Generalized Fourier transform and applications

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Day 1

1.1. Introduction

In this first talk of the reading group we will be covering the following three points:

(1) First we will introduce the geometric Fourier transform of $\mathcal{D}$-modules on an affine group scheme isomorphic to $\mathbb{G}_a^r$. The output of the geometric Fourier transform is again a $\mathcal{D}$-module, this time on the dual of our original space. We will discuss a few properties of the transform and illustrate its connection to the analytic Fourier transform.

(2) Secondly we will introduce the geometric Mellin transform of $\mathcal{D}$-modules on split tori. This time the output of the geometric Mellin transform is no longer a $\mathcal{D}$-module but a module over the ring Δ of difference operators. We will review some properties of this transform and illustrate its connection to the analytic Mellin transform.

(3) The final part consists on taking the first step towards defining a setting where the two above discussed transformations can be treated on equal footing. For this we will see that both transformations can be written as a geometric kernel.

The main reference for the following sections is [Lau85]. Other references will be indicated through the text.

1.2. Geometric Fourier transform on vector spaces

Fix a characteristic zero field k. Let V be a group k-scheme isomorphic to $\mathbb{G}_{a,k}^r$. Following [Lau85] we will use the French word "vectoriel" to denote such V. Denote $V^\vee = Hom_{k-gr}(V, \mathbb{G}_{a,k})$ the dual vectoriel of V over k.

Let $\mathcal{D}_V$ be the sheaf of differential operator of finite order on V, we will use the following notation for its global sections $D_V := H^0(V, \mathcal{D}_V)$. As k-algebra is generated by the commutative algebras $Sym_k(V)$ and $Sym_k(V^\vee)$ with the commutation rule $[v, v^\vee] = \langle v, v^\vee \rangle$. The operation $\langle\ ,\ \rangle$ is the canonical pairing evaluating $v^\vee$ at the element v. In coordinates we want to regard elements of $Sym_k(V)$ as polynomials with derivations as variables and $Sym_k(V^\vee)$ as polynomials with coordinates as variables. The sub-algebra $H^0(V, \mathcal{O}_V) \subset D_V$ is generated by $Sym_k(V^\vee)$.

Definition 1.2.1. ***(Geometric Fourier morphism)*** *Let V be a vectoriel as above and $V^\vee$ its dual. We call geometric Fourier morphism to the k-isomorphism:*

$$F : D_V \longrightarrow D_{V^\vee} \tag{1.1}$$

induced by $id_{V^\vee} : V^\vee \longrightarrow V^\vee$, $-id_V : V \longrightarrow V$ and the natural identification of V with $V^{\vee\vee}$. Taking coordinates $(x_1, \dots, x_r)$ on V and dual coordinates $(\xi_1, \dots, \xi_r)$ on $V^\vee$ the geometric Fourier morphism is given by $x_i \mapsto \partial_{\xi_i}$ and $\partial_{x_i} \mapsto -\xi_i$.

Definition 1.2.2. ***(Geometric Fourier transform)*** *Let V be a vectoriel as above and $V^\vee$ its dual. The geometric Fourier morphism (1.1) induces a functor:*

$$F : Mod_{qcoh}(\mathcal{D}_V) \longrightarrow Mod_{qcoh}(\mathcal{D}_{V^\vee}) \tag{1.2}$$

Defined by $F(M) = \widetilde{F_* H^0(V, M)}$ for M a quasi-coherent $\mathcal{D}_V$-module. This functor is called the geometric Fourier transform.

This functor is exact as it is the composition of exact functors. It extends to a functor:

$$F : D^b_{qcoh}(\mathcal{D}_V) \longrightarrow D^b_{qcoh}(\mathcal{D}_{V^\vee})$$

which by construction preserves complexes concentrated in degree 0, coherent objects and holonomic objects.

Remark 1.2.3. **(Applying the geometric Fourier transform)** Let us now check what happens when we apply the geometric Fourier transformation. What one should expect is the same underlying set with a new action, meaning a new $\mathcal{D}_{V^\vee}$-module structure. We will see that indeed this is the case.

