The Solvability Complexity Index of selected problems

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Abstract

We discuss upper and lower bounds on the Solvability Complexity Indices for several classes of problems, among them the computation of the spectrum of bounded linear operators as well as for more particular classes such as bounded self-adjoint operators.

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Contents

1 Introduction **2**

2 The SCI of spectral problems of bounded linear operators **3**
- 2.1 On lower bounds . . . 3
- 2.2 On upper bounds . . . 5
- 2.3 Controllable off-diagonal decay . . . 5
- 2.4 The self-adjoint case . . . 7
- 2.5 On the essential spectrum and discrete Schrödinger operators . . . 10

3 Decision problems **10**
- 3.1 The model . . . 10
- 3.2 A family of simple prototypic problems and their SCI . . . 11

4 Further popular problems **13**
- 4.1 Computing the norm of the inverse . . . 13
- 4.2 Solving linear systems . . . 14
- 4.3 Convergence of real sequences . . . 15
- 4.4 Deciding whether $0 \in \mathrm{sp}(A)$ . . . 16

1 Introduction

Given an infinite matrix A which defines a bounded linear operator on l^2 we are interested in computing its spectrum from the matrix entries. More precisely, we ask if there is a family of algorithms $\Gamma_{n_1,\ldots,n_i}$ $(n_1,\ldots,n_i \in \mathbb{N})$ which on the one hand provide approximations to $\mathrm{sp}(A)$, i.e. which generate compact sets $\Gamma_{n_1,\ldots,n_i}(A) \subset \mathbb{C}$ such that

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_i\to\infty} d_H(\Gamma_{n_1,\ldots,n_i}(A), \mathrm{sp}(A)) = 0, \tag{1}$$

where d_H denotes the Hausdorff distance. On the other hand, each of these algorithms shall be implementable on a computer, i.e. only uses finitely many arithmetic operations and radicals of a finite number of matrix entries of A.

Since there is an affirmative answer to that, the next question at hand asks: what is the *smallest* number i of required limits in (1) for certain classes of operators. In a sense this number then reflects the computational complexity of the problem. It was introduced by A. Hansen some years ago, and it is usually called the Solvability Complexity Index (see [5, 6]).

More generally, it could be introduced in the following abstract way: Let Ω be a set, (M,d) be a metric space and $\Sigma : \Omega \to M$ be a function. The Solvability Complexity Index $\mathrm{SCI}(\Sigma)$ of Σ is the smallest integer i for which a family $\{\Gamma_{n_1,\ldots,n_i} : n_1,\ldots,n_i \in \mathbb{N}\}$ of functions (algorithms) $\Gamma_{n_1,\ldots,n_i} : \Omega \to M$ exist with

- $\lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_i\to\infty} d(\Gamma_{n_1,\ldots,n_i}(\omega), \Sigma(\omega)) = 0$ for every $\omega \in \Omega$
- $\Gamma_{n_1,\ldots,n_i}$ are "finite" algorithms.

(Of course this is not a precise definition as long as it is not determined what "finite" means.)

For the above mentioned spectral problem we particularly have $\Omega = \mathcal{L}(l^2)$, M being the set of all compact subsets of the complex plane, equipped with the Hausdorff metric, and $\Sigma = \mathrm{sp}$ the spectral mapping. Here the algorithms Γ shall only take into account a finite number of evaluations $\langle Ae_i, e_j\rangle$ (where $\{e_i\}$ stands for the canonical basis in l^2), i.e. finitely many matrix entries of A, and perform a finite number of arithmetic operations and radicals.

In this note we give upper and lower bounds on $\mathrm{SCI}(\mathrm{sp})$ for the above spectral problem of general bounded linear operators as well as for several more particular classes of operators (i.e. subsets $\Omega' \subset \Omega$), such as self-adjoint ones or those which have a controllable off-diagonal decay. This is done in Section 2, where we also pick up analogous questions for the essential spectrum. In Section 3 We study a family of fairly simple but fundamental decision problems which shall give a deeper understanding of the concept of SCIs and reveal the open problems in that field. Section 4 is devoted to the Complexity Indices of solving linear systems, of computing the norm of the inverse of a bounded linear operator as well as of some problems around the convergence of sequences. Finally we return to the spectrum and ask how difficult (or complex) it is to determine whether $\lambda \in \mathrm{sp}(A)$ for given A and $\lambda \in \mathbb{C}$.

