Resolutions & the DuBois complex

① Simplicial & cubical resolutions

X : alg. var / $\mathbb{C}$

Def.: A "simplicial variety augmented towards X" is a sequence

$$\begin{tikzcd} \cdots\cdots & X_2 \arrow[r, bend left, "\varepsilon_{2,0}"] \arrow[r, "\varepsilon_{2,1}"] \arrow[r, bend right, "\varepsilon_{2,2}"'] & X_1 \arrow[r, shift left, "\varepsilon_{1,0}"] \arrow[r, shift right, "\varepsilon_{1,1}"'] & X_0 \arrow[r, "\varepsilon_{0,0}"] & X \end{tikzcd}$$

with $\varepsilon_{i,j} \circ \varepsilon_{i+1,j} = \varepsilon_{i,j} \circ \varepsilon_{i+1,j+1}$

$\forall i,j$ (*)

(*) implies that there is a unique morphism $\varepsilon_k : X_k \longrightarrow X$ $\forall k$ given by any composition of ε_{ij}

e.g.: $\varepsilon_2 := \varepsilon_{0,0} \circ \varepsilon_{1,0} \circ \varepsilon_{2,0}$

note that

$$\underbrace{\varepsilon_{0,0} \circ \varepsilon_{1,0}}_{\varepsilon_{0,0} \circ \varepsilon_{1,1}} \circ \varepsilon_{2,2} = \varepsilon_{0,0} \circ \underbrace{\varepsilon_{1,0} \circ \varepsilon_{2,1}}_{\varepsilon_{1,0} \circ \varepsilon_{2,0}}$$

$$\varepsilon_{0,0} \circ \underbrace{\varepsilon_{1,1} \circ \varepsilon_{2,2}}_{\varepsilon_{1,1} \circ \varepsilon_{2,1}}$$

$$\underbrace{\varepsilon_{0,0} \circ \varepsilon_{1,1}}_{\varepsilon_{0,0} \circ \varepsilon_{1,0}} \circ \varepsilon_{2,1}$$

Rem.: A sequence

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \end{tikzcd}$$

is the same as a functor

$$\Delta^{op} \longrightarrow \mathrm{Var}_{\mathbb{C}}$$

where Δ is the category with objects $\{1, \dots, k\}$, $k \in \mathbb{N}$

morphisms: order-preserving maps

Note: $$\begin{tikzcd} X_0 \arrow[r, "\varepsilon_{0,0}"] & X \end{tikzcd}$$
$$\varepsilon_{0,0} = \varepsilon_0$$

One has $\varepsilon_0^* \mathbb{Z}_X \cong \mathbb{Z}_{X_0}$

$\overset{\text{adjunction}}{\rightsquigarrow}$ $\mathbb{Z}_X \to R\varepsilon_{0*} \mathbb{Z}_{X_0}$

similarly $$\begin{tikzcd} X_1 \arrow[r, shift left, "\varepsilon_{1,0}"] \arrow[r, shift right, "\varepsilon_{1,1}"'] & X_0 \end{tikzcd}$$

one gets $\mathbb{Z}_{X_0} \to R\varepsilon_{1,0*} \mathbb{Z}_{X_1}$

$\mathbb{Z}_{X_0} \to R\varepsilon_{1,1*} \mathbb{Z}_{X_1}$

apply $R\varepsilon_{0,0*}$ to both maps

$\rightsquigarrow$ two (different) maps

$$\begin{tikzcd} R\varepsilon_{0*} \mathbb{Z}_{X_0} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & R\varepsilon_{1*} \mathbb{Z}_{X_1} \end{tikzcd}$$

define $\varphi_1 : R\varepsilon_{0*} \mathbb{Z}_{X_0} \to R\varepsilon_{1*} \mathbb{Z}_{X_1}$ to be $a - b$.

similarly: from $X_2 \substack{\to \\ \to \\ \to} X_1$ we get

$$\begin{tikzcd} R\varepsilon_{1*} \mathbb{Z}_{X_1} \arrow[r, shift left=2, "\alpha"] \arrow[r, "\beta" description] \arrow[r, shift right=2, "\gamma"'] & R\varepsilon_{2*} \mathbb{Z}_{X_2} \end{tikzcd}$$

three maps.

