

Generalized Fourier Transformation

	V vector space	$(\mathbb{G}^x)^n$ T torus	A abelian var	A abelian var
Fourier	$D_V\text{-mod} \rightarrow D_{V^v}\text{-mod}$	$D_T\text{-torus} \rightarrow A_{\mathbb{C}^n}\text{-mod}$ <small>(different mod)</small>	$D_A\text{-mod} \rightarrow G_{A^g}\text{-mod}$	$G_A\text{-mod} \rightarrow G_{\text{Pic}^0(A)}$
Dual space	V^v	\mathbb{C}^n	A^g $H^0(A, \Omega_A^1)$ -torsor above $\text{Pic}^0(A)$	$\text{Pic}^0(A)$
1-motive	$[\hat{V} \rightarrow V]$	$[\hat{T} \rightarrow T]$	$[\hat{A} \rightarrow A]$	$[0 \rightarrow A]$
dual motive	$[\hat{V}^v \rightarrow V^v]$	$[\mathbb{Z}^n \rightarrow \mathbb{C}^n]$	$[0 \rightarrow A^g]$	$[0 \rightarrow \text{Pic}^0(A)]$

1-motives

$$[G \xrightarrow{u} J]$$

↑ extension of A by $V \times T$
 $0 \rightarrow V \times T \rightarrow J \rightarrow A \rightarrow 0$

formal abelian gp scheme of finite type

$$G_{[G \xrightarrow{u} J]} \text{-mod:}$$

$G_J\text{-mod}$ w/ additional G-action compatible with u

Example

$$[0 \rightarrow J] \quad G_{[0 \rightarrow J]} \text{-mod} = G_J \text{-mod}$$

$$[\hat{J} \rightarrow J] \quad G_{[\hat{J} \rightarrow J]} \text{-mod} = D_J \text{-mod}$$

↑ formal completion of J at 0

Day 1

- ① Geometric Fourier transform on vector spaces
- ② Geometric Mellin transform on split tori
- ③ Geometric kernels

I § • Fix a field k char(k)=0

... \mathbb{G}^n ("vertical")

- I §
- Fix a field k $\text{char}(k) = 0$
 - Let V be a group k -scheme isomorphic to $G_{a,k}^n$ ("vectoriel")
 - Let $V^\vee = \text{Hom}_{V\text{-gr}}(V, G_{a,k})$ dual vectoriel over k .

$D_V := H^0(V, \mathcal{D}_V)$ is generated by $\text{Sym}_k(V)$ and $\text{Sym}_k(V^\vee)$
 w/ $[V, V^\vee] = \langle v, v^\vee \rangle$ (Note $H^0(V, \mathcal{O}_V) \subset D_V$ is generated by $\text{Sym}_k(V^\vee)$)

Definition (Geometric Fourier morphism)

Let V and V^\vee as above. We call geometric Fourier morphism:

$$F: D_V \longrightarrow D_{V^\vee}$$

induced by $\text{id}: V \rightarrow V$, $-\text{id}: V^\vee \rightarrow V^\vee$ and the natural identification $V \rightarrow V^{\vee\vee}$.
 In coords: take (x_1, \dots, x_n) on V and (y_1, \dots, y_n) dual coords. $\begin{cases} x_i \mapsto \partial_{y_i} \\ \partial_{x_i} \mapsto -y_i \end{cases}$

Definition (Geometric Fourier transform)

Let V, V^\vee, F as above. Call geo. Fourier transform to the functor:

$$F: \text{Mod}_{\text{geo}}^{\text{loc}}(D_V) \longrightarrow \text{Mod}_{\text{geo}}^{\text{loc}}(D_{V^\vee})$$

defined as $F(M) = \widehat{F_* H^0(V, M)}$

• This functor is exact. Induces $F: D_{\text{geo}}^{\text{loc}}(D_V) \longrightarrow D_{\text{geo}}^{\text{loc}}(D_{V^\vee})$

By construction preserves $\begin{cases} \text{objects in degree } 0 \\ \text{coherency} \\ \text{holonomicity.} \end{cases}$

Rmk: M a D_V -module. $F_* M = \text{Hom}_{D_V}(F^* D_V, M)$

$F^* D_V = D_{V^\vee}$ as a set but the module structure comes from

$F: D_V \rightarrow D_{V^\vee}$, since F iso $\Rightarrow F^* D_V$ is a free rank 1.

$F_* M$ has the same underlying set as M .

$$\begin{aligned} x_i \cdot m &= F(x_i) m = \partial_{y_i} \cdot m \\ -\partial_{x_i} \cdot m &= F(\partial_{x_i}) m = y_i \cdot m \end{aligned}$$

Example:

1 . . . 1.

