Hodge theory & local cohomology

classical Hodge theory: X smooth proj. $/\mathbb{C}$

$H := H^k(X,\mathbb{C})$ pure HS of weight k, i.e.

$H = \bigoplus_{p+q=k} H^{p,q}(X)$. X general, $\exists W_\bullet H$ s.t.

gr^W_m pure weight m

X	smooth qproj	proj., sing
	$m \in [k, 2k]$	$m \in [0,k]$

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construction: smooth, qproj: $X \overset{j}{\hookrightarrow} \bar{X}$ proj.

with $E := \bar{X} \setminus X$ NCD, study

$$\left(\underset{\text{stupid}}{\Omega^\bullet_{\bar{X}}(\log E), \tau^{\geq \bullet}}, \underset{\text{pole order}}{W_\bullet}, Rj_* \mathbb{Q}_X \right)$$

mixed Hodge complex of sheaves

inducing $W_\bullet$ on $H^k(X,\mathbb{Q})$, $F^\bullet$ on $H^k(X,\mathbb{C})$.

X proj., singular: Let $f: \tilde{X} \to X$ be a resolution of sing's. If:

$\mathbb{Q}_X \xrightarrow{\sim} Rf_* \mathbb{Q}_{\tilde{X}}$ ("f is of cohomological descent")

then consider $Rf_*(\Omega^\bullet_{\tilde{X}}, \tau^{\geq}; \mathbb{Q}_{\tilde{X}})$

$\rightsquigarrow$ Hodge complex of sheaves.

ex: $X = \{y^2 - x^3 = 0\} \subset \mathbb{C}^2$ ($\triangle$: not compact)

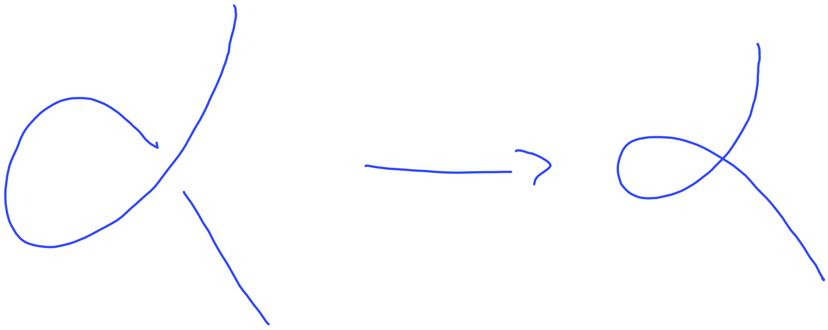
$\rightsquigarrow f: \tilde{X} \to X$ normalization, clearly $Rf_* \mathbb{Q}_{\tilde{X}} = f_* \mathbb{Q}_{\tilde{X}} = \mathbb{Q}_X$

cohomological descent almost never holds

ex: $X = \{y^2 - x^2 - x^3 = 0\} \subset \mathbb{C}^2$

more generally: X curve with nodal singularities

$$f : \tilde{X} \longrightarrow X$$



here $(f_* \mathbb{Q}_{\tilde{X}})_x = \begin{cases} \mathbb{Q}_{X,x} = \mathbb{Q} & \forall x \neq 0 \\ \mathbb{Q}^2 & x = 0 \end{cases}$

hence $f_* \mathbb{Q}_{\tilde{X}} \neq \mathbb{Q}_X$

<u>solution</u>: Let $f^{-1}(0) = \{p,q\}$, consider diagram

$$\begin{tikzcd} \{p,q\} \arrow[r, "i'"] \arrow[d, "f'"'] \arrow[dr, dashed, "g"] & \tilde{X} \arrow[d, "f"] \\ \{0\} \arrow[r, "i"] & X \end{tikzcd}$$

put: $X_0 := \tilde{X} \amalg \{0\}$, $X_1 = \{p,q\}$

Then: $X_1 \underset{f'}{\overset{i'}{\rightrightarrows}} X_0 \xrightarrow{(f,i)} X$

s.t. $(f,i) \circ i' = (f,i) \circ f' =: g$

refined version of "being of cohom. descent":

$$0 \to \underline{\mathbb{Q}}_X \xrightarrow{\alpha} (f,i)_* \underline{\mathbb{Q}}_{X_0} = f_* \underline{\mathbb{Q}}_{\tilde{X}} \oplus i_* \mathbb{Q}_{\{0\}} \xrightarrow{\beta} \underbrace{g_* \mathbb{Q}_{\{p,q\}}}_{\simeq \mathbb{Q}_{\{p\}} \oplus \mathbb{Q}_{\{q\}}} \to 0$$

where $\alpha(s) := (f^* s, s_{|\{0\}})$

β is induced from difference of maps

$$\underline{\mathbb{Q}}_{X_0} \to i_* \underline{\mathbb{Q}}_{X_1} \quad \text{and} \quad \underline{\mathbb{Q}}_{X_0} \to f_* \underline{\mathbb{Q}}_{X_1}$$