Define $\varphi_2 := \alpha - \beta + \gamma$

altogether, we obtain a sequence of morphisms in $D^b(\mathbb{Z}_X)$

$$\mathbb{Z}_X \longrightarrow R\varepsilon_{0*}\mathbb{Z}_{X_0} \xrightarrow{\varphi_1} R\varepsilon_{1*}\mathbb{Z}_{X_1} \xrightarrow{\varphi_2} \dots\dots$$

iterated cone of $R\varepsilon_{0*}\mathbb{Z}_{X_0} \to R\varepsilon_{1*}\mathbb{Z}_{X_1} \to R\varepsilon_{2*}\mathbb{Z}_{X_2} \to \dots$:

take flabby resolutions of $\mathbb{Z}_{X_i}$

$$\begin{tikzcd}
\varepsilon_{0*}\mathcal{F}_{0,1} \arrow[r] \arrow[d] & \varepsilon_{1*}\mathcal{F}_{1,1} \arrow[r] \arrow[d] & {} \\
\varepsilon_{0*}\mathcal{F}_{0,2} \arrow[r] \arrow[d] & \varepsilon_{1*}\mathcal{F}_{1,2} \arrow[r] \arrow[d] & {} \\
\vdots & \vdots &
\end{tikzcd}$$

take the total cx.

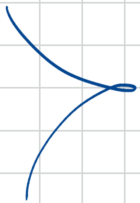
Def.: A "simplicial variety augmented towards X" is called a simplicial resolution (or hyper-resolution) if

- X_i smooth $\forall i \geq 0$
- $\varepsilon_{i,j}$ proper $\forall i,j$
- $\mathbb{Z}_X \xrightarrow{\sim} \left[R\varepsilon_{0*}\mathbb{Z}_{X_0} \xrightarrow{\varphi_1} R\varepsilon_{1*}\mathbb{Z}_{X_1} \xrightarrow{\varphi_2} \dots\right]$

iterated cone

(we say it satisfies cohomological descent)

Ex a) $X = V(y^2 - x^3) \subset \mathbb{C}^2$



$f : \tilde{X} \to X$ normalization

homeomorphism

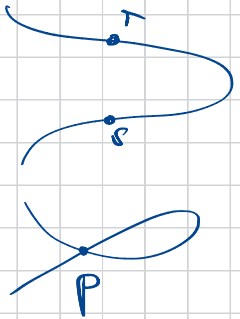
Clearly $Rf_* \mathbb{Z}_{\tilde{X}} \simeq \mathbb{Z}_X$

this is cohomological descent

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, equal, "!"'] & X \\ X_0 & \end{tikzcd}$$

⇐ hyperresolution of X

b) $X = V(y^2 - x^2 - x^3) \subset \mathbb{C}^2$



$f : \tilde{X} \to X$ normalization

$f_* \mathbb{Z}_{\tilde{X}} \xrightarrow{\not\sim} \mathbb{Z}_X$

consider

$$\begin{tikzcd}
\{r,s\} \arrow[r, "\tilde{i}"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "f"] \\
\{p\} \arrow[r, hook, "i"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 = \{r,s\} \arrow[r, shift left, "\tilde{i}"] \arrow[r, shift right, "f"'] & X_0 := \tilde{X} \sqcup \{p\} \arrow[r, "(f,i)"] & X
\end{tikzcd}$$

Claim: this is a hyperres.

$$\begin{tikzcd}
 & & & \mathbb{Z}_{\{p\}} \oplus \mathbb{Z}_{\{p\}} \arrow[d, phantom, "\shortparallel"] \\
0 \arrow[r] & \mathbb{Z}_X \arrow[r] & \cancel{R} f_* \mathbb{Z}_{\tilde{X}} \oplus \cancel{R} i_* \mathbb{Z}_{\{p\}} \arrow[r] & f_* \mathbb{Z}_{\{r\}} \oplus f_* \mathbb{Z}_{\{s\}} \\
 & & R(f,i)_* \mathbb{Z}_{X_0} & R e_{1*} \mathbb{Z}_{X_1} \arrow[d] \\
 & & & 0
\end{tikzcd}$$

is a short exact sequence

$\rightsquigarrow$ we get a distinguished triangle in $D^b(\mathbb{Z}_X)$

$$Z_X \to R\varepsilon_{0*} Z_{X_0} \xrightarrow{\varphi_1} R\varepsilon_{1*} Z_{X_1} \xrightarrow{+1}$$