Let M be a $\mathcal{D}_V$-module, it is enough to check what happens when we push forward $H^0(V, M)$ since the other operations involved give an equivalence of categories between D_V-modules and quasi-coherent $\mathcal{D}_V$-modules. Then we assume M is a D_V-module.

$$F_* M = Hom_{D_V}(F^* D_{V^\vee}, M)$$

First notice that the D_V-module structure on $F^* D_{V^\vee}$ is nothing but the D_V-module structure induced by the D_V-algebra structure $F : D_V \longrightarrow D_{V^\vee}$. In particular since F is an isomorphism $F^* D_{V^\vee}$ is free of rank 1. Thus $Hom_{D_V}(F^* D_{V^\vee}, M) \cong M$ identifying the morphisms with the image of the unit. Finally we have to check that the $D_{V^\vee}$-action is the appropriated one. Take $m \in F_* M$

$$x_i \cdot m = F(x_i)m = \partial_{\xi_i} m$$

$$\partial_{x_i} \cdot m = F(\partial_{\xi_i})m = -\xi_i m$$

Example 1.2.4. (Connecting Fourier transforms from geometry and analysis) The purpose of this example is to exhibit some connection between the geometric Fourier transform and the analytic Fourier transform. First we quickly review the analytic Fourier transform, definition and some properties that we will use. Fourier transform is part of any basic analysis undergraduate course, references are omitted.

Definition 1.2.5. ***(Fourier transform)*** *The Fourier transform of a complex-valued function $f(x)$ on $\mathbb{R}^n$ is the complex valued function:*

$$\mathcal{F}[f(x)](s) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x,s \rangle} dx \tag{1.3}$$

where $s \in \mathbb{R}^n$.

For functions that are "nice enough" the original function can be recovered from its Fourier transform using the inverse transformation:

$$\mathcal{F}^{-1}[g(s)](x) := \int_{\mathbb{R}^n} g(s) e^{2\pi i \langle x,s \rangle} ds$$

Functions that are "nice enough" are for instance rapidly decreasing functions. These functions form the so called Schwartz space. In this space the Fourier transform is an automorphism. Is one allows distributions, then the Fourier transform on the Schwartz space extends by continuity to an isomorphism of the space of temperated distributions.

Here are some properties of the transformation, we will enunciate them for the 1-dimensional case:

- It is a linear operator,
- It exchanges the roles of the operators differentiation and multiplying by the variable:

$$\mathcal{F}\left[\frac{\partial^k f(x)}{\partial x^k}\right](s) = (2\pi i s)^k \mathcal{F}[f(x)](s), \quad \mathcal{F}[x^k f(x)](s) = \left(\frac{i}{2\pi}\right)^k \mathcal{F}\left[\frac{\partial^k f(x)}{\partial s^k}\right](s) \tag{1.4}$$

Consider the affine line $\mathbb{A}^1_{\mathbb{C}}$ with coordinate x and the differential operator:

$$P := \partial_x'' + \frac{2}{\lambda} - \frac{4x^2}{\lambda^2} \qquad \lambda \in \mathbb{R}_+ \tag{1.5}$$

Denote by M the $\mathcal{D}$-module associated to P, i.e. $M = \mathcal{D}/\mathcal{D}P$. We can apply the geometric Fourier transform to M and then we obtain a $\mathcal{D}$-module, say $N := F(M)$. The question we may ask ourselves now is how can we express the relationship between the solutions of M and the solutions of N.

Let s be the coordinate in $\mathbb{A}^1_{\mathbb{C}}$ regarded as the dual of our affine line of coordinate x. Now, the isomorphism (1.1) needs to be modified a bit so everything matches exactly on both sides. This time we consider:

$$\begin{aligned} \mathbb{C}[x]\langle\partial_x\rangle &\xrightarrow{F} \mathbb{C}[s]\langle\partial_s\rangle \\ x &\longmapsto \tfrac{i}{2\pi}\partial_s \\ \partial_x &\longmapsto 2\pi i s \end{aligned}$$

This definition is motivated by properties (1.4). Now the geometric and analytic Fourier transforms operate equally on differential operators.

The differential operator we need to consider on the dual side is:

$$Q := F(P) = \partial_s'' + 2\lambda\pi^2 - 4\lambda^2\pi^4 s^2 \quad \lambda \in \mathbb{R}_+ \tag{1.6}$$

We can now rewrite $N = \mathcal{D}/\mathcal{D}Q$.