2 The SCI of spectral problems of bounded linear operators

2.1 On lower bounds

Proposition 1. *The spectrum of bounded linear operators cannot be computed in one limit.*

Proof. Consider the Hilbert space $l^2(\mathbb{N})$. For $n \in \mathbb{N}$ let

$$B_n := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

Then $\mathrm{sp}(B_n) = \{0, 2\}$ and, for every sequence $(l_n) \subset \mathbb{N}$, the operator A with matrix representation (w.r.t. the standard basis) $A := \mathrm{diag}\{B_{l_1}, B_{l_2}, B_{l_3}, \ldots\}$ is bounded, self-adjoint, and $\mathrm{sp}(A) = \{0, 2\}$.

Let $\{\Gamma_k\}$ be some spectrum-approximating algorithm that on the kth step Γ_k only reads information contained in the first $N(k) \times N(k)$ entries of A.

In order to find a counterexample we simply construct an appropriate sequence $(l_n) \subset \mathbb{N}$ by induction: For $C := \mathrm{diag}\{1, 0, 0, 0, \ldots\}$ one obviously has $\mathrm{sp}(C) = \{0, 1\}$. Choose $k_0 := 1$ and $l_1 > N(k_0) = N(1)$.

Assume that $l_1, \ldots, l_n$ are already chosen. Then the operator given by the matrix $\mathrm{diag}\{B_{l_1}, \ldots, B_{l_n}, C\}$ has $\{0, 1, 2\}$ as its spectrum, hence there exists a k_n such that $\Gamma_k(\mathrm{diag}\{B_{l_1}, \ldots, B_{l_n}, C\}) \cap B_{\frac{1}{n}}(1) \neq \emptyset$ for every $k \geq k_n$. Now, choose $l_{n+1} > N(k_n) - l_1 - l_2 - \ldots - l_n$.

Since

$$\Gamma_{k_n}(A) \cap B_{\frac{1}{n}}(1) = \Gamma_{k_n}(\mathrm{diag}\{B_{l_1}, \ldots, B_{l_n}, C\}) \cap B_{\frac{1}{n}}(1) \quad \forall\, n \in \mathbb{N}$$

we see that 1 is contained in the partial limiting set of the sequence $(\Gamma_k(A))_{k=1}^{\infty}$ which approximates $\mathrm{sp}(A) = \{0, 2\}$, a contradiction. ☐

Here is another proof:

Proof. Consider the Hilbert space $l^2(\mathbb{Z})$.[1] Let $a = (a_i) \in l^\infty(\mathbb{Z})$ be of the form

$$a = (a_i) = \left(\ldots, 1, 1, 1, \ldots, 1, \frac{1}{2}, 1, \ldots, 1, \frac{1}{3}, 1, \ldots, 1, \frac{1}{4}, 1, \ldots\right)$$

with $\frac{1}{j}$ at the l_jth position, and let $aI : (x_i) \mapsto (a_i x_i)$ denote the respective operator of multiplication by a. Further let V denote the shift operator defined by $(x_i) \mapsto (x_{i+1})$, and set $A := aV$. In analogy to the above, given a family $\{\Gamma_k\}$, we construct an appropriate sequence (l_j) of positions in order to arrive at a counterexample.

Set $C_1 := V$, for which $\mathrm{sp}(C_1) = \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ holds, and choose $k_0 := 1$ and $l_1 > N(k_0) = N(1)$.

[1] Of course, this construction could be transformed into the isomorphic $l^2(\mathbb{N})$ case, but this would damage its beauty and simplicity. Therefore we perform this proof over $l^2(\mathbb{Z})$.

Assume that the positions $l_1, \ldots, l_n$ are already chosen. By $a^s = (a_i^s)$ we denote the sequence having the entries $\frac{1}{j}$ at the positions l_j, $j = 1, \ldots, s$, and 1s elsewhere. We will show that the operator $C_n := a^n V$ has the spectrum $\mathrm{sp}(C_n) = \mathbb{T}$, hence there exists a k_n such that $\Gamma_k(C_n) \cap B_{\frac{1}{2}}(0) = \emptyset$ whenever $k \geq k_n$. Now choose $l_{n+1} > \max\{l_n, N(k_n)\}$. By this, A is determined.

Since $\Gamma_{k_n}(A) = \Gamma_{k_n}(C_n)$ for every n we see that $\lim_n \Gamma_{k_n}(A)$ cannot contain the origin. On the other hand, $0 \in \mathrm{sp}(A)$ easily follows from the observation that $\|Ae_{l_n+1}\| = \frac{1}{n} \to 0$ as $n \to \infty$, where e_k stands for the kth canonical basis element. Thus we arrive at a contradiction.

