Exercises Algebraic Geometry

Sheet 7 - solutions

1. Exercise: The Cremona transformation: Let $X \subset \mathbb{P}^2$ be given by $X = V(I)$ with $I = (xy, xz, yz)$, i.e., $X = \{(0 : 0 : 1)\} \cup \{(1 : 0 : 0)\} \cup \{(0 : 1 : 0)\}$. Denote by $\tilde{\mathbb{P}}^2$ the blowup of $\mathbb{P}^2$ in $X$. Consider the morphism

$$f : \mathbb{P}^2 \setminus X \longrightarrow \mathbb{P}^2$$

$$(x : y : z) \longmapsto (xyz : yz)$$

Show that there is no morphism $F : \mathbb{P}^2 \to \mathbb{P}^2$ with $F|_{\mathbb{P}^2 \setminus X} = f$. Show moreover that there exists an isomorphism $\tilde{F} : \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2$ which extends $f$.

(a) Suppose that there would be an extension $F$ of $f$ to the whole projective plane. In particular, on the chart $\{x \neq 0\}$, $F$ would be an extension of the map $k^2 \setminus \{(0, 0)\} \to \mathbb{P}^2$ given by $(y, z) \mapsto (yz : z : y)$ to the origin in $k^2$. But then this map would send this origin to $(0 : 0 : 0)$ which is not a point of $\mathbb{P}^2$.

(b) It is a general statement on rational maps $f = (f_0, \ldots, f_k) : X \dashrightarrow \mathbb{P}^k$ that they extend to a regular map $\tilde{F} : \tilde{X} \to \mathbb{P}^k$ where $\tilde{X}$ is the blowup of $X$ in the ideal $(f_0, \ldots, f_k)$. In our situation, this shows that there is a morphism $\tilde{F} : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$. But we can say much more: Consider the subvariety $Y$ of $\mathbb{P}^2 \times \mathbb{P}^2$ given by the equations $xc - yb, xc - za$, where $(x : y : z)$ and $(a : b : c)$ are the coordinates on the two factors.

**Lemma 1.** The two projections $p_1, p_2 : Y \to \mathbb{P}^2$ induced from the projections $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ both identify $Y$ with the blowup $\tilde{\mathbb{P}}^2$. The composition $p_2 \circ p_1^{-1}$ is well-defined on $\mathbb{P}^2 \setminus X$ and coincides with $f$, so that the identity on $Y$ gives an extension of $f$ to an automorphism of $\tilde{\mathbb{P}}^2$.

**Proof.** We will show that the preimage $p_1^{-1}(\{z \neq 0\})$ is isomorphic to the blowup of $k^2$ in the origin. This will be sufficient, as blowing up is a local construction. The affine coordinates on $\{z \neq 0\}$ are $(x, y)$ and the preimage is given by the ideal $(xc - yb, xc - za) \subset k[x, y][a, b, c]$, so that the coordinate ring of the preimage is equal to $k[x, y][c, b]/(xc - yb)$. This obviously defines the required blowup. The same is true for the other projection. As the blowup is an isomorphism outside the exceptional locus, we conclude that $p_2 \circ p_1^{-1}$ is well-defined on $\mathbb{P}^2 \setminus X$ and that it is an isomorphism of this open subset of $\mathbb{P}^2$ (i.e., a birational transformation of $\mathbb{P}^2$). Moreover, it is clear that the map $a : b : c := (xy : xz : yz)$ defined on $\mathbb{P}^2 \setminus X$ (i.e., the map $f$) is just the composition $p_2 \circ p_1^{-1}$ (because $Y$ contains the graph of $f$ as an open subset). \[\square\]

2. Exercise: Let $X \subset k^n$ be affine with $J = I(X)$. Suppose that $0 \in X$, and denote by $\overline{I} = ([x_1], \ldots, [x_n]) \subset k[X]$ the ideal of $0 \in X$. Then the coordinate ring $k[C_{X, 0}]$ of the tangent cone of $X$ at $0$ is the algebra $\bigoplus_{k \geq 0} \overline{I}^k/\overline{I}^{k+1}$.