The solution to the ODE defined by P (1.5) is given by $f(x) = Ce^{-x^2/\lambda}$ for $C \in \mathbb{C}^*$ and λ as before. We compute its Fourier transform:

$$\mathcal{F}\left[Ce^{-x^2/\lambda}\right](s) = C\int_{-\infty}^{+\infty} e^{-x^2/\lambda - 2\pi i x s}dx \overset{*}{=} \tag{1.7}$$

We want to use integration by parts:

$$\partial_x\left(e^{-x^2/\lambda-2\pi i x s}\right) = -\frac{2x}{\lambda}e^{-x^2/\lambda-2\pi i x s} - 2\pi i s e^{-x^2/\lambda-2\pi i x s}$$

Now we isolate the part that appears inside the integral sign:

$$e^{-x^2/\lambda-2\pi i x s} = -\frac{1}{2\pi i s}\partial_x\left(e^{-x^2/\lambda-2\pi i x s}\right) - \frac{x}{\pi i \lambda s}e^{-x^2/\lambda-2\pi i x s}$$

Continuing with the computation of (1.7):

$$\stackrel{*}{=} -\frac{C}{\pi i\lambda s}\int_{-\infty}^{+\infty} xe^{-x^2/\lambda-2\pi ixs}dx = -\frac{1}{\pi i\lambda s}\frac{i}{2\pi}\frac{\partial\mathcal{F}\left[Ce^{-x^2/\lambda}\right](s)}{\partial s} \tag{1.8}$$

Therefore the Fourier transform of our function satisfies the differential equation:

$$\partial_s + 2\pi^2\lambda s = 0$$

The solution to this equation is given by $g(x) := \mathcal{F}\left[Ce^{-x^2/\lambda}\right](s) = Ke^{-\pi^2\lambda s^2}$ for $K \in \mathbb{C}$. Last thing that remains to be done is to check that $Qg = 0$, which is straightforward and left to the reader.

The upshot of the example is that we have seen a motivation for calling the transformation we defined on $\mathcal{D}$-modules the Fourier transform. We have also explored one connection between both transforms, one operated at the level of linear differential operators and the other at the level of solutions to linear differential equations.

We have chosen to use explicit computations of the solutions but this can be done in a more systematic way. Let $f(x)$ be a function that solves the ODE defined by P:

$$\partial_x'' f(x) + \frac{2}{\lambda}f(x) - \frac{4x^2}{\lambda^2}f(x) = 0$$

Now we apply the Fourier transform from analysis to the whole equation and we obtain using the properties (1.4):

$$\partial_s''\mathcal{F}[f(x)](s) + 2\lambda\pi^2\mathcal{F}[f(x)](s) - 4\lambda^2\pi^4 s^2\mathcal{F}[f(x)](s) = 0$$

So $\mathcal{F}[f(x)](s)$ is a solution to the ODE defined by Q. Since the analytic and geometric Fourier transform act in the same way on differential operators, notice that we modified the definition of the geometric one to match the analytic one, we know $Q = F(P)$. Thus we reached the conclusion.

1.3. Geometric Mellin transform on split tori

Let T be a split torus of dimension n over k, i.e. a group k-scheme isomorphic to $\mathbb{G}_{m,k}^n$. Let $X_T = Hom_{k-gr}(T, \mathbb{G}_{m,k})$ the group of algebraic characters of the torus. Denote Ω_T the k-vectoriel of dimension n of invariant differential 1-forms on T.

Consider the action:

$$\begin{array}{ccc} X_T \times \Omega_T & \longrightarrow & \Omega_T \\ (\chi, \omega) & \longmapsto & \omega + \chi^*(\frac{dx}{x}) \end{array} \tag{1.9}$$

We want to consider quasi-coherent $\mathcal{O}_{\Omega_T}$-modules equipped with an action of X_T lifting the action (1.9). An example of such a module is $\mathcal{O}_{\Omega_T}$ with the natural action $\chi(f)(\omega) = f(\omega + \chi^*(\frac{dx}{x}))$.

We define the ring:

$$\Delta_{\Omega_T} := \left\{ \sum_{\chi \in X_T} f_\chi \cdot \chi \mid f_\chi \in H^0(\Omega_T, \mathcal{O}_{\Omega_T}) \text{ and } f_\chi = 0 \text{ except for finitely many } \chi \in X_T \right\}$$

together with the commutation rule $\chi \cdot f_\chi = \chi(f_\chi) \cdot \chi$.