So, it remains to prove $\mathrm{sp}(C_n) = \mathbb{T}$. For this we note that

$$\sum_{j=0}^{\infty} z^j ((a^n V)^{-1})^{j+1} = (I - z(a^n V)^{-1})^{-1}(a^n V)^{-1} = (a^n V - zI)^{-1}$$

converges for every $|z| < 1$ since the $\|((a^n V)^{-1})^{j+1}\|$ are uniformly bounded by n! □

Remark 2. Notice that the first proof even shows the assertion for self-adjoint, block-diagonal operators, whereas the second one works with banded operators.

We also point out that these proofs are based on completely different phenomena that can occur when looking at spectra of perturbed operators, although the constructions are very similar: The first one makes use of what is known as **spectral pollution**, the fact that spectral sets of restricted operators can have additional components in comparison to the unresticted one. The second proof applies the fact that parts of the spectrum can disappear under (arbitrarily small) perturbations, that is the **discontinuity** of the spectrum.

A similar outcome can be observed for pseudospectra:

Definition 3. For $N \in \mathbb{Z}_+$ and $\epsilon > 0$ the (N, ϵ)-pseudospectrum of a bounded linear operator $A \in \mathcal{L}(l^2)$ is defined as the set

$$\mathrm{sp}_{\mathrm{N},\epsilon}(A) := \{z \in \mathbb{C} : \|(A - zI)^{-2^N}\|^{2^{-N}} \geq 1/\epsilon\}.$$[2]

For $N = 0$ this is the (classical) ϵ-pseudospectrum

$$\mathrm{sp}_\epsilon(A) := \{z \in \mathbb{C} : \|(A - zI)^{-1}\| \geq 1/\epsilon\}.$$

For more information on pseudospectra we refer to [11, 1, 2, 6]. Also recall that these sets $\mathrm{sp}_{\mathrm{N},\epsilon}(A)$ are continuous w.r.t. the parameter $\epsilon > 0$, and converge to $\mathrm{sp}(A)$ as $\epsilon \to 0$ for every A.

Proposition 4. *The pseudospectra of bounded self-adjoint operators cannot be computed in one limit.*

Proof. The (N, ϵ)-pseudospectrum of the operators A from the first of the above proofs is a neighborhood of $\{0, 2\}$, for ϵ sufficiently small its intersection with $B_{\frac{1}{2}}(1)$ is empty, independently from the choice of (l_n).

Assume that there were a family of algorithm $\{\Gamma_k\}$. By exactly the same procedure as in that proof one obtains again that 1 belongs to the partial limiting set of $\{\Gamma_k(A)\}$, a contradiction. □

[2]Here we use the convention $\|B^{-1}\| = \infty$ if B is not invertible.

2.2 On upper bounds

First, we recall the groundbreaking observations of Hansen [6]:

Theorem 5. *The Solvability Complexity Index* SCI(sp) *for the spectrum of bounded linear operators on l^2 is not greater than three, and the Solvability Complexity Index* $\mathrm{SCI}(\mathrm{sp}_{\mathrm{N},\epsilon})$ *for the pseudospectrum is not greater than two.*

The basic steps in the proof and, actually, the construction of a (two-limit-) three-limit-algorithm for the (pseudo-) spectrum are as follows: Let (P_n) be the sequence of the canonical projections on $l^2(\mathbb{N})$ which send the sequence (x_i) to its finite section $P_n(x_i) := (x_1, \ldots, x_n, 0, \ldots)$, respectively. For $A \in \mathcal{L}(l^2)$ and $z \in \mathbb{C}$ one introduces the following continuous functions

$$\gamma^N(z) := \Big(\min\{\sigma_1((A - zI)^{2^N}), \sigma_1((A^* - \bar{z}I)^{2^N})\}\Big)^{2^{-N}}$$
$$\gamma^N_m(z) := \Big(\min\{\sigma_1((A - zI)^{2^N} P_m), \sigma_1((A^* - \bar{z}I)^{2^N} P_m)\}\Big)^{2^{-N}}$$
$$\gamma^N_{m,n}(z) := \Big(\min\{\sigma_1((P_n(A - zI)P_n)^{2^N} P_m), \sigma_1((P_n(A^* - \bar{z}I)P_n)^{2^N} P_m)\}\Big)^{2^{-N}}$$

where $\sigma_1(B)$ denotes the smallest singular value of B, and in the terms like $\sigma_1(P_nBP_m)$ the operator P_nBP_m is regarded as element of $\mathcal{L}(\operatorname{im} P_m, \operatorname{im} P_n)$. Then one checks that $\gamma^N_m(z) \downarrow_m \gamma^N(z)$ for every $z \in \mathbb{C}$, and $\gamma^N_{m,n}(z) \uparrow_n \gamma^N_m(z)$ for every $z \in \mathbb{C}$ and every m.[3] For the level sets

