Local Cohomological Dimension 11. 7. 2025

Definition Let X be a scheme, $Z \subset X$ closed subset. The local cohomological dimension of X along Z is

$\mathcal{O}_X$ if X regular

$$\operatorname{lcd}(X, Z) := \max \{ i \mid \mathcal{H}^i_Z \cancel{F} \neq 0 \text{ for some q.coh. } F \text{ on } X \}$$

Question (Grothendieck 1967) How to determine $\operatorname{lcd}(X, Z)$?

Ogus '73: "Complete" answer to this quest. when X regular of equicharacteristic 0.

Def 2.12 [Ogus] Z noeth. scheme of equi-char zero. Then the de Rham depth

$\operatorname{DRD}(Z)$ is the minimal d s.th.

for every $z \in Z$ (not nec. closed):

$$H^j_z(Z) = 0 \quad \forall\, j < d - \dim \overline{\{z\}}$$

if $Z/\mathbb{C}$, then

$$H^j_Z(Z,\mathbb{C}) = H^j(Z, Z\setminus\{z\}, \mathbb{C})$$

Theorem 2.13 [Ogus] X regular noeth. scheme of equichar zero and pure biequidimension n, $Z \subset X$ closed. Then

$$\mathrm{lcd}(X,Z) = n - \mathrm{DRD}(Z).$$

Thm 1 [RSW 23] If X smooth, then

$$\mathrm{lcd}(X,Z) = n - \underbrace{\min\{j \mid {}^p\mathcal{H}^j(\mathbb{Q}_Z) \neq 0\}}_{=:\, \mathrm{RHD}_{\mathbb{Q}}(Y)}$$

pf: Can replace F in defn of $\mathrm{lcd}(X,Z)$ by $\mathcal{O}_X$. (X regular)

note $\mathcal{H}^i_Z \mathcal{O}_X \neq 0 \iff \mathrm{DR}\, \mathcal{H}^i_Z \mathcal{O}_X \neq 0$.

Compute :

$$DR\ \mathcal{H}^i_Z\, \mathcal{O}_X = DR \circ \mathcal{H}^i\ R\Gamma_Z\, \mathcal{O}_X$$

$$= {}^p\mathcal{H}^i \circ DR\ \ R\Gamma_Z\, \mathcal{O}_X$$

$$\underset{\substack{\uparrow \\ \text{well known}}}{=} {}^p\mathcal{H}^i \circ \underbrace{R\Gamma_Z}_{\iota_* \iota^!} \circ \underbrace{DR\, \mathcal{O}_X}_{\mathbb{C}_X[n]}$$

$$= {}^p\mathcal{H}^i \circ \iota_* \mathbb{D}\, \iota^* \underbrace{\mathbb{D}(\mathbb{C}_X[n])}_{\mathbb{C}_X[n]}$$

$$= {}^p\mathcal{H}^i \circ \iota_* \mathbb{D}(\mathbb{C}_Z)\ [-n]$$

$$= {}^p\mathcal{H}^{i-n} \circ \iota_*\ \mathbb{D}(\mathbb{C}_Z)$$

Now $\mathbb{D}(\mathcal{D}^{\leq k}) = \mathcal{D}^{\geq -k}$

so $\quad {}^p\mathcal{H}^{i-n}\, \iota_*\, \mathbb{D}(\mathbb{C}_Z) \neq 0$

$$\Longleftrightarrow {}^p\mathcal{H}^{n-i}\, \iota_* \mathbb{C}_Z \neq 0 \underset{\text{computation}}{\Longleftrightarrow} \mathcal{H}^i_Z\, \mathcal{O}_X \neq 0$$

Thus: $\max\{ i \mid \mathcal{H}^i_Z \mathcal{O}_X \neq 0 \}$

$$\overset{j:=n-i}{=} \max\{ n-j \mid {}^p\mathcal{H}^j \mathbb{C}_Z \neq 0 \}$$

$$= n - \min\{ j \mid {}^p\mathcal{H}^j \mathbb{C}_Z \neq 0 \}.$$

□

Mustață 24 Lect. Notes

Cor 5.19 Y smooth (irred), n-diml, $E \subset Y$ snc divisor. Then

$$\forall p: \quad \mathrm{gr}^F_{-p} \mathrm{DR}(\mathcal{O}_Y(*E)) \simeq \Omega^p_Y(\log E)[n-p].$$

Let $Z \subset X$ closed, $U := X \setminus Z$

X smooth

take log res:

$$\begin{tikzcd}
 & Y \arrow[d, "\pi"] & E \arrow[l, hook'] \arrow[d] \\
U \arrow[ur, "j'"] \arrow[r, "j"] & X & Z \arrow[l, hook']
\end{tikzcd}$$

Prop 5.24 $\mathrm{gr}^F_{-p} DR(j_* \mathbb{Q}^H_U[n])$

$$\simeq R\pi_* \Omega^p_Y(\log E)[n-p]$$

pf: $\text{LHS} = \mathrm{gr}^F_{-p} DR\, \pi_* j'_* \mathbb{Q}^H_U[n]$

$$= R\pi_* \mathrm{gr}^F_{-p} \underbrace{DR(j'_* \mathbb{Q}^H_U[n])}_{DR(\mathcal{O}_Y(*E))} \underset{\text{Cor 5.19}}{=} \text{RHS}. \quad \square$$

Recall: $(\mathcal{M}, F_\bullet \mathcal{M})$ filtered $\mathcal{D}$-mod

filtration is generated at level $p :\Leftrightarrow$

$$\forall k \geq 0 \quad F_k \mathcal{D}_X \cdot F_p \mathcal{M} = F_{k+p} \mathcal{M}.$$

