Montag, 20. April 2026 12:09

Generalized Fourier Transformation

	V vector space	$\underset{\parallel}{(\mathbb{C}^\times)^n}$ T torus	A abelian var	A abelian var
Fourier	$D_V\text{-mod} \to D_{V^\vee}\text{-mod}$	$D_T\text{-torus} \to \underbrace{\Delta_{\mathbb{C}^n}}_{\text{differen mod}}\text{-mod}$	$D_A\text{-mod} \longrightarrow G_{A^\natural}\text{-mod}$	$G_A\text{-mod} \to G_{\mathrm{Pic}^\circ(A)}$
Dual space	$V^\vee$	$\mathbb{C}^n$	$A^\natural$ $H^0(A,\Omega^1_A)$-torsor above $\mathrm{Pic}^\circ(A)$	$\mathrm{Pic}^\circ(A)$
1-motive	$[\hat{V} \to V]$	$[\hat{T} \to T]$	$[\hat{A} \to A]$	$[0 \to A]$
dual motive	$[\hat{V}^\vee \to V^\vee]$	$[\mathbb{Z}^n \to \mathbb{C}^n]$	$[0 \to A^\natural]$	$[0 \to \mathrm{Pic}^\circ(A)]$

1-motives

$[G \xrightarrow{u} J]$

G: formal abelian gp scheme of finite type

J: extension of A by $V \times T$

$0 \to V \times T \to J \to A \to 0$

$G_{[G \xrightarrow{u} J]}$-mod:

G_J-mod w/ additional G-action compatible with u

Example

$[0 \to J]$ $\qquad G_{[0 \to J]}\text{-mod} = G_J\text{-mod}$

$[\hat{J} \to J]$ $\qquad G_{[\hat{J} \to J]}\text{-mod} = D_J\text{-mod}$

$\hat{J}$: formal completion of J at 0

Day 1

1. Geometric Fourier transform on vector spaces
2. Geometric Mellin transform on split tori
3. Geometric kernels

I § · Fix a field k char$(k) = 0$

("vectoriel")

I §
- Fix a field k char$(k)=0$
- let V be a group k-scheme isomorphic to $\mathbb{G}^n_{a,k}$ ("vectoriel")
- let $V^\vee = \mathrm{Hom}_{k\text{-gr}}(V, \mathbb{G}_{a,k})$ dual vectoriel over k.

$D_V := H^0(V, \mathcal{D}_V)$ is generated by $\mathrm{Sym}_k(V)$ and $\mathrm{Sym}_k(V^\vee)$ w/ $[v, v^\vee] = \underline{\langle v, v^\vee \rangle}$ (Note $H^0(V, \mathcal{O}_V) \subset D_V$ is generated by $\mathrm{Sym}_k(V^\vee)$)

Definition (Geometric Fourier morphism)

Let V and $V^\vee$ as above. We call geometric Fourier morphism:

$$F: D_V \longrightarrow D_{V^\vee}$$

induced by $id: V \to V$, $-id: V^\vee \to V^\vee$ and the natural identification $V \to V^{\vee\vee}$.

In coords: take $(x_1, \dots, x_n)$ on V and $(y_1, \dots, y_n)$ dual coords. $\left| \begin{array}{l} x_i \longmapsto \partial_{y_i} \\ \partial_{x_i} \longmapsto -y_i \end{array} \right.$

Definition (Geometric Fourier transform)

let $V, V^\vee, F$ as above. Call geo. Fourier transform to the functor:

$$F: \mathrm{Mod}_{qcoh}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}_{qcoh}(\mathcal{D}_{V^\vee})$$

defined as $F(M) = \widetilde{F_* H^0(V, M)}$

- This functor is exact. Induces $F: D^b_{qcoh}(\mathcal{D}_V) \longrightarrow D^b_{qcoh}(\mathcal{D}_{V^\vee})$

By construction preserves $\left| \begin{array}{l} \text{objects in degree } 0 \\ \text{Coherentness} \\ \text{holonomicity.} \end{array} \right.$

Rmk: M a D_V-module. $F_*M = \mathrm{Hom}_{D_V}(F^*D_{V^\vee}, M)$

$F^*D_{V^\vee} = D_{V^\vee}$ as a set but the module structure comes from $\underline{F: D_V \to D_{V^\vee}}$, since F iso $\Rightarrow$ $F^*D_{V^\vee}$ is a free rank 1.