$$\begin{array}{ccccccc} 0 \to \underline{\mathbb{Q}}_X & \xrightarrow{\alpha} & f_* \underline{\mathbb{Q}}_{\tilde{X}} \oplus i_* \mathbb{Q}_{\{0\}} & \xrightarrow{\beta} & \mathbb{Q}_{\{p\}} \oplus \mathbb{Q}_{\{q\}} \to 0 \\ s & \longmapsto & (f^* s, s_{|\{0\}}) & & s \mapsto (f^* s_{|\{p\}} - s_{|\{0\}}, f^* s_{|\{q\}} - s_{|\{0\}})^{f^* s_{|\{p\}} = f^* s_{|\{q\}} = s_{|\{0\}}} = (0,0) \\ & & (t, x) & \longmapsto & (t_{|\{p\}} - x, t_{|\{q\}} - x) \end{array}$$

Talk 2: $X_\bullet := [X_1 \rightrightarrows X_0] \xrightarrow{\varepsilon} X$ is simplicial resolution of X and we have $\underline{\mathbb{Q}}_X \xrightarrow{\sim} R\varepsilon_* \underline{\mathbb{Q}}_{X_\bullet}$

Def: X arbitrary, $X_\bullet \xrightarrow{\varepsilon} X$ simplicial resolution

then $\underline{\Omega}^\bullet_X := R\varepsilon_* \underline{\Omega}^\bullet_{X_\bullet}$ filtered de Rham complex of X

filtered diff. complex, $F^p \underline{\Omega}^\bullet_X := R\varepsilon_{\bullet *} \tau^{\geq p} \Omega^\bullet_{X_\bullet}$

Then $\underline{\Omega}^p_X := gr^p_F \underline{\Omega}^\bullet_X \in D^b(Coh(X))$ is

the p-th du Bois complex (indep. of choice $X_\bullet \to X$)

<u>clear</u>: X smooth $\Rightarrow \underline{\Omega}^p_X = \Omega^p_X$ (a sheaf)

actually: $\forall X$: $\exists$ morphism $\Omega^p_X \to \underline{\Omega}^p_X$ Kähler diff

many more properties of $\underline{\Omega}^p_X$ to be discussed in Talk 3

<u>Def</u>: X is m-du Bois $\Leftrightarrow \forall p \leq m: \Omega^p_X \simeq \underline{\Omega}^p_X$

(i.e. $\Omega^p \simeq \mathcal{H}^0(\underline{\Omega}^p)$ and $\mathcal{H}^i(\underline{\Omega}^p) = 0 \; i > 0$)

0-du Bois $\hat{=}$ du Bois-sing's. (Talk 4+7)

more down-to-earth description: $X \subset Y$, Y smooth

Let $(\tilde{Y}, E) \xrightarrow{f} (Y, X)$ be strong log resolution

i.e. $E = f^{-1}(X)_{red.} \Rightarrow \underline{\Omega}^0_X \overset{\text{Steenbrink triangle Talk 3}}{=} Rf_* \underline{\Omega}^0_E \overset{\underline{\Omega}^p_E = \Omega^p_{\tilde{Y}} / \Omega^p_{\tilde{Y}}(\log E)(-E), \text{ in part: } E \text{ du Bois}}{=} Rf_* \mathcal{O}_E$. So: X du Bois

$\Leftrightarrow \mathcal{O}_X \xrightarrow{\sim} Rf_* \mathcal{O}_E$ (recall: X rational $\Leftrightarrow$ $\mathcal{O}_X \xrightarrow{\simeq} Rf_* \mathcal{O}_{\tilde{X}}$, $\tilde{X} \to X$ res. of sing's)

<u>examples</u>: $-$ $X = V(y^2 - x^3) \subset \mathbb{C}^2$, then normalization $f : \tilde{X} \to X$ is simplicial resolution of coh. descent $\rightsquigarrow$ $\underline{\Omega}^\bullet_X = f_* \Omega^\bullet_{\tilde{X}}$, in particular $\underline{\Omega}^0_X = f_* \mathcal{O}_{\tilde{X}}$, but $f_* \mathcal{O}_{\tilde{X}} \neq \mathcal{O}_X$ (i.e. $\dim \left(f_*\mathcal{O}_{\tilde{X}} / \mathcal{O}_X \right) = \delta = 1$) not du Bois (also $\underline{\Omega}^1_X = f_* \omega_{\tilde{X}} \neq \omega_X$: not rational)

$-$ $X = V(y^2 - x^2 - x^3)$. Recall simp. res.

$$\begin{tikzcd} \{p,q\} \arrow[r, "i'"] \arrow[d, "f'"'] \arrow[dr, dotted, "g"] & \tilde{X} \arrow[d, "f"] \\ \{0\} \arrow[r, hook, "i"] & X \end{tikzcd}$$

$$\{p,q\} \underset{f'}{\overset{i'}{\rightrightarrows}} \tilde{X} \sqcup \{0\} \xrightarrow{(f,i)} X$$

<u>check</u>: $\underline{\Omega}^0_X \simeq \left[f_* \mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\{0\}} \xrightarrow[\gamma]{} g_* \mathcal{O}_{\{p,q\}} \right]$

$\gamma(h, x) = (h_{|\{p\}} - x, h_{|\{q\}} - x)$. γ surj.