$\Rightarrow$ Z_X is the cone of φ_1

c) In general, given a proper $f : \tilde{X} \to X$ and $Z \subset X$ with

$$\tilde{X} \setminus E \simeq X \setminus Z \qquad \left(E := f^{-1}(Z)_{red}\right)$$

then we get a square

$$\begin{tikzcd} E \arrow[r, hook, "\tilde{i}"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "f"] \\ Z \arrow[r, hook, "i"'] & X \end{tikzcd}$$

set $X_0 = \tilde{X} \sqcup Z$

$X_1 = E$

get:

$$X_1 \rightrightarrows X_0 \to X \qquad (\Delta)$$

which satisfies cohomological descent

Therefore, (Δ) is a hyperresolution of X whenever E, Z, $\tilde{X}$ are smooth.

d) If E is not smooth, but a snc divisor

$\overset{\cap}{X}$ smooth

$E = E_1 + \dots + E_n$

set $X_k := \bigsqcup_{i_0 < \dots < i_k} E_{i_0} \cap \dots \cap E_{i_k}$

$\varepsilon_{kj} : X_k \to X_{k-1}$ induced by inclusions

simplest case: $E = E_1 + E_2$

$X_1 = E_1 \cap E_2$ $X_0 = E_1 \sqcup E_2$

cohomological descent follows from Mayer-Vietoris sequence

$$0 \to \mathbb{Z}_E \to \mathbb{Z}_{E_1} \oplus \mathbb{Z}_{E_2} \to \mathbb{Z}_{E_1 \cap E_2} \to 0$$

$\longrightarrow X_1 \rightrightarrows X_0 \longrightarrow E$ is a hyperresolution

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<u>Def:</u> A (n-) cubical variety is a functor

$$\square_n^{op} \longrightarrow Var_{\mathbb{C}}$$

category of subsets of $\{0, \dots, n-1\}$
+ inclusions

$n=2$

$$\begin{tikzcd}
X_{\{0,1\}} \arrow[r] \arrow[d] & X_{\{1\}} \arrow[d] \\
X_{\{0\}} \arrow[r] & X_{\emptyset}
\end{tikzcd}$$

$n=3$

$$\begin{tikzcd}
 & X_{012} \arrow[rr] \arrow[dl] \arrow[dd] & & X_{12} \arrow[dl] \arrow[dd] \\
X_{0,1} \arrow[rr] \arrow[dd] & & X_{1} \arrow[dd] & \\
 & X_{02} \arrow[rr] \arrow[dl] & & X_{2} \arrow[dl] \\
X_{0} \arrow[rr] & & X_{\emptyset} &
\end{tikzcd}$$

It is called a cubical resolution of X

if

- $X_{\emptyset} = X$

- X_I smooth $\forall\, I \subset \{0,\dots,n-1\}$, $I \neq \emptyset$
- $\varepsilon_{I\subseteq J}$ proper $\forall\, I, J$
- the associated hyperresolution satisfies cohomological descent

Rk: A cubical variety determines a simplicial variety augmented towards $X_\emptyset$

ex n=3

$$\begin{tikzcd} X_0 := X_{\{0\}} \sqcup X_{\{1\}} \sqcup X_{\{2\}} \arrow[r] & X_\emptyset \\ X_1 := X_{\{0,1\}} \sqcup X_{\{0,2\}} \sqcup X_{\{1,2\}} \arrow[u, shift right, "\varepsilon_{1,0}"] \arrow[u, shift left, "\varepsilon_{1,1}"'] & \end{tikzcd}$$

clear

$\varepsilon_{1,0}$: $X_{\{0,1\}} \to X_{\{0\}}$, $X_{\{0,2\}} \to X_{\{0\}}$, $X_{12} \to X_1$

$\mathcal{E}_{1,1}$: $X_{0,1} \to X_1$, $X_{02} \to X_2$, $X_{12} \to X_2$

Existence of resolutions

1982

Thm (Deligne, Guillén - Navarro-Aznar - Pascal-Guainza - Puerta)
A variety X of dimension n admits a hyperresolution with $\dim X_i \leq n-i$ (of length $\leq n$).

Proof sketch: One proves the existence of a $(n+1)$-cubical resolution with $\dim X_I \leq n - |I| + 1$.

Step 1: Choose a resolution $\pi : \tilde{X} \to X$, $D \subset X$ minimal closed

$$\tilde{X} \setminus \pi^{-1}(D) \cong X \setminus D$$

and set $X^{(1)}_{\emptyset} = X$, $X^{(1)}_{\{0\}} = \tilde{X}$, $X^{(1)}_{\{1\}} = D$,
$X^{(1)}_{\{0,1\}} = \pi^{-1} D$.

