Exercises Algebraic Geometry
Sheet 8 - solutions

1. Exercise: Calculate the divisor of $x/y$ on the Segre quadric $X = V(xy - zw) \subset \mathbb{P}^3$.
   It is easy to see that $(x, xy - zw) = (x, zw) = (x, z) \cap (x, w)$ and similarly $(y, xy - zw) = (y, zw) = (y, z) \cap (y, w)$. If we denote by $X_1 = V(x, z) \subset X$, $X_2 = V(x, w) \subset X$ and by $Y_1 = V(y, z) \subset X$, $Y_2 = V(y, w) \subset X$, then the divisor of the rational function $x/y$ is given as
   \[
   \left(\frac{x}{y}\right) = X_1 + X_2 - Y_1 - Y_2 \in \text{Div}(X)
   \]

2. Exercise: Determine the divisor of $x_1/x_0 - 1$ on the circle $X = V(x_1^2 + x_2^2 - x_0^2) \subset \mathbb{C}^3$.
   The function $x_1/x_0 - 1$, written as $\frac{x_1 - x_0}{x_0}$, gives the divisor $2X_2 - X_{12} - X_{21}$ where we write $X_2 = V(x_2, x_1 - x_0) \subset X$, $X_{12} = V(x_1 + ix_2, x_2) \subset X$ and $X_{21} = V(x_1 + ix_0, x_2)$. To see the multiplicities, we argue as above, e.g., we have that $x_0 - x_1 = x_2^2(x_0 + x_1)^{-1} \in k[X]_{(x_1-x_0,x_2)}$ where $x_2$ is a generator of the maximal ideal of the local ring $k[X]_{(x_1-x_0,x_2)}$.

3. Exercise: Calculate the divisor of $y$ on the cone $X = V(xy - z^2) \subset k^3$.
   Let us first describe the set-theoretic vanishing locus of the function $y$ on $X$: Obviously, we have $(xy - z^2, y) = (y, z^2)$. This means that the divisor of $y$ is supported by the irreducible variety $Y = V(y, z) \subset X$. In order to determine the multiplicity of $y$, let us consider the local ring $\mathcal{O}_{X,Y}$, i.e., the localization of the coordinate ring $k[X] = k[x, y, z]/(xy - z^2)$ at the prime ideal $(y, z) \subset k[X]$. This localization has a maximal ideal generated by $z$, because in this localization, we have that $y = x^{-1}z^2 \subset (z)$. The very same equation tells us that the multiplicity $\nu_Y(y)$ we are looking for is two: $y$ is a $z^2$ times a unity $(x-1)$. We conclude that
   \[ (y) = 2Y \in \text{Div}(X) \]

4. Exercise: Prove that for any smooth variety $X$, $\text{Cl}(X \times k) \cong \text{Cl}(X)$.
   We define the map
   \[ \pi^*: \text{Div}(X) \rightarrow \text{Div}(X \times k) \]
   \[ D = \sum_i a_i X_i \mapsto \pi^* D = \sum_i a_i \pi^{-1}(X_i) \]
   where $\pi: X \times k \rightarrow X$ denotes the projection. If $D = (f/g) \in k(X)$, then $\pi^* D$ is just the divisor of the same $f/g$, this time seen as an element in $k(X)(t)$ ($t$ being the coordinate on $k$ in $X \times k$). Therefore, the map $\pi^*$ sends divisors of rational functions on $X$ to divisors of rational functions on $X \times k$ and thus descends to a map $\pi^*: \text{Cl}(X) \rightarrow \text{Cl}(X \times k)$.
   Next we would like to show that $\pi^*$ is both injective and surjective. In order to do that, we need to discuss the possible irreducible codimension one subvarieties of $X \times k$: Two types of such prime divisors $C$ can occur: Either $C$ is dominant over $X$, i.e., $\pi(C)$ is a dense subset of $X$, or the closure of $\pi(C)$ is a prime divisor of $X$. There cannot be any other type of prime divisors on $X \times k$, because if $\pi(C)$ would be of dimension strictly smaller then $\dim(X) - 1$, then one could find a chain $\pi(C) \subseteq \bar{C} \subseteq X$, and $\pi^{-1}(\bar{C})$ would lie between $C$ and $X \times k$ so that $C$ would not be a divisor on $X \times k$. 


Let us show that $\pi^*$ is injective: Suppose that $D \in \text{Div}(X)$ and that $\pi^*(D) = (f/g)$ for some $f, g \in k[X][t]$ with $f, g$ relatively prime. Then $f$ and $g$ are necessarily elements in $k[X]$, as otherwise $\pi^*(D)$ would have components which are dominant over $X$. These cannot be of the form $\pi^{-1}(C_i)$ for some prime divisor $C_i \in \text{Div}(X)$, so that $(f/g)$ would not be of the form $\pi^*(D)$.

It follows from this discussion that any prime divisor $\tilde{C}$ on $X \times k$ which projects to (a dense subset of) a divisor $C$ on $X$ is of the form $\tilde{C} = \pi^*(C)$, in particular, $\tilde{C}$ is in the image of $\pi^*$. In order to prove surjectivity of $\pi^*$, we need to show that any prime divisor $C$ on $X \times k$ dominant over $X$ is linearly equivalent to a divisor which projects to a divisor on $X$. Let $I \subset k[X][t]$ be the defining ideal of $C$. Consider the map $k[X][t] \to k(X)[t]$ and let $\tilde{I} \subset k(X)[t]$ be the image of $I$. The ring $k(X)[t]$ is a principal ideal domain (because $k(X)$ is a field) so that $\tilde{I} = (f)$ for some $f \in k(X)[t] \subset k(X)(t)$. This means that we can consider the divisor $(f) \in \text{Div}(X \times k)$ of $f$, and the fact that $f$ is an element in $k(X)[t]$ shows that this divisor contains $C$ and perhaps some other divisor of type $\pi^{-1}(D)$ with $D \in \text{Div}(X)$, but no other divisor dominant over $X$. This shows that $C$ is linearly equivalent to a divisor in the image of $\text{Div}(X) \to \text{Div}(X \times k)$, so that $\pi^*$ on the class groups is surjective.

Remark: The statement just proved is valid in a more general context, namely, it is sufficient to suppose that $X$ is regular in codimension one, that is, that the (closed) subset of points $x$ such that $X$ is singular at $x$ is of codimension at least two in $X$.  

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