Exercises Algebraic Geometry

Sheet 3

1. Check whether some of the following complex algebraic sets are isomorphic.
   (a) $V(y) \subset \mathbb{C}^2$
   (b) $V(xy) \subset \mathbb{C}^2$
   (c) $V(xy - 1) \subset \mathbb{C}^2$
   (d) $V(xy(x - y)) \subset \mathbb{C}^2$
   (e) $V(x^2 + y^2) \subset \mathbb{C}^2$
   (f) $V(y^3 - x) \subset \mathbb{C}^2$
   (g) $V(xy, yz, xz) \subset \mathbb{C}^3$
   (h) $V(y - x^2, z - x^3) \subset \mathbb{C}^3$

2. Consider a regular map $\varphi : k^n \to k^m$. Are the following statements true or false? Give a short proof (or counterexample).
   (a) For any algebraic set $X \subset k^n$, the image $f(X)$ is algebraic in $k^m$.
   (b) For any algebraic set $Y \subset k^m$, the inverse image $f^{-1}(Y)$ is algebraic in $k^n$.
   (c) For any algebraic set $X \subset k^n$, the graph $\Gamma_{X, f} := \{(x, \varphi(x)) \mid x \in X\}$ is algebraic in $k^{n+m}$.

3. (a) Show that the polynomial ring $k[x_1, \ldots, x_n]$ can be equipped with a different grading by fixing $\alpha := (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and putting
   $$k[x_1, \ldots, x_n]_d := \bigoplus_{i_1, \ldots, i_n} k x_1^{i_1} \cdots x_n^{i_n}$$
   with $\sum_{i, j} j a_j = d$.
   Is the usual grading a particular case of this (i.e., for some specific $\alpha \in \mathbb{Z}^n$)? We call a polynomial in $k[x]_d$ (for fixed $\alpha \in \mathbb{Z}$) quasi-homogenous (or weighted homogenous) of degree $d$.
   (b) Let $\text{char}(k) = 0$. Show that $f$ is quasi-homogenous of degree $d$ if and only if $\sum_{i=1}^n a_i x_i \partial x_i f = d \cdot f$.

4. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring. Show the equivalence of the following statements.
   (a) $R$ is noetherian.
   (b) $R_i$ is noetherian and $R_+ = \bigoplus_{i \geq 0} R_i$ is a finitely generated ideal in $R$.
   (c) $R_0$ is noetherian and $R$ is a finitely generated $R_0$-algebra.

5. Show that the projective morphism (this means: a regular map between projective varieties) given by
   $$\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$$
   $$(s : t) \mapsto (s^2 : st : t^2)$$
   is an isomorphism (i.e., a biregular map) between $\mathbb{P}^1$ and its image (in particular, show that the image is a projective subvariety of $\mathbb{P}^2$).