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_0 \arrow[r] & X_{\emptyset} \\
Y^{(1)} \arrow[u, Rightarrow] \arrow[r] & Z^{(1)} \arrow[u, Rightarrow]
\end{tikzcd}$$

1-cubical var. 1-cubical var.

$$\begin{tikzcd}
E^{(1)} \arrow[r] \arrow[d] & \tilde{Y}^{(1)} \arrow[d, "\text{choose a resolution}"] \arrow[dr] & \\
D^{(1)} \arrow[r] & Y^{(1)} \arrow[r] & Z^{(1)}
\end{tikzcd}$$

The outher square <u>is</u> a 3-cubical var.

etc.

in every step:

- morphisms proper
- dim $X_I \leq n-|I|+1$
- X_I is smooth if I does not contain highest index

finally: the other X_I are 0-dim, hence smooth also.

- cohomological descent is preserved during this procedure

$\square$

② The filtered De Rham complex

<u>Recall:</u> $f : Z \to Y$ proper of smooth var's

$\rightsquigarrow \quad f^* \Omega^\bullet_Y \longrightarrow \Omega^\bullet_Z$ (pullback of forms)

compatible with stepped filtration

$\overset{\text{adj.}}{\rightsquigarrow} \quad \Omega^\bullet_Y \longrightarrow Rf_* \Omega^\bullet_Z$

Let X be a cx. var., $\dim X = n$, $\varepsilon_\bullet : X_\bullet \longrightarrow X$ hyperresolution

<u>Def:</u>

$$\underline{\Omega}^\bullet_X := \left[R\varepsilon_{0*} \Omega^\bullet_{X_0} \xrightarrow{\varphi_1} R\varepsilon_{1*} \Omega^\bullet_{X_1} \xrightarrow{\varphi_2} R\varepsilon_{2*} \Omega^\bullet_{X_2} \longrightarrow \dots \right]$$

(as before with $\mathbb{Z}_{X_i}$)

$$=: R\varepsilon_{\bullet *} \Omega^\bullet_{X_\bullet}$$

iterated cone

Filtration: $F^p \Omega^\bullet_{X_i} = \Omega^{\geq p}_{X_i}$

$$\rightsquigarrow \quad F^p R\varepsilon_{i*} \Omega^\bullet_{X_i} := R\varepsilon_{i*} \Omega^{\geq p}_{X_i}$$

this leads to a distinguished triangle

$$F^{p-1} R\varepsilon_{i*} \Omega^\bullet_{X_i} \to F^p R\varepsilon_{i*} \Omega^\bullet_{X_i} \to R\varepsilon_{i*} \Omega^p_{X_i}[-p] \xrightarrow{+1}$$

Define $F^p \underline{\Omega}^\bullet_X := R\varepsilon_{\bullet *} \Omega^{\geq p}_{X_\bullet}$

$$:= \left[R\varepsilon_{0*} \Omega^{\geq p}_{X_0} \xrightarrow{\varphi_1} R\varepsilon_{1*} \Omega^{\geq p}_{X_1} \to \dots \right]$$

$$\underline{\Omega}^p_X := gr^p_F \underline{\Omega}^\bullet_X [p]$$

$$\simeq R\varepsilon_{\bullet *}\,\Omega^p_{X_\bullet}$$

$$= \left[R\varepsilon_{0*}\,\Omega^p_{X_0} \xrightarrow{\varphi_1} R\varepsilon_{1*}\,\Omega^p_{X_1} \to \cdots \right]$$

pth Du Bois complex of X.

Thm (Du Bois) $X_\bullet' \to X_\bullet$ morph. of hyperresolutions of X induces a quasi-isomorphism

$$\underline{\Omega}^p_{X'} \longrightarrow \underline{\Omega}^p_X .$$

In part, $\underline{\Omega}^p_X$ and hence $\underline{\Omega}^\bullet_X$ are well-defined.

explictly: Ω^p_X is the total complex of

$$\begin{tikzcd}
\mathcal{E}_{0*}\mathcal{A}^{p,0}_{X_0} \arrow[r, "\varphi_1"] \arrow[d, "\bar\partial"] & \mathcal{E}_{1*}\mathcal{A}^{p,0}_{X_1} \arrow[r, "\varphi_2"] \arrow[d] & \cdots \\
\mathcal{E}_{0*}\mathcal{A}^{p,1}_{X_0} \arrow[r] \arrow[d] & \mathcal{E}_{1*}\mathcal{A}^{p,1}_{X_1} \arrow[d] & \cdots \\
\mathcal{E}_{0*}\mathcal{A}^{p,2}_{X_0} \arrow[r] \arrow[d] & \mathcal{E}_{1*}\mathcal{A}^{p,2}_{X_1} \arrow[d] & \cdots \\
\vdots & \vdots &
\end{tikzcd}$$

