Higher du Bois singularities

recall from Yichen's talk: Z has du Bois singularities $\iff$ $\mathcal{O}_Z \to \underline{\Omega}^0_Z$ iso in $D^b(\mathrm{Coh}\, Z)$, i.e. $\mathcal{O}_Z \simeq \mathcal{H}^0 \underline{\Omega}^0_Z$ & $\mathcal{H}^i \underline{\Omega}^0_Z = 0 \quad \forall i > 0$

geometric characterization: $\mathcal{O}_Z \simeq Rf_* \mathcal{O}_E$, if $Z \subset X^{\text{smooth}}$ and $(\tilde{X}, E) \to (X, Z)$ log resolution

Hodge characterization (Henry's talk): $Z \subset X$ pure of codim r. Assume Z is C.M. Then Z du Bois $\iff F_0 \mathcal{H}^q_Z \mathcal{O}_X = E_0 \mathcal{H}^q_Z \mathcal{O}_X \ \forall q$. Recall that

$$E_k \mathcal{H}^q_Z \mathcal{O}_X = \mathrm{Im}\left[\mathcal{E}xt^q(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{O}_X) \to \lim_k \left(\mathcal{E}xt^q(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{O}_X)\right) = \mathcal{H}^q_Z \mathcal{O}_X \right]$$

not shown in Henry's talk. We'll see generalization today (but only for l.c.i case). Recall from Paul's talk:

Given variety Z, let $Z_\bullet \to Z$ be hyperresolution. $\exists$ canonical morphism: adjoint to $\varepsilon_0^* \Omega^p_Z \to \Omega^p_{Z_0}$

$$\underset{\text{Kähler diff's}}{\Omega^p_Z} \longrightarrow \varepsilon_{0,*} \Omega^p_{Z_0} \longrightarrow \left[R\varepsilon_{0,*} \Omega^p_{Z_0} \to \cdots \right] = \underline{\Omega}^p_Z$$

by double complex

<u>Def</u> (Saito - Jung - Kim - Yoon) Z has (only) p-du Bois singularities ("Z is p-du Bois") iff $\forall k \in \{0, \dots, p\}$, the natural map $\Omega_Z^k \to \underline{\Omega}_Z^k$ is iso in $D^b(\mathrm{Coh}\, Z)$, i.e.:

$\Omega_Z^k \simeq \mathcal{H}^0 \underline{\Omega}_Z^k$, $\forall 0 \le k \le p$ & $\underbrace{\mathcal{H}^i \underline{\Omega}_Z^k = 0, \ \forall 0 \le k \le p}_{Z \text{ is pre-}p\text{-du Bois}}$ $\forall i > 0$

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some examples: 1.) C curve with cusps as sing's

<u>seen</u>: $f: \hat{C} \to C$ is hyp.-res $\leadsto \underline{\Omega}_C^0 \simeq f_* \mathcal{O}_{\hat{C}}$, $\underline{\Omega}_C^1 \simeq f_* \omega_{\hat{C}}$

<u>hence</u>: C is pre-0-du Bois & pre-1-du Bois

recall from Yichen's talk: $\mathcal{H}^0 \underline{\Omega}_X^0 = \mathcal{O}_{X^{sn}}$: semi-normalization, but $\mathcal{O}_C \neq \mathcal{O}_{C^{sn}} = \mathcal{O}_{\hat{C}} \Rightarrow$ not 0-du Bois (also $f_* \omega_{\hat{C}} \neq \omega_C$ by duality: not 1-du-Bois)

<u>rk</u>: X is du Bois $\Leftrightarrow$ X is 0-du Bois $\Leftrightarrow$ X is pre-0-du Bois & semi-normal

Z toric $\Longrightarrow$ Z pre-p-du Bois $\forall p$

(no ref. in Popa's note), not p-du Bois in general

2.) Z with quotient singularities

$\xRightarrow[\text{talk}]{\text{Paul's}}$ $\underline{\Omega}^p_X \simeq (\Omega^p_X)^{\vee\vee}$, in part. pre-$p$-du Bois $\forall p$