$$\mathrm{sp}_{\mathrm{N},\epsilon}(A) = \{z \in \mathbb{C} : \gamma^N(z) \le \epsilon\}$$
$$\Delta^N_{\epsilon,m,n}(A) := \{z \in \mathbb{C} : \gamma^N_{m,n}(z) \le \epsilon\}$$

one derives the convergence

$$\lim_{m\to\infty} \lim_{n\to\infty} d_H(\mathrm{sp}_{\mathrm{N},\epsilon}(A), \Delta^N_{\epsilon,m,n}(A)) = 0,$$

hence

$$\lim_{\epsilon\to 0} \lim_{m\to\infty} \lim_{n\to\infty} d_H(\mathrm{sp}(A), \Delta^N_{\epsilon,m,n}(A)) = 0.$$

Finally, one introduces the set $\Theta_m := \{s/m + \mathbf{i}t/m : s, t = -m^2, \ldots, m^2\}$. Then

$$\Gamma^N_{\epsilon,m,n}(A) := \Delta^N_{\epsilon,m,n}(A) \cap \Theta_m = \{z \in \Theta_m : \gamma^N_{m,n}(z) \le \epsilon\}$$

can be computed with finitely many arithmetic operations and radicals of finitely many entries of A, using Choleskys decomposition, and it still holds

$$\lim_{m\to\infty} \lim_{n\to\infty} d_H(\mathrm{sp}_{\mathrm{N},\epsilon}(A), \Gamma^N_{\epsilon,m,n}(A)) = 0.$$

2.3 Controllable off-diagonal decay

Definition 6. We say that the dispersion of the bounded linear operator A is bounded by the function $f : \mathbb{N} \to \mathbb{N}$ if

$$D_{f,m}(A) := \max\{\|(I - P_{f(m)})AP_m\|, \|P_mA(I - P_{f(m)})\|\} \to 0 \quad \text{as } m \to \infty.$$

[3] Here $\uparrow_n$ denotes monotone convergence as n tends to infinity.

Remark 7. Note that for every operator A there is always a function f which is a bound for its dispersion since AP_m, P_mA are compact and P_n converges strongly to the identity. But there is no function f which acts as a uniform bound for all operators. Nevertheless, there are important (sub)classes of operators having well known uniform bounds, which should be mentioned:

- band operators with bandwidth less than d: $f(k) = k + d$.
- band-dominated and weakly band-dominated operators: $f(k) = 2k$. For definitions and properties of band and band-dominated operators see [10, 8]. Weakly band-dominated operators can be found in [9].
- Laurent/Toeplitz operators with piecewise continuous generating function: $f(k) = k^2$ (cf. [3] and [7, Proposition 5.4]).
- Let $\mathcal{M}$ be a family of bounded operators with a common bound f. Then g, given by $g(k) = f(k) + k$, is a common bound for all operators in the Banach algebra which is generated by $\mathcal{M}$.

Proposition 8. *Let $f : \mathbb{N} \to \mathbb{N}$. The pseudospectra of operators whose dispersion is bounded by f can be computed in one limit.*

Proof. We give the proof only for the case $N = 0$ and omit the super- and subscripts N. For $N > 0$ one proceeds analogously.

Let A be such that f is a bound for its dispersion, and $\epsilon > 0$. In order to simplify the notation we choose a sequence (δ_m) which converges antitonically to zero such that $\delta_m \geq D_{f,m}(A)$ for every m. Define functions $\alpha_m : \mathbb{C} \to \mathbb{C}$

$$\alpha_m(z) := \min\{\sigma_1(P_{f(m)}(A - zI)P_m), \sigma_1(P_{f(m)}(A^* - \bar{z}I)P_m)\}.$$

Since $|\sigma_1(B + C) - \sigma_1(B)| \leq \|C\|$ holds for arbitrary bounded operators B, C, we further conclude that

$$\gamma_m(z) \geq \alpha_m(z) \geq \gamma_m(z) - \delta_m \geq \gamma_m(z) - \delta_k$$

for every $z \in \mathbb{C}$ and $m \geq k$, and moreover, for all $y, z \in \mathbb{C}$ and $m \in \mathbb{N}$,

$$|\alpha_m(y) - \alpha_m(z)| \leq |y - z|. \tag{2}$$

With $\Delta_{\epsilon,m}(A) := \{z \in \mathbb{C} : \gamma_m(z) \leq \epsilon\}$ and $\Psi_{\epsilon,m}(A) := \{z \in \mathbb{C} : \alpha_m(z) \leq \epsilon\}$ we conclude for all $m \geq k$ that