**Solution:** First note the following obvious ring isomorphism:

$$\bigoplus_{k \geq 0} \overline{I}^k/\overline{I}^{k+1} \cong \frac{k[x]}{J} \oplus \frac{I}{J + I} \oplus \frac{I^2 + J}{J^3 + J} \oplus \ldots \oplus \frac{I^k + J}{J^{k+1} + J} \oplus \ldots$$
where this time \( I = (x_1, \ldots, x_n) \subset k[x] \) (note that 0 \( \in X \) implies \( J \subset I \)). Define the following isomorphism of graded \( k \)-algebras (on the left hand side, the grading is the one induced from the usual grading of \( k[x] \) on the right hand side, the grading is by the given decomposition)

\[
\frac{k[x]}{LT(J)} \cong \frac{k[x]}{J + 1} \oplus J + 2 \oplus J + 3 \oplus \cdots \oplus I_{k^1 + 1} \oplus \cdots
\]

1 \( \mapsto \) \((1), 0, \ldots\)

\( x_i \mapsto (0, [x_i], \ldots) \)

Here \( LT(g) \) is the leading term of \( g \) with respect to the partial order given by the degree (i.e., this is not a monomial ordering as \( LT(g) \) is not a monomial). We first show that this is well-defined, i.e., that it sends any \( f^{(i)} \in LT(J) \) to zero: obviously \( f^{(i)} \) has zero image in \((I^k + J)/(I^{k+1} + J)\) for any \( k < i \) or \( k > i \), but also for \( k = i \): the difference \( f - f^{(i)} \) lies in \( I^{k+1} \subset I^{k+1} + J \), so that \( f^{(i)} \equiv f \) in \((I^k + J)/(I^{k+1} + J)\) and thus \([f^{(i)}] = 0\). Moreover, the above map is obviously surjective, and injectivity follows by the same argument: suppose that for any homogenous \( g \in k[x] \), the image is zero in \( \oplus_{k \geq 0} I^k/(I^{k+1} + J) \), then it is zero on each factor, in particular in \( I^k/(I^{k+1} + J) \) with \( k = \deg(g) \), so that \( g \in I^{k+1} + J \) so that there is \( \tilde{g} \in I^{k+1} \) with \( g + \tilde{g} \in J \), this implies \( g \in LT(J) \).

3. **Exercise:** For an affine variety \( X \subset k^n \) (with \( J = I(X) \)) containing the origin, let \( \tilde{X} \) be the blowup of \( X \) in the origin (i.e., in the ideal \( \tilde{I} = (x_1, \ldots, [x_n]) \)). Show that the exceptional divisor \( E \subset \tilde{X} \), seen as an algebraic set in \( \mathbb{P}^{n-1} = (0) \times \mathbb{P}^{n-1} \subset k^n \times \mathbb{P}^{n-1} \) is contained in the projective zero locus \( V_p(LT(J)) \) of the initial ideal \( LT(J) \) (recall that for a homogenous ideal \( J \subset k[y_1, \ldots, y_n] \), we denote by \( V_p(J) \subset \mathbb{P}^{n-1} \) the projective variety given the vanishing of the elements in \( J \) and by \( V_a(J) \subset k^n \) its affine cone. Note further that \( E \) is actually equal to \( V_a(LT(J)) \), but we do not prove this here).

**Solution:** \( \tilde{X} \) is by definition the closure in \( k^n \times \mathbb{P}^{n-1} \) of the graph \( \Gamma \) of \( X \setminus \{0\} \to \mathbb{P}^n \) given by \( y_i = x_i \) where \((x_1, \ldots, x_n), (y_1 : \ldots : y_n) \) are coordinates on \( k^n \times \mathbb{P}^{n-1} \). Let \( J = (f_1, \ldots, f_k) \), with \( f_i = LT(f_i) + \text{Tail}(f_i) \). Then the functions \( \tilde{f}_i := \left(\frac{y_i}{x_i}\right)^{d_i} \cdot f_i \) (with \( d_i = \deg(LT(f_i)) \)) are also zero on \( \Gamma \) and thus on \( \tilde{X} \). But \( \tilde{f}_i \in LT(f_i)(y_1, \ldots, y_n) + (x_1, \ldots, x_n) \), because the relations \( \frac{y_i}{x_i} = \frac{y_i}{x_i} \) allows to rewrite the term \( \left(\frac{y_i}{x_i}\right)^{d_i} \) such that each monomial of \( LT(f_i) \) gets its \( x \)-variables replaced by \( y \)'s. This implies that \( (\tilde{f}_i)_{x_i=0} = LT(f_i)(y_1, \ldots, y_n) \) is zero. This shows that \( I(E) \supseteq LT(J) \subset k[y_1, \ldots, y_n] \). The other direction can actually be shown by the last exercise, it is possible to prove that the coordinate ring of \( \tilde{X} \) is the \( k[X] \)-algebra \( k[X] / I k[X] \cong \oplus_{k \geq 0} I^k / I^{k+1} \).