REMINDERS ON (MIXED) HODGE MODULES AND HODGE IDEALS

D-Modules

X = smooth alg var $/\mathbb{C}$

$\mathcal{D}_X$ = sheaf of diff oper. on $X \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$

gen. by $\mathcal{O}_X, T_X$

Have a filtr.

$F_k \mathcal{D}_X = \{$ diff. oper of order $\leq k \}$

D-Module = quasi-coh. sheaf $\mathcal{M}$ on X st $\mathcal{M}$ is a $\mathcal{D}_X$-module

FILTERED D-Module : $\mathcal{M}$ D-Module + $F_\bullet \mathcal{M}$

- $F_i \mathcal{M} \subseteq F_{i+1} \mathcal{M}$
- $\bigcup_i F_i \mathcal{M} = \mathcal{M}$

+ compatible wt $F_\bullet \mathcal{D}_X$:

$$F_i \mathcal{D}_X \cdot F_j \mathcal{M} \subseteq F_{i+j} \mathcal{M} \quad \forall i,j \geq 0$$

Def A Filtr $F_\bullet \mathcal{M}$ is <u>good</u> if
$\forall i$ $F_i\mathcal{M}$ is coh. over $\mathcal{O}_X$ and

$$\left.\begin{array}{l}\exists\, K>>0 \text{ st} \\ F_i D_X \cdot F_j\mathcal{M} = F_{i+j}\mathcal{M} \quad \begin{array}{l}\forall i>0 \\ \forall j\geq K\end{array}\end{array}\right\} (*)$$

$K_0 :=$ smallest K st $(*)$ holds
↑
generating level of the filtr.

Def A D-Module $\mathcal{M}$ is called <u>coherent</u> if it is locally fin. gen as D_X-module

$\mathcal{M}$ coherent $\rightsquigarrow \exists\, F_\bullet\mathcal{M}$ good
" $\longleftarrow$ $(\mathcal{M}, F_\bullet\mathcal{M})$

If $(\mathcal{M}, F_\bullet\mathcal{M})$ good

$\mathrm{gr}^F \mathcal{M} = \bigoplus_i \mathrm{gr}^F_i \mathcal{M}$ module over $\mathrm{gr}^F D_X$

coherent over $\mathrm{gr}^F D_X$

$\dim(\mathrm{supp}(\mathrm{gr}^F \mathcal{M})) \geq n$

$(\mathcal{M}, F_\bullet \mathcal{M})$ is hol. if $=$ holds.

Ex ① $(\mathcal{E}, \nabla)$ flat v.b. $=$ D-modules that are locally free

$\nabla : T_X \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{E})$

② $X \supseteq D =$ red. hypersurf.

$$\mathcal{O}_X(*D) = \bigcup_{m \geq 1} \mathcal{O}_X(mD)$$

locally $D = (h = 0)$

$$= \mathcal{O}_X\left[\tfrac{1}{h}\right]$$

it is a D-module (holonomic)

(MIXED) HODGE MODULES.

Phil: flat v.b ⇝ D-mod

(loc. syst ⇝ per. sheav)

P-(M) VHS ⇝ P-(M) Hodge modules

$(\mathcal{M}, F^{\bullet}\mathcal{M}, W_{\bullet}\mathcal{M}$, Per. sheaf, pol.)

hol D-module

good

st these satisfy many properties.

Rmk ① $F_i D_X \cdot F_j \mathcal{M} \subseteq F_{i+j} \mathcal{M}$

Griffiths transvers.

② have 6 functor formalism.

Ex $X \supseteq D$ $\mathcal{O}_X(*D)$ D-module

Actually, it is a Hodge module

$U := X \setminus D \xhookrightarrow{j} X \qquad \dim X = n$

On U have trivial Hodge module

$$\left(\mathcal{O}_U, F^\bullet \mathcal{O}_U, \underline{\mathbb{Q}}_U[n]\right) =: \underline{\mathbb{Q}}_U^H[n]$$

$F^\bullet\mathcal{O}_U$ $=$ trivial

$F^0 \mathcal{O}_U = 0$

$F^1 \mathcal{O}_U = \mathcal{O}_U$

$j_* \underline{\mathbb{Q}}_U^H[n]$ Mixed Hodge mod on X

$=$

$\left(j_* \mathcal{O}_U, \text{filtration}, j_* \underline{\mathbb{Q}}_U[n]\right)$

$j_* \mathcal{O}_U = \mathcal{O}_X(*D)$

__Goal__: Understand $F^\bullet \mathcal{O}_X(*D)$

__Rmk__: Have a filtration

$P_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \ \forall k \geq 0$

$\underline{\underline{\text{Thm}}}$ (Saito) If D is smooth

$$\Rightarrow F_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \quad \forall k \geq 0$$