F_*M has the same underlying set as M.

$$x_i \cdot m = F(x_i)m = \partial_{y_i} \cdot m$$
$$-\partial_{x_i} \cdot m = F(\partial_{x_i})m = y_i m$$

Example:

Example:

Recall: $\mathcal{F}[f(x)](y) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x y}\, dx$

- is linear
- $\mathcal{F}[\partial_x^k f(x)](y) = (2\pi i s)^k \mathcal{F}[f(x)](y)$
- $\mathcal{F}[x^k f(x)](y) - \left(\frac{i}{2\pi}\right)^k \partial_y^k \mathcal{F}[f(x)](y)$

$\mathbb{A}^1_{\mathbb{R}}$ coordinate x, dual coordinate y.

$$P := \partial_x'' + \frac{2}{\lambda} - \frac{4x^2}{\lambda^2} \quad \lambda \in \mathbb{R}_+ \qquad f(x) = C e^{-x^2/\lambda} \quad C \in \mathbb{R}^*$$

$$M = \mathcal{D}/\mathcal{D}P \xrightarrow{\text{Fourier}} N := F(M) = \mathcal{D}/\mathcal{D}Q \qquad Q = F(P)$$

$g(x)$ solution of $Q \Rightarrow g = \mathcal{F}[f](y)$

$$\partial_x'' + \frac{2}{\lambda} f - \frac{4x^2}{\lambda^2} f = 0$$

(2) Geometric Mellin transform

- Let T_k be a split torus of dimension n.
- $X_T = \mathrm{Hom}_{k\text{-gr}}(T, \mathbb{G}_{m,k})$
- Ω_T the k-vectorial of invariant 1-forms on T.

Consider
$$X_T \times \Omega_T \longrightarrow \Omega_T$$
$$(\chi, \omega) \longmapsto \omega + \chi^*\left(\frac{dx}{x}\right)$$
$$\chi : T \to \mathbb{G}_{m,k} \leftarrow \text{coord } x$$

Consider q-coh $\mathcal{O}_{\Omega_T}$-modules with an action of X_T lifting with

example: $\mathcal{O}_{\Omega_T}$ w/ $X_T \times \mathcal{O}_{\Omega_T} \longrightarrow \mathcal{O}_{\Omega_T}$ s.t. $(\chi \cdot f)(\omega) = f(\omega + \chi^*(\frac{dx}{x}))$.

$$\Lambda_{\Omega_T} := \left\{ \sum_{\chi \in X_T} f_\chi \chi \;\middle|\; f_\chi \in H^0(\Omega_T, \mathcal{O}_{\Omega_T}) \text{ s.t. } f_\chi = 0 \text{ almost always} \right\}$$

Commutation rule $\chi \cdot f_\chi = \chi(f_\chi) \cdot \chi$

$D_T = H^0(T, \mathcal{D}_T)$ is generated by $H^0(T, \mathcal{O}_T)$ and $H^0(\Omega_T, \mathcal{O}_{\Omega_T})$ w/ $[D, \chi] = \langle \chi^*(\frac{dx}{x}), D \rangle \chi$

$D_T = H^0(T, \mathcal{D}_T)$ is generated by $\underbrace{H^0(T, \mathcal{O}_T)}_{k[X_T]}$ and $H^0(\Omega_T, \mathcal{O}_{\Omega_T})$ w/ $[D, \chi] = \langle \chi^*(\frac{dx}{x}), D \rangle \chi$

$\langle , \rangle : \Omega_T \times \Omega_T^\vee \to \mathbb{G}_{a,k}$

<u>Definition</u> (Geometric Mellin morphism)

Let $T, X_T, \Omega_T, \Delta_{\Omega_T}$ as above. Call geo. Mellin morphism

$$M : D_T \longrightarrow \Delta_{\Omega_T}$$

induced by $id : X_T \to X_T$ and $-id : \Omega_T \to \Omega_T$

<u>Definition</u> (Geometric Mellin transform)

