Local cohomology and Hodge theory

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§1. Local cohomology [ILL⁺07]

Throughout, R comm Noeth & $\mathfrak{a} \subseteq R$ ideal.

Defⁿ For M f.g. R-module.

$$\Gamma_{\mathfrak{a}}(M) := \{ m \in M \mid \mathfrak{a}^t \cdot m = 0 \text{ for some } t \geq 0 \} \subseteq M$$

fact $\Gamma_{\mathfrak{a}} : R\text{-mod} \to R\text{-mod}$ is left-exact.

Defⁿ $H^i_{\mathfrak{a}}(M) := H^i(\Gamma_{\mathfrak{a}}(I^\bullet))$ for $I^\bullet \leftarrow M$ an injective resolution of M.

"i-th local coh module of M with support in $\mathfrak{a}$".

Propⁿ 1) $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$. $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion.

2) $H^i_{\mathfrak{a}}(M) = H^i_{\sqrt{\mathfrak{a}}}(M)$. 3) $\oplus \circ H^i_{\mathfrak{a}} = H^i_{\mathfrak{a}} \circ \oplus$

4) s.e.s. $0 \to M' \to M \to M'' \to 0$

$\rightsquigarrow$ l.e.s. $\cdots \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M'') \to H^{i+1}_{\mathfrak{a}}(M') \to \cdots$

Thm $\exists$ iso $H^i_{\mathfrak{a}}(M) = \varinjlim_t \operatorname{Ext}^i_R(R/\mathfrak{a}^t, M)$.

Proof given R-mod I

$$\operatorname{Hom}_R(R/\mathfrak{a}^t, I) \xrightarrow{\sim} \{ m \in I \mid \mathfrak{a}^t \cdot m = 0 \} \subseteq \Gamma_{\mathfrak{a}}(I).$$

$$\Rightarrow \varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \xrightarrow{\sim} \Gamma_{\mathfrak{a}}(I).$$

Cohomology functors commute with direct limit. ∎

Theorem Let $f_1, \dots, f_r$ generators of $\mathfrak{a}$. Consider Čech complex

$$\check{C}(f_1,\dots,f_r) := 0 \to \overset{0}{R} \to \overset{1}{\bigoplus R_{f_i}} \to \overset{2}{\bigoplus R_{f_i f_j}} \to \cdots \to \overset{r}{R_{f_1 \cdots f_r}} \to 0.$$

Then

$$H^i_{\mathfrak{a}}(M) = \check{H}^i(f_1, \dots, f_r, M) = H^i(\check{C}(f_1, \dots, f_r) \otimes_R M).$$

Theorem

$$\min\{ i \mid H^i_{\mathfrak{a}}(M) \neq 0 \} = \operatorname{depth}_R(\mathfrak{a}, M)$$
$$\left(:= \min\{ i \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0 \}\right)$$

Proof Write $l = \min\{ i \mid H^i_{\mathfrak{a}}(M) \neq 0 \}$.

Then

$$\operatorname{Ext}^i(R/\mathfrak{a}, M) \overset{\Gamma_{\mathfrak{a}}(M) \text{ support on } \mathfrak{a}}{\simeq} H^i(\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I^\bullet))) \simeq \begin{cases} 0 & : i < l \\ \operatorname{Hom}_R(R/\mathfrak{a}, H^l_{\mathfrak{a}}(M)) & : i = l. \end{cases}$$

$H^l_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion

$\Rightarrow l = \operatorname{depth}_R(\mathfrak{a}, M)$. ∎

Defⁿ $\operatorname{cd}_R(\mathfrak{a}, M) := \min\{ k \mid H^i_{\mathfrak{a}}(M) = 0 \ \forall i > k \}$.

"cohomological dimension of $\mathfrak{a}$". $\operatorname{cd}_R(\mathfrak{a}) := \max_M \{ \operatorname{cd}_R(\mathfrak{a}, M) \}$.

Lemma $\operatorname{cd}_R(\mathfrak{a}) = \operatorname{cd}_R(\mathfrak{a}, R)$.