The non-commutative k-algebra $D_T := H^0(T, \mathcal{D}_T)$ is generated by $H^0(T, \mathcal{O}_T)$ and $H^0(\Omega_T, \mathcal{O}_{\Omega_T})$ with the commutation rule $[D, \chi] = \langle \chi^* \left(\frac{dx}{x}\right), D \rangle \cdot \chi$. Here $\langle\ ,\ \rangle$ is the canonical pairing $\Omega_T \times \Omega_T^\vee \longrightarrow \mathbb{G}_{a,k}$.

Definition 1.3.1. ***(Geometric Mellin morphism)*** *Let T, X_T, Ω_T and Δ_{Ω_T} be as above. We call geometric Mellin morphism to the k-isomorphism:*

$$M : D_T \longrightarrow \Delta_{\Omega_T} \tag{1.10}$$

induced by $id : X_T \longrightarrow X_T$, $-id : \Omega_T \longrightarrow \Omega_T$.

Definition 1.3.2. ***(Geometric Mellin transform)*** *Let T, Ω_T and Δ_{Ω_T} be as above, the geometric Mellin morphism (1.10) induces a functor:*

$$M : Mod_{qcoh}(\mathcal{D}_T) \longrightarrow Mod_{qcoh}(\Delta_{\Omega_T}) \tag{1.11}$$

Defined by $M(N) = \widetilde{M_* H^0(T, N)}$ for N a quasi-coherent $\mathcal{D}_T$-module.

The functor is exact as it is a composition of exact functors. It extends to a functor:

$$M : D^b_{qcoh}(\mathcal{D}_T) \longrightarrow D^b_{qcoh}(\Delta_{\Omega_T})$$

which by construction preserves objects concentrated in degree 0.

Remark 1.3.3. **(Applying the geometric Mellin transform)** One can check, in the same way we did in Remark 1.2.3, how the output module looks after applying the geometric Mellin transform. The key point is that since we are pushing forward through an isomorphism $F^*\Delta_{\Omega_T}$ will be a free module of rank 1 so the underlying set of the module we are pushing forward will be unchanged. The reader can check what is the new action.

Example 1.3.4. Connecting Mellin transforms from geometry and analysis) This example will follow closely the example in the case of the Fourier transform. We start again with some review of the analytic Mellin transform. The Mellin transform is equivalent to the Fourier transform and to the bilateral Laplace transform. If one proves a theorem for the Fourier transform then one has an analogous theorem for the other two transforms. Interesting enough Fourier analysis and Laplace analysis are treated in dozens of books as independent subjects but Mellin was for long time relegated as a consequence of one of the other two. Only recently it has been treated from a point of view that does not rely in any of the other two aforementioned transforms. A very nice and kind introduction to the subject can be found in [BJ97].

Definition 1.3.5. ***(Mellin transform)*** *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{C}$ be a function such that $f(x)x^{s-1}$ is integrable for some $s \in \mathbb{C}$. The Mellin transform of such a function is given by the formula:*

$$\mathcal{M}[f(x)](s) := \int_0^{+\infty} f(x)x^s \frac{dx}{x} \tag{1.12}$$

If the Mellin transform is defined for some $s \in \mathbb{C}$, then it is defined for all points in the vertical line through s. This observation is what leads to the definition of the local and global Mellin transforms whose difference relays on the space of functions where the transform is defined. The local and the global transform behave differently, for instance the global Mellin transform always generates holomorphic functions but this is no longer true for the local Mellin transform. We will not discuss further this matter.

In a similar fashion as in Fourier analysis, there is an inversion theorem in Mellin analysis. The inverse operator being given by:

$$\mathcal{M}^{-1}[g(s)](x) := \int_{c-i\infty}^{c+i\infty} g(s)x^{-s}ds \qquad x \in \mathbb{R}$$

We will not enter into details about when $\mathcal{M}^{-1}$ is indeed the inverse of the Mellin transform, but notice that the local and global case need to be treated separately. We enunciate now some properties satisfied by the Mellin transform:

- It is a linear operator,

- it converts the operator multiplying by the variable into a shift $s \mapsto s+1$ that we will denote τ
$$\mathcal{M}[xf(x)](s) = \mathcal{M}[f(x)](s+1) =: \tau\mathcal{M}[f(x)](s)$$

- it converts the operator $-x\partial_x$ into multiplication by the variable:
$$\mathcal{M}[-x\partial_x f(x)](s) = s\mathcal{M}[f(x)](s)$$

- the Mellin transform of e^{-x} is the Gamma function:
$$\mathcal{M}[e^{-x}](s) = \Gamma(s)$$

Consider T the punctured complex affine line with coordinate x and the differential operator:

$$P := \partial_x'' + \frac{2}{\lambda} - \frac{4x^2}{\lambda^2} \qquad \lambda \in \mathbb{R}_+ \tag{1.13}$$

This is the same operator we used in Example 1.2.4. We call $N = \mathcal{D}/\mathcal{D}P$ the associated $\mathcal{D}$-module.