$\ker(\gamma) = \mathcal{O}_X \Rightarrow \mathcal{O}_X \simeq \underline{\Omega}^0_X$: du Bois

again: $\underline{\Omega}^1_X = f_* \omega_{\tilde{X}} \neq \omega_X$ not rational

$\llcorner$ always have $\underline{\Omega}^{top}_X = f_* \omega_{\tilde{X}}$ (since not normal)

Mixed Hodge modules & local cohomology

$U \to \Delta^{*k}$ VPHS $\xrightarrow[\text{Sabbah-Schnell}]{\text{Schmid}}$ $\exists$ limit MHS on $\Psi_t(V)$

The 3 situations:
- X smooth proj
- X singular
- $U \to \Delta^{*k}$

can all be handled by M. Saito's theory of MHM: $(\mathcal{M}, F_\bullet\mathcal{M}, W_\bullet\mathcal{M}, K)$ ......

+ 6 functors formalism.

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particular aspect: Hodge ideals (Talk 5)

Let $D \subset Y$ be (reduced) divisor, $U = Y \setminus D \overset{j}{\hookrightarrow} Y$

$\rightsquigarrow j_* {}^p\underline{\mathbb{Q}}^H_U$ MHM on Y with underlying $\mathcal{D}_Y$-module $\mathcal{O}_Y(*D)$

<u>Saito</u>: $F_k\mathcal{O}_Y(*D) \subset \mathcal{O}_Y((k+1)D)$; " = " $\Longleftrightarrow$ D smooth [MP]

<u>Def.</u>: $I_k(D) \subset \mathcal{O}_Y$ s.t.: $F_k\mathcal{O}_Y(*D) = I_k(D)\cdot\mathcal{O}_Y((k+1)D)$

k'th Hodge ideal

notice: $I_0(D) = \mathcal{J}((1-\varepsilon)D)$, $\varepsilon \ll 1$, multiplier ideal

Philosophy: Birational geometry of (Y, D) governed by $F_\bullet\mathcal{O}_Y(*D)$ resp. $I_\bullet(D)$, e.g.

<u>Theorem (Popa-Mustata)</u>: Let $(\tilde{Y}, E) \xrightarrow{f} (Y, D)$ be strong log. resolution, then

$R^i f_* \Omega_{\tilde{Y}}^{\dim Y - i}(\log E) = 0 \quad \forall i > k$ "local vanishing"

iff $F_\bullet\mathcal{O}_Y(*D)$ is generated at level k,

i.e. $\forall \ell \geq k \quad F_\ell\mathcal{O}_Y(*D) = F_{\ell-k}\mathcal{D}_Y \cdot F_k\mathcal{O}_Y(*D)$

$\exists$ exact sequence: $0 \to \mathcal{O}_Y \to \mathcal{O}_Y(*D) \to \underbrace{\mathcal{O}_Y(*D)/\mathcal{O}_Y}_{H^1_D\mathcal{O}_Y} \to 0$

SES in MHM(Y): $0 \to \underbrace{j_{!*}\,{}^p\mathbb{Q}^H_U}_{= {}^p\mathbb{Q}^H_Y} \to j_*\,{}^p\mathbb{Q}^H_U \to i_* i^!\,{}^p\mathbb{Q}^H_Y \to 0$

Since $F_\bullet$ and $W_\bullet$ are trivial on $\mathcal{O}_Y$,

$$(\text{Hodge data})(j_* \mathbb{Q}_U^H) \longleftrightarrow (\text{Hodge data})(i_* i^! \mathbb{Q}_Y)$$

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more generally (Talk 6): $X \overset{i}{\hookrightarrow} Y$, Y smooth

have triangle: $i_* i^! \, {}^p\mathbb{Q}_Y^H \to {}^p\mathbb{Q}_Y \to j_* {}^p\mathbb{Q}_U \xrightarrow{+1}$ in $D^b MHM(Y)$

$\mathrm{Dmod}(\mathcal{H}^q(i_* i^! \, {}^p\mathbb{Q}_Y^H)) = \mathcal{H}^q_X \mathcal{O}_Y$ q-th loc. coh.