Notation as above. Geo. Mellin transform is the functor:

$$M : \mathrm{Mod}_{qcoh}(\mathcal{D}_T) \longrightarrow \mathrm{Mod}_{qcol}(\Delta_{\Omega_T})$$

defined as $M(N) = \widetilde{M_* H^0(T, N)}$

<u>example</u>: $\mathcal{M}[f(x)](s) = \int_0^{+\infty} f(x) x^s \frac{dx}{x}$ | $\mathcal{M}[x f(x)](s) = \mathcal{M}[f(x)](s+1) =: T \mathcal{M}[f(x)](s)$

$\mathcal{M}[x\partial_x f](s) = s \mathcal{M}[f](s)$

$P = \partial_\lambda'' + \frac{2}{\lambda} - \frac{4x^2}{\lambda^2}$ $\qquad Q = (s+1)s + \frac{2T^2}{\lambda} - \frac{4T^4}{\lambda^2}$ $\qquad \lambda \in \mathbb{R}_+$

3 Geometric kernels

<u>Definition</u>

Let V be a k-vectoriel of dim n and $V^\vee$ its dual.

$$\begin{tikzcd} & V \times_k V^\vee \arrow[dl, "p"'] \arrow[dr, "q"] & \\ V & & V^\vee \end{tikzcd}$$

take L a line bundle on $V \times_k V^\vee$. Call geometric Fourier transform associated to L to:
with flat connection

$$F_* := q_+ (p^*(-) \otimes_{\mathcal{O}_{V \times_k V^\vee}} L)[n] : \mathrm{Mod}_{qcoh}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}_{qcoh}(\mathcal{D}_{V^\vee})$$

<u>Theorem</u> Both definitions of geo. Fourier coincide for suitable L.

$$L = (\mathcal{O}_{V\times V^\vee}, d + dk -, -, -)$$

<u>Rmk</u>: M be a $\mathcal{D}_V$-module.

- $p^*(M) = M \otimes_k k[y]$ as $\mathcal{D}_{V\times V^\vee}$-modules $x_i \partial_{x_i}$ act on M, $y_i \partial_{y_i}$ act on $k[y]$
- $p^*(M) \otimes L[n]$ is equal to $p^*(M)[n]$ as set but the actions of $x_i \longleftrightarrow y_i$

$$\partial_{x_i}(m \otimes L) = (\partial_{x_i} + y_i) m \otimes L$$
$$\longrightarrow \partial y_i (m \otimes L) := (\partial y_i + x_i) m \otimes L$$

- q_+ is the relative de Rham w.r.t variables x_i
- $F_* M$ is the koszul complex $K(\partial_{x_i}, M \otimes k[y] \otimes L)$

<u>Claim</u>: only has 0-th cohomology and is identified with M.

- $k[y]\langle \partial_y \rangle$ $\partial_{y_i}(m \otimes 1 \otimes L) = x_i \otimes 1 \otimes L$

$\partial_{x_i}(m \otimes 1 \otimes L)$ is a coboundary

Upshot: is the action.

$$F_* : D^b_{qcoh}(\mathcal{D}_V) \longrightarrow D^b_{qcoh}(\mathcal{D}_{V^\vee})$$

<u>Rmk</u>: Same is true for geometric Mellin transform

$$M_* : q_+(p^*(-) \otimes k)[n] : Mod_{qcoh}(\mathcal{D}_T) \longrightarrow Mod_{qcoh}(\Lambda_{\Omega_T})$$

$$\begin{tikzcd} & T \times_k \Omega_T \arrow[dl, "p"'] \arrow[dr, "q"] & \\ T & & \Omega_T \end{tikzcd}$$

<u>Rmk</u>: (4-extensions)

Grothendieck defined the so called 4-extensions for objects that include vectoriels and split tori.

- If we consider $V^\natural$ the universa $\natural$-extension for $V \Rightarrow V^\natural \cong V^u$.
- 〃 $T^\natural$ 〃 〃 for $T \Rightarrow T^\natural \cong \Omega_T / \sim_{X_T}$

Fin