3.) $Y \subset \mathbb{P}^N$ projective, $X := C_Y \subset \mathbb{A}^{N+1}$ cone of Y

$$\left(X \simeq C(Y,L) := \mathrm{Spec}\left(\bigoplus_{m\geq 0} H^0(Y, L^{\otimes m})\right), \ L = \mathcal{O}_Y(1) \right)$$

Paul's talk: $\Gamma(X, \mathcal{H}^i \underline{\Omega}^p_X) \simeq \bigoplus_{m\geq 1} H^i(Y, \Omega^p_Y \otimes L^{\otimes m}) \oplus H^i(Y, \Omega^{p-1}_Y \otimes L^{\otimes m})$

for $(i,p) \neq (0,0)$ and $\Gamma(X, \mathcal{H}^0 \underline{\Omega}^0_X) = \mathcal{O}_X$

Def.: Y projective satisfies Bott vanishing if

$H^i(Y, \Omega^p_Y \otimes A) = 0$ $\forall i > 0, p \geq 0$, A very ample

(true for abelian varieties, toric varieties, ...)

Cor.: Y satisfies Bott vanishing $\Rightarrow$ C_Y pre-p-du Bois $\forall p$.

moral: pre-p-du Bois is (sometimes) easy to check

much harder: prove that $\Omega^p_Z \xrightarrow{\simeq} \mathcal{H}^0 \underline{\Omega}^p_Z$

recall from construction: $\mathcal{H}^0\underline{\Omega}^p_Z \hookrightarrow \varepsilon_{0,*}\Omega^p_{Z_0}$,

hence $\mathcal{H}^0\underline{\Omega}^p_Z$ torsion free (and reflexive if Z rational)

hence: Z p-du Bois $\Rightarrow \Omega^k_Z$ torsion free $\forall k \in \{0,\dots,p\}$.

<u>Th. (P. Graf)</u>: $Z \subset X$ hypersurface, $\mathrm{tors}(\Omega^p_Z) = 0$

$\Longleftrightarrow \; p \leq \mathrm{codim}_Z(Z^{sing}) - 1$

Torsion freeness is just a necessary criterion.

<u>Thm</u> (MP, Saito, ...) $Z \subset X$ hypersurface. TFAE:

1.) Z p-du Bois
2.) $\tilde{\alpha}(Z) \geq p+1$ $\quad \left(\tilde{\alpha}(X) = \min(\alpha \mid \tilde{b}_f(-\alpha) = 0) \atop Z = f^{-1}(0)\right)$
3.) $\mathcal{I}_k(Z) = 0 \quad \forall k \in \{0,\dots,p\}$
4.) $\mathcal{I}_p(Z) = 0$

sketch of pf later in more gen. situation

when $Z \subset X$ is l.c.i.

Higher rational singularities: recall: X smooth

$\Rightarrow \omega_X = \Omega_X^{\dim X}$ and $R\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \omega_X) = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \omega_X) \cong \Omega_X^{n-p}$

Z general: $\omega_Z^\bullet$ dualizing complex, put $\mathbb{D}_Z(-) := R\mathcal{H}om_{\mathcal{O}_Z}(-, \omega_Z^\bullet[-n])$

Proposition: Z irreducible of $\dim = n$, then $\exists$ morphism

$\psi_p : \underline{\Omega}_Z^p \to \mathbb{D}_Z(\underline{\Omega}_Z^p) = R\mathcal{H}om_{\mathcal{O}_Z}(\underline{\Omega}_Z^{n-p}, \omega_Z^\bullet[-n])$ in $D^b(\mathrm{Coh}\, Z)$

Proof: idea: pull everything back to resolution

$f : \tilde{Z} \to Z$, use $(*)$ on $\tilde{Z}$. Notation:

1.) Functoriality (Paul's talk): $\exists\, \alpha_p : \underline{\Omega}_Z^p \to Rf_* \Omega_{\tilde{Z}}^p$

2.) $Rf_*((*)$ on $\tilde{Z})$: $Rf_* \Omega_{\tilde{Z}}^p \to Rf_* \mathbb{D}_{\tilde{Z}} \Omega_{\tilde{Z}}^{n-p}$

3.) Commutation of Rf_* & $\mathbb{D}$:

$Rf_* \mathbb{D}_{\tilde{Z}} \Omega_{\tilde{Z}}^{n-p} \cong \mathbb{D}_Z Rf_* \Omega_{\tilde{Z}}^{n-p} \xRightarrow{2.)+3.)}$

$$\exists\, \beta_p : Rf_* \Omega_{\tilde{Z}}^p \to \mathbb{D}_Z Rf_* \Omega_{\tilde{Z}}^{n-p}$$

4.) Consider $\mathbb{D}_Z \alpha_{n-p} = R\mathcal{H}om_{\mathcal{O}_Z}(\alpha_{n-p}, \omega_Z^\bullet[-n])$:

$$\gamma_p : \mathbb{D}_Z(Rf_* \underline{\Omega}^{n-p}_{\hat{Z}}) \to \mathbb{D}_Z \underline{\Omega}^{n-p}_Z$$

Define $\psi_p := \gamma_p \circ \beta_p \circ \alpha_p : \underline{\Omega}^p_Z \to \mathbb{D}_Z \underline{\Omega}^{n-p}_Z$ $\square$

<u>Rk</u>: to be shown that this is independent of choice of $f : \hat{Z} \to Z$ . . . .

<u>Def</u>: Z irred. of dim n. Let $p \geq 0$, then we say that Z is p-rational iff $\forall k \in \{0, \dots, p\}$ the composition

$$\Omega^k_Z \to \underline{\Omega}^k_Z \xrightarrow{\psi_k} \mathbb{D}_Z \underline{\Omega}^{n-k}_Z$$

is an iso in $D^b(\mathrm{Coh}\, Z)$, i.e.

$$\underline{\Omega}^k_Z \simeq \mathcal{H}^0 \mathbb{D}_Z \underline{\Omega}^{n-k}_Z \ \& \ \underbrace{\mathcal{H}^i \mathbb{D}_Z \underline{\Omega}^{n-k}_Z = 0}_{\text{pre-}p\text{-rational}} \quad \forall k \in \{0,\dots,p\}, \ \forall i > 0$$

$\forall k \in \{0,\dots,p\}$

Rk: By definition of p-th du Bois complex:

$$\underline{\Omega}^n_Z \simeq f_* \omega_{\tilde{Z}} \quad \text{(for any resolution)}.$$

By Grauert-Riemenschneider: $f_* \omega_{\tilde{Z}} \simeq R f_* \omega_{\tilde{Z}}$

hence: $\mathbb{D}_Z \underline{\Omega}^n_Z \simeq \mathbb{D}_Z R f_* \omega_{\tilde{Z}} \underset{\text{Grom.}}{\simeq} R f_* \mathbb{D}_{\tilde{Z}} \omega_{\tilde{Z}} \simeq R f_* \mathcal{O}_{\tilde{Z}}$

Hence:

$$Z \ 0\text{-rat.} \iff \mathcal{O}_Z = \underline{\Omega}^0_Z \xrightarrow{\simeq} \mathbb{D}_Z \underline{\Omega}^n \simeq R f_* \mathcal{O}_{\tilde{Z}} \iff Z \text{ rat.}$$

Theorem (Saito, MP): $Z \subset X$ hypersurface, then TFAE: 1.) Z p-rational; 2.) $\tilde{\alpha}(Z) > p+1$

more generally

Theorem (Shen-Venkatesh-Vo '23): assume Z e.c.i

Then: Z p-rational $\Rightarrow$ Z p du Bois

<u>Rk</u>: apparently, for Z non-l.c.i, p-du Bois and p-rat need to be defined differently.