In general,

coherent
torsion free $\subseteq \mathcal{O}_X(*D)$

$\underline{\underline{\text{Thm}}}$ (Saito) $F_k \mathcal{O}_X(*D) \subseteq \mathcal{O}_X((k+1)D)$

Let $V \overset{i}{\subseteq} X$ open st

- $\operatorname{codim}_X(X \setminus V) \geq 2$
- $D|_V$ is smooth

$$F_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \text{ on } V$$

Get

$$F_k \mathcal{O}_X(*D) \hookrightarrow i_* F_k \mathcal{O}_X(*D) = i_* \mathcal{O}_X((k+1)D)$$
$$\| $$
$$\mathcal{O}_X((k+1)D)$$

We have $I_k(D) \subseteq \mathcal{O}_X$ st

$$F_k \mathcal{O}_X(*D) = I_k(D) \otimes \mathcal{O}_X((k+1)D)$$

$\subset$ Hodge ideals

<u>Rewrite</u>: if D is smooth

$$\Rightarrow I_k(D) = \mathcal{O}_X \quad \forall k \geq 0$$

• <u>Thm</u> (M-P) if $I_k(D) = \mathcal{O}_X$ $\forall k \geq 0$

$\Rightarrow D$ smooth

$I_k(D)$ detect smooth of D

• <u>Thm</u> (Saito) $I_0(D) = \mathcal{J}(\underline{(1-\varepsilon)D})$ mult. ideal

$\mathbb{Q}$-div

$0 < \varepsilon \ll 1$

$$\mathcal{J}((1-\varepsilon)D) = \left\{ g \in \mathcal{O}_X \;\middle|\; \frac{g}{h^{1-\varepsilon}} \in L^2_{loc} \right\}$$

$D = (h=0)$

related to study of birational geom (X, D)

<u>PROP</u>: $I_0(D) = \mathcal{O}_X \Leftrightarrow (X, D)$ log-canonical

(X, D) l.c: $\pi : \tilde{X} \to X$ res of sing

$$\pi^*(K_X + D) \sim K_{\tilde{X}} + \pi^{-1}D + \sum a_i E_i$$

$a_i \geq -1$, E_i: exp. divis.

$a_i \in \mathbb{Q}$

(X, D) l.c if $a_i \geq -1$

D has du Bois sing.

$I_0(D) = \mathcal{O}_X$: log can

$I_k(D) = \mathcal{O}_X$ for some k

$I_k(D) = \mathcal{O}_X \ \forall k \geq 0$: D smooth

PROP We have

$m_x \supseteq I_0(D) \supseteq I_1(D) \supseteq I_2(D) \cdots \cdots$

in $\mathcal{O}_X$

in $m_x \subsetneq \mathcal{O}_X$

Thm (M-D) If $m = \mathrm{mult}_x D \geq 2$ and

$m(k+1) > n = \dim X \Rightarrow I_k(D) \subseteq m_x$

$\boxed{k=0}$ if $m > n \Rightarrow I_0(D) \subseteq m_x$

~~Idea~~

$k > 0$ eg $k = 2$ $m > \frac{n}{3}$ $\rightsquigarrow$ $I_2(D) \subseteq m_x$

"Birational" description of Hodge Ideals.

$\boxed{k=0}$ ~~to~~ $X \supseteq D$

consider a log-resolution

$$\begin{array}{ccc} f^{-1}(D)_{red} = E & \longrightarrow & D \\ \text{SNC} \quad \downarrow & & \cap \\ Y & \xrightarrow{f} & X \\ \| & & \\ \text{smooth} & & \end{array}$$

f iso over $U = X \setminus D$

$$f_* \underline{\omega_Y(E)} \subseteq \omega_X(D)$$

$$\|$$

$$\mathcal{J} \otimes \omega_X(D)$$

$$\|$$

$$I_0(D)$$

$\Omega^p(\log E)$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & D \arrow[d, hook] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$ log-res.

Consider

$$A^\bullet : 0 \to \underset{-n}{f^* D_X} \to \underset{-n+1}{\Omega^1_Y(\log E) \otimes_Y f^* D_X} \to \cdots$$

$$\cdots \to \underset{-1}{\Omega^{n-1}_Y(\log E) \otimes f^* D_X} \to \underset{0}{\omega_Y(E) \otimes f^* D_X} \to 0$$

it is a complex of right $f^{-1}D_X$-modules

and it is exact except in deg. 0.