The geometric Mellin transform will give us a module $R := M(N)$ over the ring Δ_{Ω_T} of difference operators.
$$\Delta_{\Omega_T} := \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$$

If we apply the geometric Fourier transform to P we obtain:

$$Q := M(P) = (s+1)s + \frac{2\tau^2}{\lambda} - \frac{4\tau^4}{\lambda^2}$$

where we used that $[s, \tau^{-1}] = \tau^{-1}$. Thus, $R = \Delta_{\Omega_T}/\Delta_{\Omega_T}Q$ and we can once again ask ourselves about the relationship of the solutions of N and R.

The solution to the ODE defined by P is $f(x) = Ce^{-x^2/\lambda}$ for $C \in \mathbb{C}^*$. Now we compute its Mellin transform:

$$\mathcal{M}[Ce^{-x^2/\lambda}](s) = C\int_0^{+\infty} e^{-x^2/\lambda}x^{s-1}dx \underset{x^2/\lambda = t}{=} C\int_0^{+\infty} e^{-t}\sqrt{t}^{s-1}\sqrt{\lambda}^{s-1}\frac{\sqrt{\lambda}}{2\sqrt{t}}dt = \frac{C\sqrt{\lambda}^s}{2}\Gamma\left(\frac{s}{2}\right)$$

Finally we check that the function we just obtained is annihilated by Q.

$$Q\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right) = (s+1)s\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right) + \tfrac{2\tau^2}{\lambda}\left(\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right)\right) - \tfrac{4\tau^4}{\lambda^2}\left(\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right)\right) =$$

$$s^2\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right) + s\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right) + \sqrt{\lambda}^s\Gamma\left(\tfrac{s}{2}+1\right) - 2C\sqrt{\lambda}^s\Gamma\left(\tfrac{s}{2}+2\right) =$$

$$s^2\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right) + s\tfrac{C\sqrt{\lambda}^s}{2}\Gamma\left(\tfrac{s}{2}\right) + \sqrt{\lambda}^s\tfrac{s}{2}\Gamma\left(\tfrac{s}{2}\right) - 2C\sqrt{\lambda}^s\tfrac{s}{2}\left(\tfrac{s}{2}+1\right)\Gamma\left(\tfrac{s}{2}\right) =$$

$$\left(\tfrac{s^2}{2} + \tfrac{s}{2} + \tfrac{s}{2} - \tfrac{s^2}{2} - s\right)C\sqrt{\lambda}^s\Gamma\left(\tfrac{s}{2}\right) = 0$$

In this example we have seen a relationship between the geometric and analytic Mellin transforms. The geometric Mellin transform operates at the level of modules and the analytic Mellin transform operates at the level of solutions.

As with the Fourier transform, a more systematic approach is possible just applying the analytic Mellin transform to the equation:

$$\partial_x'' f(x) + \frac{2}{\lambda} f(x) - \frac{4x^2}{\lambda^2} f(x) = 0$$

and noticing that the geometric and analytic Mellin transforms operate in the same way on differential operators. Working out the details is left to the reader.

1.4. Geometric Fourier and Mellin transforms as geometric kernels

In this last section the goal is to express both the geometric Fourier transform and the geometric Mellin transform as geometric kernels. This is the first step towards a unified definition for these geometric transformations.