$\mathrm{Dmod}(\mathcal{H}^q(j_* {}^p\mathbb{Q}_U)) = R^q j_* \mathcal{O}_U$ $(\cong \mathcal{H}^{q+1}_X(\mathcal{O}_Y)\ \forall q>0)$

<u>Fact 1:</u> $\mathcal{H}^q_X \mathcal{O}_Y = 0 \quad \forall\, q < \mathrm{codim}_Y(X)$

<u>Fact 2:</u> $X \subset Y$ is l.c.i.: $\mathcal{H}^q_X \mathcal{O}_Y = 0 \ \forall q \neq \mathrm{codim}_Y X$

<u>Def. (Talk 8):</u> $\mathrm{lcd}(X,Y) := \max\{q \mid \mathcal{H}^q_X \mathcal{O}_Y \neq 0\}$

$\mathrm{lcdef}(X,Y) := \mathrm{lcd}(X,Y) - \mathrm{codim}_Y X$

<u>Th (Ogus, RSW):</u> $\mathrm{lcdef}(X,Y) = \max\{j \mid {}^p\mathcal{H}^{-j}({}^p\mathbb{Q}_X) \neq 0\}$

intrinsic to X

Th (M.P.): The following are equiv.

1.) $\operatorname{lcd}(X,Y) \leq c$

2.) $\exists$ strong log. res. $(\tilde{Y}, E) \xrightarrow{f} (Y, X)$ s.t.

$$R^p f_* \Omega^q_{\tilde{Y}}(\log E) = 0 \quad \forall\, p+q \geq n+c$$

technology to prove this: determine generating level of $F_\bullet \mathcal{H}^q_X \mathcal{O}_Y$:

naively: $F_p \mathcal{H}^q_X \mathcal{O}_Y \subset \mathcal{O}_p \mathcal{H}^q_X \mathcal{O}_Y := \{u \in \mathcal{H}^q_X \mathcal{O}_Y \mid \mathcal{I}^p_X \cdot u = 0\}$

Problem: $\mathcal{O}_p \mathcal{H}^q_X \mathcal{O}_Y$ not coherent in general (Lyubeznik)

Recall: $\mathcal{H}^q_X \mathcal{O}_Y = \varinjlim_p \mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{O}_Y / \mathcal{I}^p_X, \mathcal{O}_Y)$, define

$E_p \mathcal{H}^q_X \mathcal{O}_Y := \operatorname{Im}\left(\mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{O}_Y / \mathcal{I}^p_X, \mathcal{O}_Y)\right)$

$Y \subset X$ l.c.i $\rightsquigarrow E_\bullet = \mathcal{O}_\bullet$. In general, have: $F_0 \mathcal{H}^q_X \mathcal{O}_Y \subset E_0 \mathcal{H}^q_X \mathcal{O}_Y$

For $k > 0$, $F_k \mathcal{H}^q_X \mathcal{O}_Y \subset E_k \mathcal{H}^q_X \mathcal{O}_Y$ is unclear in gen.

local cohomology & du Bois complex:

general philosophy: $M \in MHM(Y)$, study $gr_p^F DR^\cdot(M) \in D^b(Coh(Y))$ (e.g. Schnell-Kebekus: extendability of diff. forms to resolutions)

Theorem (MP): $X \overset{i}{\subset} Y$ as above, then

$$\underline{\Omega}_X^p \overset{!}{\simeq} R\mathcal{H}om_{\mathcal{O}_Y}\left(gr_p^F DR(i_* i^! \,{}^p\underline{\mathbb{Q}}_Y^H), \omega_X\right)[p]$$

$$\underset{\text{duality}}{\simeq} gr_{-p}^F DR({}^p\underline{\mathbb{Q}}_X^H)[p]$$

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local cohomology & higher du Bois sing's:

supp. $X \subset Y$ l.c.i., $r := \mathrm{codim}_Y(X)$, let

$$p(X) := \max\{p \mid F_p \mathcal{H}_X^r \mathcal{O}_Y = O_p \mathcal{H}_X^r \mathcal{O}_Y\}.$$

Th [M.P.]: $\forall p \geq 0$: $p(X) \geq p \Leftrightarrow X$ is p-du Bois

case $r=1$: X p du Bois $\Leftrightarrow I_k(X) = 0 \ \forall k \leq p \Leftrightarrow \tilde{\alpha}(X) \geq p+1$
minimal exp.

to finish: (cohom.) motivation for p-du Bois

<u>Theorem</u> (Friedman-Laza): $f: X \to S$ proper flat, fibres l.c.i and p-du Bois. Then $R^i f_* \Omega^j_{X/S}$ is $\mathcal{O}_S$-locally free and compatible with base change $\forall\, i \geq 0, \forall\, j \leq p$.

(case $p=0$ known due to du Bois, Talk 4)