Prf $0 \to K \to F \to M \to 0$ with F free. ∎

Proposition $\operatorname{cd}_R(\mathfrak{a}) \leq \dim R$.

upshot $H^i_{\mathfrak{a}}(R)$ vanishes unless $\operatorname{depth}_R(\mathfrak{a}) \leq i \leq \dim_R$.

theorem $(R, \mathfrak{m})$ local ring. Then R is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(R) = 0 \ \forall i \neq \dim R$.

in particular $R/\mathfrak{a}$ complete intersection in $g \Rightarrow H^i_{\mathfrak{a}}(R) = 0 \ \forall i \neq \operatorname{depth}_R(\mathfrak{a}) = \dim R/\mathfrak{a}$.

§2. Hodge filtration on local cohomology [MP22]

X smooth cplx alg var. $Z \subseteq X$ proper closed subvariety.

Write $\mathcal{I}_Z$ ideal sheaf, $i : Z \to X$, $j : U := X \setminus Z \to X$.

Defⁿ $\mathcal{M}$ qc $\mathcal{O}_X$-mod. $\Gamma_Z(V, \mathcal{M}) \subseteq \Gamma(V, \mathcal{M})$ sections of $\mathcal{M}$ on V with support on $Z \cap V$. $\mathcal{H}^q_Z(\mathcal{M})$ for higher derived functors.

Lemma $\mathcal{I}_Z = (f_1, \dots, f_r)$. Then the complex

$$0 \to \mathcal{O}_X \to \bigoplus \mathcal{O}_X\left[\tfrac{1}{f_i}\right] \to \bigoplus \mathcal{O}_X\left[\tfrac{1}{f_i f_j}\right] \to \cdots$$

represents (locally) the derived $\mathcal{D}_X$-module $i_* i^! \mathcal{O}_X$.

$\therefore \mathcal{H}^q i_* i^! \mathcal{O}_X \simeq \mathcal{H}^q_Z(\mathcal{O}_X)$ as $\mathcal{O}_X$-modules.

So we induce a MHM str on $\mathcal{H}^q_Z(\mathcal{O}_X)$, since $\mathcal{H}^q i_* i^!$ preserves MHM.

in $D^b \mathrm{MHM}(X)$ we have exΔ $i_* i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+1}$.

$\Rightarrow 0 \to \mathbb{Q}^H_X[n] \to j_* j^* \mathbb{Q}^H_X[n] \to \mathcal{H}^1_Z(\mathbb{Q}^H_X[n]) \to 0$

& $\mathcal{H}^{q+1}_Z(\mathbb{Q}^H_X[n]) \simeq R^q j_* j^* \mathbb{Q}^H_X[n] \ (= R^q j_* \mathcal{O}_U)$

Example Z divisor. $\mathcal{H}^1_Z \mathcal{O}_X \simeq \dfrac{\mathcal{O}_X(*D)}{\mathcal{O}_X}$. & filtrations are compatible.

Rmk Z lci $\Rightarrow \mathcal{H}^q_Z(\mathcal{O}_X) = 0$ if $q \neq r := \operatorname{codim}(Z, X)$.

Fact the Hodge filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$ is equal to the one induced by Čech complex.

Example Z smooth.

$$F_p \mathcal{H}^r_Z(\mathcal{O}_X) = \{ u \in \mathcal{H}^r_Z(\mathcal{O}_X) \mid \mathcal{I}_Z^{p+1} \cdot u = 0 \}.$$

theorem $\pi : (Y, E) \to (X, Z)$ l.r.

$$A^\bullet = \{ 0 \to \pi^* \mathcal{D}_X \to \Omega^1_Y(\log E) \otimes \pi^* \mathcal{D}_X \to \cdots \}$$
$$\simeq \omega_Y(*E) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{Y \to X}$$

then $R^q \pi_* F_{p-n} A^\bullet \hookrightarrow R^q \pi_* A^\bullet \simeq R^q j_* \omega_U \simeq \mathcal{H}^q_Z \omega_X$

& image of $F_{p-n} \mathcal{H}^q \omega_{XZ}$ for $q \geq 1$

Proposition $\mathcal{I}_Z \cdot F_p \mathcal{H}_Z(\mathcal{O}_X)^q \subseteq F_{p-1} \mathcal{H}^q_Z(\mathcal{O}_X)$.

