(1 \otimes m) = 1 \otimes \tau^{\lambda_i} m$

In particular $L^* M$ is holonomic.

<u>Def.</u> Let $\mathcal{H}_{P,Q,L} = L^* \mathcal{H}_{P,Q}$

Th...

Def: Let $H_{p,Q,L} = L^* \mathcal{H}_{p,Q}$

Thm (Sabbah, Loeser)

Let M be holonomic $D_{\mathbb{G}_m^p}$-module.

1. $\chi(\mathbb{G}_m^p, M) = \dim_{\mathbb{C}(s)} M$

2. Let $\dim_{\mathbb{C}(s)} M = 1$. TFAE:
   a) M is irreducible
   b) M has no quotient and no submodule that is $\mathbb{C}[s]$-torsion.
   c) M is isomorphic to the unique irreducible holonomic submodule of a tensor product of fin. many $H_{p,Q,L}$.

Rmk: Let M' be this tensor product. M' is hol., has finite length
$\Rightarrow$ has irred submodule. Let M_1, M_2 be irred submodules
$M_1 \cap M_2 = \{0\}$, thus $\exists\ M_1 \hookrightarrow M'/M_2$, $M'(s) = M_2(s)$
so M'/M_2 has $\mathbb{C}[s]$ torsion, so $M_1 = \{0\}$

Prf: a$\Leftrightarrow$b as in $p=1$ case.
c)$\Rightarrow$a) trivial
a)$\Rightarrow$c). m generator of M.

Let $\tau_i \cdot m = \varphi_i \cdot m$, $\varphi_i(s) \in \mathbb{C}(s)$

$\tau_i \tau_j \cdot m = \tau_i \varphi_j(s) \cdot m = \varphi_j(s+1_i) \tau_i \cdot m = \varphi_j(s+1_i)\varphi_i(s) \cdot m$

$\tau_j \tau_i m = \varphi_i(s+1_j)\varphi_j(s) \cdot m$

Integrability condition: $\frac{\varphi_i(s+1_j)}{\varphi_i(s)} = \frac{\varphi_j(s+1_i)}{\varphi_j(s)}$.

Lemma Up to base change

$$\varphi_i = c_i \prod_{L \in \mathcal{L}} \prod_{\alpha \in \mathbb{C}/\mathbb{Z}} \prod_{\lambda \in \mathbb{Z}} (L(s) - \alpha + \lambda)^{n_i(\alpha,\lambda)}$$

$\mathcal{L}$ integral linear functions

Rmk: $m' = h(s) \cdot m$ Then for $\tau_i m' = \psi_i \cdot m'$ we have

$$\psi_i = \frac{h(s+1_i)}{h(s)} \varphi_i$$

Prf: $p \geq 2$, Let $P(s)$ be some irreducible factor of φ_i, Let $\prod_{\sigma \in \mathbb{Z}^p} P(s+\sigma)^{n_i(\sigma)}$ be all traslates of P appearing in φ_i.

$\prod_{\sigma \in \mathbb{Z}^p} P(s+\sigma)^{n_i(\sigma)}$ be all traslates of P appearing in φ_i.

If P is not invariant under translation by any $\sigma \in \mathbb{Z}^p$.

Then this expression is unique.

From the integrability condition: $n_i(\sigma - 1_j) - n_i(\sigma) = n_j(\sigma - 1_i) \cdot n_j(\sigma) = n_{ij}(\sigma)$

Let $m(\sigma) = -\sum_{k \geq 0} n_i(\sigma - k \cdot 1_i)$

Then $m(\sigma - 1_i) - m(\sigma) = n_i(\sigma)$

$-n_i(\sigma - k 1_i) = \sum_{l \geq 0} n_{ij}(\sigma - k 1_i - l 1_j) = \sum n_{ik} = \sum n_{ki}$

$\Rightarrow m$ does not depend on i.

m has finite support.

$-\sum_{k \in \mathbb{Z}} n_i(\sigma - k \cdot 1_i) = -\sum_{k \in \mathbb{Z}} n_i(\sigma - k 1_i - 1_j) = -\sum_{k \in \mathbb{Z}} n_i(\sigma - k 1_i - l 1_j) = 0$

for some l

$h(s) = \prod_{\sigma \in \mathbb{Z}^p} P(s+\sigma)^{m(\sigma)}$

If P is stable under translation by some sublattic. $\Gamma \subset \mathbb{Z}^p$

it decends through $\pi: \mathbb{C}^p \to \mathbb{C}^p / \mathbb{C} \otimes \Gamma \cong \mathbb{C}^q$

Let $q = 1$, then $\pi: \mathbb{C}^p \to \mathbb{C}$, given by $L(s) = \lambda_1 s_1 + \dots + \lambda_p s_p$ $\lambda_i \in \mathbb{Z}$

So we reduced to $P(t) = t - \alpha$

Pullback under L is than $L(s) - \alpha$

Lem... [illegible]

... under L is than $L(s)-a$

Lemma $M(s)$ is isomorphic to tensor product of $\mathcal{H}_{x,a,L}$

Fix L, a, exponent of $L(s)-a_{t,i}$ in φ_i is $n_i(\lambda)$

everything as above, except $n(\lambda) \neq 0$ for $\lambda > 0$ in general.

$$L^* \mathcal{H}_{1,t-a} = D \Big/ \tau_i + \left((L(s)-a) \cdot (L(s)-a+1) \cdots (L(s)-a+\lambda_i-1) \right)$$

corresponds to $n_i(\lambda) = \mathbb{1}_{[0,\lambda_i-1]}$

$M(s) = M'(s)$