$$\Delta_{\epsilon+\delta_k,m}(A) \supset \Delta_{\epsilon+\delta_m,m}(A) \supset \Psi_{\epsilon,m}(A) \supset \Delta_{\epsilon,m}(A),$$

where the sets on the left and on the right are known to converge to $\mathrm{sp}_{\epsilon+\delta_k}(A)$ and $\mathrm{sp}_\epsilon(A)$, resp., as $m \to \infty$. Since $\mathrm{sp}_{\epsilon+\delta_k}(A) \to \mathrm{sp}_\epsilon(A)$ as $k \to \infty$, this yields $\lim_{m\to\infty} \Psi_{\epsilon,m}(A) = \mathrm{sp}_\epsilon(A)$.

Finally, we introduce the desired approximations $\Gamma_m(A)$ for $\mathrm{sp}_\epsilon(A)$. Recall that by the Cholesky decomposition, for given $z \in \mathbb{C}$ and $m \in \mathbb{N}$, only finitely many arithmetic operations and radicals of entries of A, i.e. of elements in $\{\langle Ae_i, e_j\rangle : i, j = 1, \ldots, \max\{m, f(m)\}\}$, are required in order to determine whether $\alpha_m(z) \leq \epsilon$ or not. Set

$$\Gamma_m(A) := \{z \in \Theta_m : \alpha_m(z) \leq \epsilon\}.$$

Then it is clear that on the one hand $\Psi_{\epsilon,m}(A) \supset \Gamma_m(A)$. On the other hand, for sufficiently large m it holds true that for every point $x \in \Psi_{\epsilon-1/m,m}(A)$ there is a point $y_x \in \Theta_m \cap U_{1/m}(x)$ [4] and from (2) we get $|\alpha_m(y_x) - \alpha_m(x)| < 1/m$ that is y_x even belongs to $\Gamma_m(A)$. Thus $\Gamma_m(A) + B_{1/m}(0) \supset \Psi_{\epsilon-1/m,m}(A)$ for sufficiently large m. Combining this, we arrive at

$$\Psi_{\epsilon,m}(A) + B_{1/m}(0) \supset \Gamma_m(A) + B_{1/m}(0) \supset \Psi_{\epsilon-1/m,m}(A) \supset \Psi_{\epsilon-1/k,m}(A),$$

for $m \geq k$ large. By the above, the sets on the left tend to $\mathrm{sp}_\epsilon(A)$ as $m \to \infty$, and the sets on the right converge to $\mathrm{sp}_{\epsilon-1/k}(A)$ for every k. Since the latter converge to $\mathrm{sp}_\epsilon(A)$ as $k \to \infty$ this provides $\lim_{m\to\infty} \Gamma_m(A) = \mathrm{sp}_\epsilon(A)$. □

Corollary 9. *Let $f : \mathbb{N} \to \mathbb{N}$. The spectrum of operators whose dispersion is bounded by f can be computed in two limits.*

Proof. Take the above one-limit-algorithm for the pseudospectrum and the additional limit $\epsilon \to 0$. This is the best possible outcome for every family of operators which includes all band operators, as the 2nd proof of Proposition 1 reveals. □

2.4 The self-adjoint case

Let A be bounded and self-adjoint. Then the spectrum of A is a compact subset of $\mathbb{R}$, hence its complement in $\mathbb{R}$ is a finite or countable union of disjoint open intervals U_i. One of them, let's say U_0, is of the form $(-\infty, a)$, another one, let's say U_1, is (b, ∞). All other U_i are bounded. Moreover, $\gamma(x) = \sigma_1(A - xI) = \|(A - xI)^{-1}\|^{-1}$ coincides with $\mathrm{dist}(x, \mathrm{sp}(A))$ in every point $x \in \mathbb{R}$ and is a continuous piecewise linear function which is zero on the spectrum of A, equals $\gamma(x) = -x + a$ on U_0, $\gamma(x) = x - b$ on U_1, and is a "hat function" on each of the other intervals U_i: zero at the end points, $l_i/2$ at the center point of U_i, where l_i denotes the length of U_i and (affine) linear on both subintervals laying left/right of the center.

Let $K \subset \mathbb{R}$ be a compact set and $\delta > 0$. We introduce a δ-grid for K by $G^\delta(K) := (K + B_\delta(0)) \cap (\delta\mathbb{Z})$. Obviously, this set is finite.

For a given function $f : \mathbb{R} \to [0, \infty)$ we define sets $\Delta_K^\delta(f)$ as follows: For $x \in G^\delta(K)$ consider the points $x_1 := x - f(x)/2$, $x_2 := x + f(x)/2$ and

- If $f(x) \geq 1$ let $M_x := \emptyset$, otherwise
- If $f(x_1) \leq f(x_2)$ then $M_x := \{x - f(x)\}$, otherwise
- If $f(x_2) \leq f(x_1)$ then $M_x := \{x + f(x)\}$.