Lemma 5.29 Given $(\mathcal{M}, F_\bullet \mathcal{M})$. Then

$F_\bullet \mathcal{M}$ generated at level $p \Leftrightarrow$

$$\mathcal{H}^0(\mathrm{gr}^F_{i-n} DR(\mathcal{M})) = 0 \quad \forall i \geq p+1$$

pf: consider the map in $\mathrm{gr}^F_{i-n} DR(\mathcal{M})$:

$$\Omega_X^{n-1} \otimes gr^F_{i-1}(\mathcal{M}) \to \Omega_X^n \otimes gr^F_i(\mathcal{M})$$

($\mathcal{H}^0$ = coker of this map)

in local coordinates $x_1, \ldots, x_n$:

$$dx_1 \wedge \ldots \widehat{dx_i} \ldots \wedge dx_n \otimes \bar{u}$$

$$\longmapsto (-1)^{i-1} dx_1 \wedge \ldots \wedge dx_n \otimes \overline{\partial_i u}$$

Then $\mathcal{H}^0(gr^F_{i-n} DR(\mathcal{M})) = 0$

$$\Longleftrightarrow \quad F_i\mathcal{M} \subseteq F_{i-1}\mathcal{M} + \sum_{j=1}^{n} \partial_j F_{i-1}\mathcal{M}$$

i.e. $F_i\mathcal{M} = F_1\mathcal{D}_X \cdot F_{i-1}\mathcal{M}$.

$\square$

Theorem 5.30 Let $q \geq 1$ be such that $\mathcal{H}^i_Z \mathcal{O}_X = 0$ $\forall i > q$. Then for $p \geq 0$ the following are equivalent:

(1) Hodge filtr. on $\mathcal{H}^q_Z(\mathcal{O}_X)$ gen. at level p

(2) $R^{q-1+i}\pi_* \Omega_Y^{n-i}(\log E) = 0 \quad \forall i \geq p+1$

If $q \geq 2$, same equivalence also holds for $p < 0$.

proof: for $q \geq 2$: $\mathcal{H}^q_Z \mathcal{O}_X \cong R^{q-1} j_* \mathcal{O}_U$

(and $0 \to \mathcal{O}_X \to j_* \mathcal{O}_U \to \mathcal{H}^1_Z \mathcal{O}_X \to 0$)

for $q = 1$ $\rightsquigarrow$ need $p \geq 0$

generation level can be checked for $R^{q-1} j_* \mathcal{O}_U$

Lemma 5.29

$\Longleftrightarrow$ $\mathcal{H}^0 (gr^F_{i-n} DR(R^{q-1} j_* \mathcal{O}_U)) = 0$

$\forall\, i \geq p+1$

Consider "hypercohomology" spectral sequence

$$E_2^{ab} = \mathcal{H}^a (gr^F_{i-n} DR(\underbrace{R^b j_* \mathcal{O}_U}_{= R^b j_* \mathcal{O}^H_U[n]}))$$

$$\Rightarrow \mathcal{H}^{a+b} (gr^F_{i-n} DR(j_* \mathcal{O}^H_U [n]))$$

Since $DR(..)$ concentrated in non pos. degrees

have $E_2^{ab} = 0 \quad \forall\, a > 0$

Assumption on q $\Rightarrow$ $R^b j_* \mathbb{Q}^H_U[n] = 0$

$\forall\, b \neq q$

$$\Rightarrow \quad E_2^{ab} = 0 \quad \forall\, b \geq q$$

Hence if $a+b = q-1$, then $E_2^{ab} = 0$

unless $(a,b) = (0, q-1)$

Moreover $E_2^{0,q-1} = E_\infty^{0,q-1}$

$$\Longrightarrow \quad \mathcal{H}^0\left(\mathrm{gr}^F_{i-n} \mathrm{DR}\, R^{q-1} j_* \mathbb{Q}^H_U[n]\right)$$

$$\simeq \mathcal{H}^{q-1}\left(\mathrm{gr}^F_{i-n} \mathrm{DR}(j_* \mathbb{Q}^H_U[n])\right)$$

$$\underset{\text{Cor 5.24}}{\simeq} R^{q-1+i} \pi_* \Omega_Y^{n-i}(\log E) .$$

$\square$

Corollary 5.33 $\mathrm{lcd}(X,Z)$ is the min c

s.th. $R^{j+i} \pi_* \Omega_Y^{n-i}(\log E) = 0$

$\forall\, i \geq 0 \quad \forall\, j \geq c$.

proof: $\mathrm{lcd}(X,Z) \leq c \quad \Longleftrightarrow$

$$\mathcal{H}^q_Z \mathcal{O}_X = 0 \quad \forall q \geq c+1$$

$$\Longleftrightarrow \quad \forall q \geq \underbrace{c+1}_{\geq 2} : \mathcal{H}^\ell_Z \mathcal{O}_X = 0 \ \forall \ell > q$$

Hodge filtr. starts at zero

and filtr. on $\mathcal{H}^q_Z \mathcal{O}_X$ gen. at

level $p = -1$.

Thm 5.30 $\Longrightarrow$

$$R^{\overbrace{q-1}^{j}+i} \pi_* \Omega^{n-i}_Y (\log E) = 0$$

$$\forall i \geq 0 \quad \forall j := q-1 \geq c$$

$\square$

MP22: $\quad \mathrm{lcd}(X, Z) \leq n-1$

$$\Longleftrightarrow \quad R^{n-1} \pi_* \omega_Y(E) = 0$$

Cor 11.22 $\quad Z$ codim r w/ quot sings

$$\Longrightarrow \quad \mathrm{lcd} = n - \mathrm{depth}(\mathcal{O}_Z) = r.$$