$D_T = H^0(\mathbb{T}, \mathcal{D}_T)$ is generated by $\underbrace{H^0(\mathbb{T}, \mathcal{O}_T)}_{k[x_T]}$ and $H^0(\Omega_T, \mathcal{O}_{\Omega_T})$ w/ $[D, \chi] = \langle x^s (\frac{dx}{x}) \rangle, D \times X$
 $\langle \cdot \rangle: \Omega_T \times \Omega_T^v \rightarrow \mathbb{G}_{m,k}$

Definition (Geometric Mellin morphism)

Let $T, X_T, \Omega_T, \Delta_{\Omega_T}$ as above. Call geo. Mellin morphism:

$$M: D_T \longrightarrow \Delta_{\Omega_T}$$

induced by $\text{id}: X_T \rightarrow X_T$ and $-\text{id}: \Omega_T \rightarrow \Omega_T$

Definition (Geometric Mellin transform)

Notation as above. Geo. Mellin transform is the functor:

$$M: \text{Mod}_{\text{geom}}(\mathcal{D}_T) \longrightarrow \text{Mod}_{\text{geom}}(\Delta_{\Omega_T})$$

defined as $M(N) = \widetilde{M_x H^0(\mathbb{T}, N)}$

Example, $M[f(x)](s) = \int_0^{+\infty} f(x) x^s \frac{dx}{x} \quad \left| \quad \begin{aligned} M[x f(x)](s) &= M[f(x)](s+1) =: \tau M[f(x)](s) \\ M[x^2 f(x)](s) &= s M[f(x)](s) \end{aligned} \right.$

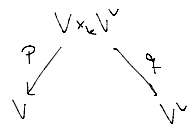
$$P = \partial_x^4 + \frac{2}{\lambda} - \frac{4x^2}{x^2}$$

$$Q = (s+1)s + \frac{2\tau^2}{\lambda} - \frac{4\tau^4}{\lambda^2} \quad \lambda \in \mathbb{R}_+$$

③ Geometric kernels

Definition

Let V be a k -vectoriel of $\dim n$ and V^v its dual.



take L a line bundle on $V \times_k V$ with flat connection. Call geometric Fourier transform associated to L to:

$$\mathcal{F}_* := q_+ \left(p^* (-) \otimes_{V \times_k V} L \right) [n]: \text{Mod}_{\text{geom}}(\mathcal{D}_V) \longrightarrow \text{Mod}_{\text{geom}}(\mathcal{D}_{V^v})$$

Theorem Both definitions of geo. Fourier coincide for suitable L .

$$L = (\mathcal{O}_{V \times V}, d+k, -, -)$$

Rmk. M be a \mathcal{D}_V -module.

• $\mathcal{P}^*(M) = M \otimes_k k[y]$ as $\mathcal{D}_{V \times V}$ -modules $x_i \partial_{x_i}$ act on M
 $y_i \partial_{y_i}$ act on $k[y]$

• $\mathcal{P}^*(M) \otimes L[n]$ is equal to $\mathcal{P}^*(M)[n]$ as set but the actions of $x_i \leftrightarrow y_i$

$$\begin{aligned} \partial_{x_i}(m \otimes L) &:= (\partial_{x_i} + y_i)m \otimes L \\ \rightarrow \partial_{y_i}(m \otimes L) &:= (\partial_{y_i} + x_i)m \otimes L \end{aligned}$$

• \mathcal{F}_+ is the relative de Rham w.r.t variables x_i

• $\mathcal{F}_* M$ is the Koszul complex $K(\partial_{x_i}, M \otimes_k k[y] \otimes L)$

Claim: only has 0-th cohomology and is identified with M .

• $k[y] \langle \partial_{y_i} \rangle \quad \partial_{y_i}(m \otimes L) = x_i \otimes L$

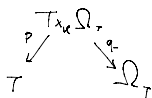
$\partial_{x_i}(m \otimes L)$ is a coboundary

Upshot: is the action.

$$\mathcal{F}_* : \mathcal{D}_{\text{qcoh}}^b(\mathcal{D}_V) \longrightarrow \mathcal{D}_{\text{qcoh}}^b(\mathcal{D}_V)$$

Rmk. Same is true for geometric Mellin transform

$$\mathcal{M}_* : \mathcal{F}_+(\mathcal{P}^*(-) \otimes k)[n] : \text{Mod}_{\text{qcoh}}(\mathcal{D}_T) \longrightarrow \text{Mod}_{\text{qcoh}}(\Delta_{-n, T})$$



Rmk: (\mathcal{F} -extensions)

Grothendieck defined the so called \mathcal{F} -extensions for objects that include vectoriels and split tori.

• If we consider V^{\natural} the universal \mathfrak{g} -extension for $V \Rightarrow V^{\natural} \cong V^{\vee}$.

• \parallel T^{\natural} \parallel \parallel for $T \Rightarrow T^{\natural} \cong \Omega_T / \sim_{T^{\natural}}$

(Fin)