Now define $\Delta_K^\delta(f) := \bigcup_{x \in G^\delta(K)} M_x$. Notice that for the computation of $\Delta_K^\delta(f)$ only finitely many evaluations of f are required.

Proposition 10. *Let K be a compact set containing the spectrum of A and $0 < \delta < \epsilon < 1/2$. Further assume that f is a function with $\|f - \gamma\|_{\infty,\hat{K}} < \epsilon$ on $\hat{K} := (K + B_{2\epsilon+\mathrm{diam}(K)}(0))$. Then $d_H(\Delta_K^\delta(f), \mathrm{sp}(A)) < 7\epsilon$.*

[4] Here $U_r(x)$ and $B_r(x)$ denote the open/closed ball with radius r centered at x

Proof. We firstly show that every point of $\Delta_K^\delta(f)$ is closer to $\mathrm{sp}(A)$ than 7ϵ.

Let $x \in G^\delta(K)$, and notice that $x_1, x_2 \in \hat{K}$. First, assume that $\gamma(x) \leq 3\epsilon$. Then $f(x) < 4\epsilon$, hence M_x is empty or contains one of the points $x \pm f(x)$ which are closer to $\mathrm{sp}(A)$ than 7ϵ, since $\mathrm{dist}(x, \mathrm{sp}(A)) \leq 3\epsilon$ and $\mathrm{dist}(x \pm f(x), x) < 4\epsilon$.

Now assume that $\gamma(x) > 3\epsilon$, hence $f(x) > 2\epsilon$. If $x \in U_0$ then it follows that $f(x_1) > f(x_2)$ and the resulting set M_x is empty, or contains $x + f(x)$ which is closer to $\mathrm{sp}(A)$ than ϵ. The same arguments work in the case $x \in U_1$. So, assume that x belongs to a bounded U_i and let y denote the center of U_i. If the distance between x and y is less than ϵ then both $x - f(x)$ and $x + f(x)$ are closer to $\mathrm{sp}(A)$ than 3ϵ. Otherwise, $x \leq y - \epsilon$ ($x \geq y + \epsilon$) yields $\gamma(x_1) \leq \gamma(x_2) - 2\epsilon$ ($\gamma(x_1) \geq \gamma(x_2) + 2\epsilon$), thus $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$, resp.), and the resulting M_x is empty or closer to $\mathrm{sp}(A)$ than ϵ.

Secondly, we show that for every point in $\mathrm{sp}(A)$ the distance to $\Delta_K^\delta(f)$ is smaller than 3ϵ. Let $x \in \mathrm{sp}(A) \subset K$. Then there is a point $y \in G^\delta(K)$ with $|x - y| < \delta < \epsilon$, hence $f(y) < \gamma(y) + \epsilon < 2\epsilon < 1$. Thus, M_y is not empty and contains a point which is closer to x than 3ϵ. □

Our next goal is the definition of suitable approximating functions for γ. Here we have to take into account the following aspects:

- The functions shall approximate γ locally uniformly.
- There shall be a compact set which contains $\mathrm{sp}(A)$ such that all of these functions are greater than 1 outside that set.
- The evaluation of the functions shall require only finitely many arithmetic operations and radicals.

The functions $\gamma_m(x) := \sigma_1((A - xI)P_m)$ are known to converge antitonically to γ as $m \to \infty$, and with Dini's Theorem the convergence is uniform on every compact set, in particular on the ball/interval $B_{2\|A\|+4}(0) \subset \mathbb{R}$.

The functions $\gamma_{m,n}(x) := \sigma_1(P_n(A - xI)P_m)$ are known to converge isotonically to γ_m as $n \to \infty$ for every m, hence again uniformly on the interval $B_{2\|A\|+4}(0)$.

Outside that interval we have, for $n > m$, by a Neumann argument

$$\begin{aligned}\gamma_{m,n}(x) &= \sigma_1(P_n(A - xI)P_nP_m) \geq \sigma_1(P_n(A - xI)P_n) \\ &= \|(P_n(A - xI)P_n)^{-1}\|^{-1} = |x|\|(P_n - x^{-1}P_nAP_n)^{-1}\|^{-1} \\ &\geq 2.\end{aligned}$$

Finally, applying the Cholesky decomposition, it is possible to compute the closest value $f_{m,n}(x) \geq \gamma_{m,n}(x)$ from the set $\{k/m : k \in \mathbb{N}\}$ using only a finite number of operations, as desired: For k/m apply the Cholesky decomposition to the matrix $P_m(A - xI)P_n(A - xI)P_m - (k/m)^2P_m$ in order to test whether k/m is smaller than $\sigma_1(P_n(A - xI)P_m)$ or not. Take $f_{m,n}(x)$ as the smallest of the values k/m for which the test fails.