Have: quasi-iso $A^\bullet \underset{\#}{\simeq} H^0(A^\bullet)$

Can define a filtr $F_{n-k} A^\bullet$ $\forall k \geq 0$:

$$F_{n-k} A^\bullet : 0 \to f^* F_{n-k} D_X \to \Omega^1_Y(\log E) \otimes f^* F_{n-k+1} D_X$$

$$\cdots \to \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_k D_X \to 0$$

<u>Def</u> $F_{n-k}\,\omega_X(*D) :=$

$$\operatorname{Im}\left[R^0 f_* F_{n-k} A^\bullet \to R^0 f_* A^\bullet\right] \hookrightarrow \omega_X(*D)$$

PROP $\underline{F_{n-k}\,\omega_X(*D)} \subset \omega_X((k+1)D)$

$\|$

$\mathcal{J}_k \otimes \omega_X((k+1)D)$

THM $\|$

$I_n(D)$

duality left $\to$ right

$M \to M \otimes \omega_X$

$\mathcal{O}_X(*D) \to \omega_X(*D)$

Goal: understand $\underline{F_0\,\mathcal{O}_X(*D)}$.

$\uparrow$

$\exists\, k \gg 0$ st

$F_i D_X \cdot F_j\,\mathcal{O}_X(*D) = F_{i+j}\,\mathcal{O}_X(*D)$

$k_0 :=$ smallest $\Big]$ generating level of the filtr.

$\underline{\text{Thm}}$ (MP) The Filtration $F_\bullet \mathcal{O}_X(*D)$ is generated at level k

$$\Delta = D \qquad \begin{array}{ccc} E & \to & D \\ \downarrow & & \downarrow \text{log-res} \\ Y & \xrightarrow{f} & X \end{array}$$

We have that

$$\boxed{R^q f_* \Omega_Y^{n-q}(\log E) = 0}$$

$\forall q > k$.

Bernstein-Sato poly and minimal exponent.

$\underline{\text{Thm}}$ (B-S) If $f \in \mathcal{O}_X(X)$ non-zero $\exists!$ monic poly of minimal degree $b_f(s) \in \mathbb{C}[s]$ and $P \in \mathcal{D}_X[s]$ st

$$P \cdot f^{s+1} = b_f(s) \cdot f^s$$

in $\underline{\underline{\mathcal{D}_X[\tfrac{1}{f}, s]}} \cdot \underline{\underline{f^s}}$ formal symbol

$\mathcal{D}_X[\tfrac{1}{f}] \otimes \mathbb{C}[s]$

f = poly P = diff. oper

$P f^{s+1} = b_f(s) \cdot f^s$

$f(x) = x$

$\partial_x f^{m+1} = (m+1) f^m$

FACTS: • if f is non-inv.

$\hookrightarrow b_f(-1) = 0$

$b_f(-1) \cdot \frac{1}{f} \in \mathcal{D}_X \cdot 1 \subseteq \mathcal{O}_X$

• roots of $b_f(s)$ are related to bir. geom of (X, D), $D = (f=0)$

• roots $s \in \mathbb{Q}_{<0}$

$\alpha_f = -(\text{greatest root of } b_f(s))$

$\alpha_f = \operatorname{lct}(f)$ log-can threshold

$\alpha_f = 1 \iff (X, D)$ is log-can.

If f is non-invertible $\Rightarrow (s+1) \mid b_f(s)$

$$\tilde{b}_f(s) = \frac{b_f(s)}{(s+1)} \quad \text{reduced B-S}$$

If $\alpha_f = 1$: (X, D) log-can.

$\tilde{\alpha}_f = \alpha_f = 1$

$\tilde{\alpha}_f > 1$ new information

minimal exp. of f / microlocal lgt.

<u>**THM**</u> The Hodge filtration on $\mathcal{O}_X(*D)$ is generated at level

$$k_0 := n - 1 - \lceil \tilde{\alpha}_f \rceil$$

We have: $R^q f_* \Omega_Y^{n-q}(\log E) = 0$

$\forall q > n-1-\lceil \hat{\partial}_f \rceil$

VANISHING THEOREM:

Fix L = line bundle on X

We have

$$H^i(X, \omega_X \otimes \underbrace{I_k(D) \otimes \mathcal{O}_X((k+1)D)}_{\substack{F_k \mathcal{O}_X(*D) \\ \|}} \otimes L) = 0$$

$\forall i > 0$

if i) $L(jD)$ is ample for $0 \leq j \leq k$

ii) $I_{k-1}(D) = \mathcal{O}_X$

iii) $H^j(X, \Omega_X^{n-j} \otimes L((k-j+1)D)) = 0$

$\forall\ 1 \leq j \leq k$

In the cases where (iii) is ~~$\not\!\!\mathcal{O}$~~ out.
satisf. $\rightsquigarrow$ get ~~on~~ consequences
on study of sing of hyper.

- $\mathbb{P}^n$
- (A, Θ)

$U = X \setminus D$

$F^k H^i(U, \mathbb{C}) \leftarrow\!\!\rightsquigarrow$ Recovered $F_\bullet \mathcal{O}_X(*D)$