Definition 1.4.1. ***(Geometric Fourier transform)*** *Consider a vectoriel V over k of dimension n and $V^\vee$ its dual vectoriel over k. Let p, q be the canonical projections of $V \times_k V^\vee$ and take L a line bundle with flat connection on $V \times_k V^\vee$. The geometric Fourier transform associated to L is defined to be the functor:*

$$\mathcal{F}_* := q_+(p^*(-) \otimes_{\mathcal{O}_{V\times_k V^\vee}} L)[n] : Mod_{qcoh}(\mathcal{D}_V) \longrightarrow Mod_{qcoh}(\mathcal{D}_{V^\vee})$$

Theorem 1.4.2. *The geometric Fourier transform functors from Definition 1.2.2 and Definition 1.4.1 are canonically isomorphic for a suitable choice of L.*

The proof of this statement can be found in [KL85]. Following [Mal88] I will describe in a sketchy way the steps of the proof. Let M be a $\mathcal{D}_V$-module. Applying $\mathcal{F}_*$ has the following effect on M:

- $p^*(M) = M \otimes_k k[\xi]$ as $\mathcal{D}_{V\times_k V^\vee}$-modules where x_i, ∂_{x_i} acts on M and ξ_i, ∂_{ξ_i} acts on $k[\xi]$ in the obvious ways.
- $p^*(M)\otimes L[n]$ is equal to $p^*(M)[n]$ as a set but now the actions of x_i and ξ_i are exchanged and we define the actions of the partials as:

$$\partial_{x_i}(m \otimes L) := (\partial_{x_i} + \xi_i)m \otimes L$$

$$\partial_{\xi_i}(m \otimes L) := (\partial_{\xi_i} + x_i)m \otimes L$$

- q_+ is the relative de Rham complex only with respect to the variables x_i.
- $\mathcal{F}_*M$ is finally the Koszul complex $K(\partial_{x_i}, M \otimes k[\xi] \otimes L)$. Its cohomology vanishes except in degree zero where it can be identified with $M \otimes 1$. So far the image of both geometric Fourier functors have the same underlying set object.
- Lastly the action of $k[\xi]\langle\partial_\xi\rangle$ is given by:

$$\partial_{\xi_i}(m \otimes 1 \otimes L) = x_i m \otimes 1 \otimes L$$

$$\partial_{x_i}(m \otimes 1 \otimes L) \in \{\text{coboundaries}\} \Longrightarrow \partial_{x_i} m \otimes 1 \otimes L = -m \otimes \xi_i \otimes L$$

Remark 1.4.3. The above isomorphism is not canonical due to the identification of the relative de Rham complex with the Koszul complex. See [Mal88, Remarque 1.2] for more details about this issue.

The geometric Fourier transform is then exact and extends to a functor:

$$\mathcal{F}_* := q_+(p^*(-) \otimes_{\mathcal{O}_{V\times_k V^\vee}} L)[n] : D^b_{qcoh}(\mathcal{D}_V) \longrightarrow D^b_{qcoh}(\mathcal{D}_{V^\vee})$$

Definition 1.4.4. ***(Geometric Mellin transform)*** *Consider a split torus T over k of dimension n and Ω_T the k-vectoriel of its invariant 1-forms. Let p, q be the canonical projections of $T \times_k \Omega_T$ and take K a line bundle with flat connection relative to Ω_T on $T \times_k \Omega_T$. The geometric Mellin transform associated to K is defined to be the functor:*

$$\mathcal{M}_* := q_+(p^*(-) \otimes_{\mathcal{O}_{T\times_k \Omega_T}} K)[n] : Mod_{qcoh}(\mathcal{D}_T) \longrightarrow Mod_{qcoh}(\Delta_{\Omega_T})$$

Theorem 1.4.5. *The geometric Mellin transform functors from Definition 1.3.2 and Definition 1.4.4 are canonically isomorphic for a suitable choice of K.*

The proof of this statement is claimed in [Lau85] to be mutatis mutandis the same proof given in [KL85] for Theorem 1.4.2. The geometric Mellin transform is then exact and extends to a functor:

$$\mathcal{M}_* := q_+(p^*(-) \otimes_{\mathcal{O}_{T\times_k \Omega_T}} K)[n] : D^b_{qcoh}(\mathcal{D}_T) \longrightarrow D^b_{qcoh}(\Delta_{\Omega_T})$$

Remark 1.4.6. **(Natural extensions)** Grothendieck defined the so called $\natural$-extensions for objects that include vectoriels and split tori. These $\natural$-extensions form a category. For the precise definition one can look in [MM74]. If we consider the universal object for $\natural$-extensions of a vectoriel V, this object is isomorphic to $V^\vee$. If we consider the universal object for $\natural$-extensions of a split torus T, this object is isomorphic to the quotient of Ω_T by the action of X_T described in (1.9). The upshot is that for both the geometric Fourier and Mellin transforms the target category can be thought as the category of a certain type of quasi-coherent sheaves of modules over the corresponding universal $\natural$-extension.

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