The algorithm Let A be a bounded self-adjoint operator. The (m,n)th step $\Gamma_{m,n}(A)$ of the algorithm is now as follows: For given m, n consider the compact set $K_m := B_m(0)$, the (finite) grid $G^{1/m}(K_m)$, the (finitely computable) function $f_{m,n}$ on the grid, and compute the set $\Gamma_{m,n}(A) := \Delta_{B_m(0)}^{1/m}(f_{m,n})$.

Theorem 11. *The spectrum of bounded self-adjoint operators can be computed in two limits.*

Proof. Let $m_0 > 2\|A\| + 4$. For all $n > m \geq m_0$, the points in the finite set $G^{1/m}(K_m) \setminus B_{m_0}(0)$ lead to function values of $f_{m,n}$ being larger than 1, hence $\Gamma_{m,n}(A) = \Delta^{1/m}_{B_{m_0}(0)}(f_{m,n})$. Fix $\epsilon \in (0, 1/2)$. Then there is an $m_1 > m_0$ with $m_1 > 3/\epsilon$ such that $\|\gamma - \gamma_m\|_{\infty,\hat{K}} < \epsilon/3$ on $\hat{K} := B_{1+3m_0}(0)$ for all $m > m_1$. Moreover, for every m there is an $n_1(m)$ such that $\|\gamma_m - \gamma_{m,n}\|_{\infty,\hat{K}} < \epsilon/3$ for all $n > n_1(m)$. This yields

$$\begin{aligned}\|\gamma - f_{m,n}\|_{\infty,\hat{K}} &\leq \|\gamma - \gamma_m\|_{\infty,\hat{K}} + \|\gamma_m - \gamma_{m,n}\|_{\infty,\hat{K}} + \|\gamma_{m,n} - f_{m,n}\|_{\infty,\hat{K}} \\ &\leq \epsilon/3 + \epsilon/3 + 1/m < \epsilon\end{aligned}$$

whenever $m > m_1$ and $n > n_1(m)$. Hence, by Proposition 10 with $K := B_{m_0}(0)$, it holds that $d_h(\Gamma_{m,n}(A), \mathrm{sp}(A)) < 7\epsilon$ whenever $m > m_1$ and $n > n_1(m)$. Thus, it is proved that

$$\lim_{m\to\infty} \limsup_{n\to\infty} d_H(\Gamma_{m,n}(A), \mathrm{sp}(A)) = 0.$$

To ensure that $(\Gamma_{m,n}(A))_{n\in\mathbb{N}}$ already converges w.r.t. the Hausdorff distance for every fixed m we just mention that the sequence of functions $(\gamma_{m,n})_n$ is isotonic, hence $(f_{m,n})_n$ is isotonic as well. Moreover these $f_{m,n}$ are effectively evaluated only in finitely many points and can take only finitely many values, where the bounds do not depend on n. Thus the sets $\Gamma_{m,n}(A)$ change only finitely many times as n grows. Consequently, there is an $n_2(m)$ such that all $\Gamma_{m,n}(A)$ with $n \geq n_2(m)$ coincide. □

Theorem 12. *Let $f : \mathbb{N} \to \mathbb{N}$. The spectrum of bounded self-adjoint operators whose dispersion is bounded by f can be computed in one limit, using the above algorithm with $n = f(m)$.*

Proof. Take the idea from the previous theorem with $n = f(m)$. Then there is an $m_2 > m_1$ such that $\|\gamma_m - \gamma_{m,f(m)}\|_{\infty,\hat{K}} < \epsilon/3$ for all $m > m_2$, hence

$$\begin{aligned}&\|\gamma - f_{m,f(m)}\|_{\infty,\hat{K}} \\ &\quad\leq \|\gamma - \gamma_m\|_{\infty,\hat{K}} + \|\gamma_m - \gamma_{m,f(m)}\|_{\infty,\hat{K}} + \|\gamma_{m,f(m)} - f_{m,f(m)}\|_{\infty,\hat{K}} < \epsilon\end{aligned}$$

and the assertion follows analogously. □

Summarizing the above observations we get the following picture:

$\mathrm{SCI}(\mathrm{sp}_\epsilon)$	bound known	bound unknown
self-adjoint	1	2
general	1	2