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from here on: assume $Z \subset X$ is l.c.i.

aim: generalize [Z p-du Bois $\iff I_p(Z) = 0$] from hypersurfaces to l.c.i. varieties

recall from Henry's talk: $\mathcal{H}^q_Z \mathcal{O}_X = 0 \quad \forall q \neq r := \operatorname{codim}_X Z$

and $F_k \mathcal{H}^r_Z \mathcal{O}_X \subset E_k \mathcal{H}^r_Z \mathcal{O}_X = O_k \mathcal{H}^r_Z \mathcal{O}_X$

<u>Def</u>: MP'22: $p(Z) = \max\{k \mid F_k \mathcal{H}^r_Z \mathcal{O}_X = E_k \mathcal{H}^r_Z \mathcal{O}_X\}$ is called the singularity level of Z.

<u>Th [MP'22]</u>: $Z \subset X$ reduced, l.c.i, then $\forall p \geq 0$:

$$p(Z) \geq p \iff Z \text{ is } p\text{-du Bois}$$

<u>Rk</u>: for $r = 1$, $p(Z) \geq p \iff I_p(Z) = 0$

<u>Sketch of proof</u>: put $I := I_Z$ $\quad Z$ l.c.i $\Rightarrow Z$ CM $\Rightarrow$ sheaf

1.) $gr_k^E \mathcal{H}_Z^r \mathcal{O}_Z \simeq Sym^k\left(\left(I/I^2\right)^\vee\right) \otimes \omega_Z \otimes \omega_X^{-1}$

$\ulcorner$ if $Z \subset X$ is general, consider

$$0 \to I^k/I^{k+1} \to \mathcal{O}_X/I^{k+1} \to \mathcal{O}_X/I^k \to 0$$

notice (Tschebeck): $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = 0 \quad \forall i < \operatorname{codim} \mathcal{F}$

hence ($r := \operatorname{codim}_X Z$): $\mathcal{E}xt^{r-1}_X(I^k/I^{k+1}, \mathcal{O}_X) = 0$

$$\Rightarrow \mathcal{E}xt^r(\mathcal{O}_X/I^k, \mathcal{O}_X) \hookrightarrow \mathcal{E}xt^r(\mathcal{O}_X/I^{k+1}, \mathcal{O}_X)$$

hence $\mathcal{E}xt^r(\mathcal{O}_X/I^k, \mathcal{O}_X) \simeq \underbrace{\operatorname{Im}\left(\mathcal{E}xt^r(\mathcal{O}_X/I^k, \mathcal{O}_X) \hookrightarrow \mathcal{H}_Z^r \mathcal{O}_X\right)}_{E_k H_Z^r \mathcal{O}_X}$

therefore: $gr_k^E \mathcal{H}_Z^r \mathcal{O}_X \simeq \mathcal{E}xt^r_{\mathcal{O}_X}(I^k/I^{k+1}, \mathcal{O}_X)$ $\lrcorner$

Claim (MP '22, Lemma 9.1): $Z \subset X$ l.c.i

$$\Rightarrow \mathcal{E}xt^r(\underbrace{I^k/I^{k+1}}, \mathcal{O}_X) \simeq Sym^k\left(\left(I/I^2\right)^\vee\right) \otimes \omega_Z \otimes \omega_X^{-1}$$

$Sym^k(I/I^2)$: $\mathcal{O}_Z$-locally free

<u>notice</u>: $\mathcal{F}$ $\mathcal{O}_Z$-loc. free: $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = 0 \quad \forall i < r$

$$\mathcal{E}xt^r(\mathcal{F}, \omega_X) = \mathcal{F}^\vee \otimes \omega_Z$$

2.) from inclusion $F_\bullet \mathcal{H}^r_Z \mathcal{O}_X \subset E_\bullet \mathcal{H}^r_Z \mathcal{O}_X$ we get morphism of complexes $\forall p$

$$\varphi_p : Gr^F_{p-n} DR_X \mathcal{H}^r_Z \mathcal{O}_X \longrightarrow Gr^E_{p-n} DR_X \mathcal{H}^r_Z \mathcal{O}_X \quad (*)$$