Defⁿ order filtration $O_k \mathcal{H}^q_Z(\mathcal{O}_X) = \{ u \in \mathcal{H}^q_Z(\mathcal{O}_X) \mid \mathcal{I}_Z^{k+1} \cdot u = 0 \}$.

Facts 1) Z divisor, $= P_k / \mathcal{O}_X$.

2) $F_k \mathcal{H}^q_Z(\mathcal{O}_X) \subseteq O_k \mathcal{H}^q_Z(\mathcal{O}_X)$.

Another fact $q = r$. Z smooth $\Leftrightarrow$ equality $\forall k$.

Defⁿ recall

$$\mathcal{H}^q_Z(\mathcal{O}_X) \simeq \varinjlim_k \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Z^k, \mathcal{O}_X).$$

$$E_k \mathcal{H}^q_Z(\mathcal{O}_X) = \operatorname{im}\left(\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Z^k, \mathcal{O}_X) \to \mathcal{H}^q_Z(\mathcal{O}_X) \right).$$

"Ext functor"

qns when does $F_k \subseteq E_k$? $=$?

Prop $E_k \mathcal{H}^r_Z(\mathcal{O}_X) = O_k \mathcal{H}^r_Z(\mathcal{O}_X)$.

Prop $\forall q$, $F_0 \mathcal{H}^q_Z(\mathcal{O}_X) \subseteq E_0 \mathcal{H}^q_Z(\mathcal{O}_X)$.

thm - Z du Bois $\Rightarrow F_0 \mathcal{H}^q_Z(\mathcal{O}_X) = E_0 \mathcal{H}^q_Z(\mathcal{O}_X) \ \forall q$

- Z CM pure dim n. Then $F_0 \mathcal{H}^q_Z(\mathcal{O}_X) = E_0 \mathcal{H}^q_Z(\mathcal{O}_X) \ \forall q$ $\Leftrightarrow$ Z du Bois.

§3. relation to du Bois complex [MuS24] 6 §

$\pi : (Y, E) \to (X, Z)$

recall "Steenbrink triangle" $\mathbb{R}\pi_* \Omega^p_Y(\log E)(-E) \to \underline{\Omega}^p_X \to \underline{\Omega}^p_Z \xrightarrow{+1}$

Propⁿ $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}^p_Z, \omega_X) \to \Omega^{n-p}_X \to \mathbb{R}\pi_* \Omega^{n-p}_Y(\log E) \xrightarrow{+1}$

Proof apply $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$ to Steenbrink triangle.

$\mathbb{R}\mathcal{H}om(\mathbb{R}\pi_* \Omega^p_Y(\log E)(-E), \omega_X)$

$\simeq \mathbb{R}\pi_* \mathbb{R}\mathcal{H}om(\Omega^p_Y(\log E)(-E), \omega_Y)$

$\simeq \mathbb{R}\pi_* \Omega^{n-p}_Y(\log E)$. ∎

$\Omega^n_Y(\log E) = \omega_Y(E)$

Cor $\underline{\Omega}^p_Z = \mathbb{R}\mathcal{H}om(\operatorname{gr}^F_{p-n} \mathrm{DR}_X(i_* i^! \mathbb{Q}^H_X[n]), \omega_X)[p]$.

Proof apply $\operatorname{gr}^F_{p-n} \mathrm{DR}_X$ to $i_* i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+1}$

$$\begin{array}{ccccc} \operatorname{gr}^F_{p-n} \mathrm{DR}_X(i_* i^! \mathbb{Q}^H_X[n]) \to & \operatorname{gr}^F_{p-n} \mathrm{DR}_X(\mathbb{Q}^H_X[n]) & \to & \operatorname{gr}^F_{p-n} \mathrm{DR}_X(j_* \mathbb{Q}^H_U[n]) & \xrightarrow{+1} \\ & \wr| & & \wr| \ \text{Prop}^n \ 5.24 & \\ & \Omega^{n-p}_X[p] & \longrightarrow & \mathbb{R}\pi_* \Omega^{n-p}_Y(\log E)[p] & \end{array}$$