Ex : X smooth, $\varepsilon = id$

$$\underline{\Omega}^{\circ}_{X} = \Omega^{\circ}_{X} \ , \ \underline{\Omega}^{p}_{X} = \Omega^{p}_{X}$$

Ex : $X = V(y^2 - x^3 = 0)$

$f : \tilde{X} \to X$ normalization is hyperresolution

$$\Rightarrow \underline{\Omega}^{0}_{X} \cong f_* \Omega^{0}_{\tilde{X}} \cong f_* \mathcal{O}_{\tilde{X}} \not\cong \mathcal{O}_X$$

$$\left(\mathbb{C}[t] \quad \text{vs.} \quad \frac{\mathbb{C}[x,y]}{(y^2-x^3)} \right)$$

$$\underline{\Omega}^{1}_{X} \cong f_* \omega_{\tilde{X}} \not\cong \omega_X$$

$\llcorner$ by dualizing the above non-isomorphism

Ex: $X = V(y^2 - x^2 - x^3)$ $\quad f: \tilde{X} \to X$ normalization

recall hyperres. coming from

$$\begin{tikzcd}
\{r,s\} \arrow[r, "\tilde{i}"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "f"] \\
\{p\} \arrow[r, "i"] & X
\end{tikzcd}$$

$\underline{\Omega}^0_X$ = cone of

$$\begin{tikzcd}
R(f,i)_* \mathcal{O}_{X_0} \arrow[r] \arrow[d, equal] & Rf_* X_1 \arrow[d, equal] \\
f_* \mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\{p\}} & f_* \mathcal{O}_{\{r,s\}}
\end{tikzcd}$$

observe that one has a short exact sequence

$$0 \to \mathcal{O}_X \to f_* \mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\{p\}} \to f_* \mathcal{O}_{\{r,s\}} \to 0$$

(exercise)

$$\Rightarrow \underline{\Omega}^0_X \cong \mathcal{O}_X$$

i.e. here one gets a <u>nicer</u> result for $\underline{\Omega}^0_X$ but the hyperresolution is more complicated than "just" normalization

$$\underline{\Omega}^1_X \cong f_* \omega_{\tilde{X}} \neq \omega_X$$

Ex: $X = C(Y) \subset \mathbb{C}^3$ cone over a smooth conic in $\mathbb{P}^2$

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "f"] \\
\{0\} \arrow[r, hook] & X
\end{tikzcd}$$

blow-up of the vertex $\{0\}$

$$\underline{\Omega}^0_X = \Big[\underbrace{Rf_* \mathcal{O}_{\tilde{X}}}_{\overset{\cong}{\mathcal{O}_X}} \oplus \mathcal{O}_{\{0\}} \longrightarrow \underset{\overset{\cong}{\mathcal{O}_{\{p\}}}}{Rf_* \mathcal{O}_C}\Big]_{\text{cone}}$$

(X has rational singularities)

($C \cong \mathbb{P}^1$)

$$\Rightarrow \quad \underline{\Omega}^0_X \cong \ker\left(\begin{array}{ccc} \mathcal{O}_X \oplus \mathcal{O}_{\{p\}} & \longrightarrow & \mathcal{O}_{\{p\}} \\ (f, g) & \longmapsto & f|_p - g \end{array} \right)$$

$$\cong \mathcal{O}_X$$

$$\underline{\Omega}^2_X \cong Rf_* \Omega^2_{\tilde{X}} \cong f_* \omega_{\tilde{X}} \underset{\substack{| \\ \text{X has rational} \\ \text{singularities}}}{\cong} \omega_X$$

$$\underline{\Omega}^1_X \cong \underbrace{\left[Rf_* \Omega^1_X \longrightarrow Rf_* \Omega^1_C \right]}_{\text{cone}}$$

consider the long exact sequence

$$0 \to \mathcal{H}^0\underline{\Omega}^1_X \to f_*\Omega^1_{\tilde{X}} \to \overbrace{f_*\omega_C}^{=0\ (C\simeq\mathbb{P}^1)} \to \mathcal{H}^1\underline{\Omega}^1_X$$

$$\to R^1 f_*\Omega^1_{\tilde{X}} \to R^1 f_*\omega_C \to \mathcal{H}^2\underline{\Omega}^1_X \to 0$$

isomorphism
(exercise?)

one can show that

$$\Longrightarrow \quad \underline{\Omega}^1_X \cong f_*\Omega^1_{\tilde{X}} \quad (\neq \Omega^1_X)$$