<u>Henry's talk:</u> $\underline{\Omega}^p_Z \simeq R\mathcal{H}om_{\mathcal{O}_X}\Big(Gr^F_{p-n} DR\big(\underbrace{i_* i^! \mathbb{Q}^H_X[n]}_{\overset{Z \text{ l.c.i}}{\simeq} \mathcal{H}^r_Z \mathcal{O}_X[r]}, \omega_X\big)\Big)[p]$

hence: $\mathcal{H}^i \underline{\Omega}^p_Z \simeq \mathcal{E}xt^{p+r+i}\left(Gr^F_{p-n} DR\, \mathcal{H}^r_Z \mathcal{O}_X, \omega_X\right)$

$$\psi^i_p : \mathcal{E}xt^{i+p+r}\left(Gr^E_{p-n} DR_X \mathcal{H}^r_Z \mathcal{O}_X\right) \longrightarrow \mathcal{H}^i \underline{\Omega}^p_Z$$

from 1.) (local freeness of I/I^2):

$$\mathcal{E}xt^{i+p+r}\left(Gr^E_{p-n} DR_X \mathcal{H}^r_Z \mathcal{O}_X\right) = \begin{cases} 0 & \forall i > 0 \\ \Omega^p_Z & i = 0 \\ \text{something} & \forall i < 0 \end{cases}$$

Eagon-Northcott-cplx.

moreover: $\psi^0_p : \Omega^p_Z \to \mathcal{H}^0 \underline{\Omega}^p_Z$ is the morphism coming from $\Omega^p_Z \to \underline{\Omega}^p_Z$ in $D^b(\mathrm{Coh}\, Z)$ (apparently because $\mathcal{H}^0 \underline{\Omega}^p_Z$ is torsion free)

3.) Let $p \leq p(Z) \Rightarrow E_k \mathcal{H}^r_Z \mathcal{O}_X = F_k \mathcal{H}^r_Z \mathcal{O}_X$ $\forall k \leq p$

$\Rightarrow \varphi_k$ iso $\Rightarrow \varphi^i_k$ iso $\quad \forall i, \forall 0 \leq k \leq p$

$\Rightarrow \mathcal{H}^i \underline{\underline{\Omega}}^k_Z = 0 \quad \forall i > 0, \forall 0 \leq k \leq p$ pre-p-du Bois

and $\varphi^0_k : \Omega^k_Z \to \mathcal{H}^0 \underline{\Omega}^k_Z$ is iso $\forall 0 \leq k \leq p \Rightarrow$ p-du Bois

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Assume that Z is p-du Bois.

Need to show: $p(Z) \geq p$. Induction on p:

$p = -1$: obvious: no assumption: (-1)-du Bois

no conclusion: $p(Z) \geq -1$

assume $p(Z) \geq p-1 \Longrightarrow \varphi_p : gr^F_p DR \mathcal{H}^r_Z \mathcal{O}_X \to gr^E_p DR \mathcal{H}^r_Z \mathcal{O}_X$

coker $= \omega_X \otimes E_p \mathcal{H}^r_Z \mathcal{O}_X / F_p \mathcal{H}^r_Z \mathcal{O}_X$: sheaf

show coker $= 0$ by: 1.) $\mathcal{E}xt^i(gr^F_{p-1} DR \mathcal{H}^r_Z \mathcal{O}_X, \omega_X) = 0$
$\forall i \neq p+r$ (by $\Omega^p_Z \xrightarrow{\cong} \underline{\Omega}^p_Z$)

2.) $\operatorname{codim}_Z Z^{\text{sing}} \geq p$