$\mathrm{SCI}(\mathrm{sp})$	bound known	bound unknown
self-adjoint	1	2
general	2	2 or 3

2.5 On the essential spectrum and discrete Schrödinger operators

Discrete Schrödinger operators with potential $V = (V_i) = (V(i))$ in l^∞

$$(Ax)_i = x_{i+1} + x_{i-1} + V_i x_i, \quad x = (x_i),$$

are band operators hence, by the previous sections, the pseudospectra can be computed in one limit, the spectrum can be computed in two limits. In case of a real potential the operator is self-adjoint and one can compute the spectrum in one limit. Surprisingly, this is not true for the essential spectrum:

Proposition 13. *The essential spectrum of self-adjoint discrete Schrödinger operators cannot be computed in one limit.*

Proof. From [4] and [10] it is known that the essential spectrum of a discrete Schrödinger operator on $l^2(\mathbb{Z})$ with potential V is $[-2, 2]$ if and only if $V_i \to 0$ as $|i| \to \infty$.

Moreover, if we consider now a potential V, with $V(m_n) = 1$ for a certain strictly increasing sequence $(m_n) \subset \mathbb{N}$ and $V(m) = 0$ otherwise, then the essential spectrum of the respective Schrödinger operator A is not contained in $[-2, 2]$.

Assume that we have a family of algorithms $\{\Gamma_k\}$ which provides the approximation of the essential spectrum of such operators A in one limit, again only reading information contained in the first $N(k) \times N(k)$ entries of A in the kth step, resp. Set $k_1 := 1$ and $m_1 := N(1)+1$. We construct a counterexample as usual: Given the numbers $m_1, \ldots, m_n$, we choose m_{n+1} as follows: Introduce the operator A_n, which is the Schrödinger operator with the potential having 1s at the positions $m_1, \ldots, m_n$ and 0s elsewhere. Then $\mathrm{sp}_{\mathrm{ess}}\, A_n = [-2, 2]$, hence there is a k_{n+1} such that $\Gamma_k(A_n) \subset [-2 - 1/n, 2 + 1/n]$ for every $k \geq k_{n+1}$. Set $m_{n+1} := N(k_{n+1}) + 1$.

This yields $\lim_n \Gamma_{k_n}(A) = \lim_n \Gamma_{k_n}(A_n) \subset [-2, 2]$, a contradiction. □

Remark 14. This particularly shows that for the larger classes of Jacobi operators, band operators, band-dominated operators, self-adjoint operators, and of course bounded linear operators in general, there is no one-limit-algorithm for the essential spectrum.

3 Decision problems

3.1 The model

Within this section we exclusively deal with problems (functions)

$$\Sigma : \Omega \to M := \{\textit{Yes}, \textit{No}\},$$

where M is equipped with the discrete metric. This means that for such problems we search for algorithms $\Gamma_{n_1,\ldots,n_k} : \Omega \to M$ which, for a given input $\omega \in \Omega$, answer *Yes* or *No*. Clearly, a sequence $(m_i) \subset M$ of such "answers" converges to $m \in M$ if and only if finitely many m_i are different from m.

3.2 A family of simple prototypic problems and their SCI

Sequences Let Ω denote the collection of all sequences $(a_i)_{i\in\mathbb{N}}$ with entries $a_i \in \{0,1\}$. For $(a_i) \in \Omega$ we want to answer the question

Q1 Does (a_i) have a non-zero entry?

That is, we search for algorithms $\Gamma : \Omega \to M := \{\mathit{Yes}, \mathit{No}\}$ that are allowed to look at a finite number of entries of (a_i) and give the output *Yes* or *No*. Of course, there is no such algorithm which can do that for all such sequences. But, defining a family of algorithms Γ_n by

$$\Gamma_n(a_i) = \mathit{Yes} \quad \Leftrightarrow \quad \sum_{i=1}^{n} a_i > 0,$$

the limit $\lim_{n\to\infty} \Gamma_n(a_i)$ will give the correct answer for arbitrary (a_i). Thus

$$\mathrm{SCI}(\mathbf{Q1}) = 1.$$

Q2 Does (a_i) have infinitely many non-zero entries?

Assume that there is a one-limit-algorithm which can answer this question. Then there exist strictly increasing sequences $(n_k), (i_k) \subset \mathbb{N}$ such that the algorithms Γ_{n_k} only consider entries of (a_i) with index $i < i_k$, resp., and for the sequence (a_i) which has 1s exactly at the positions i_k the algorithms answers are $\Gamma_{n_k}(a_i) = \mathit{No}$. This is again proved by induction: Set $(a_i^1) := (0)$. Then there is an n_1 such that $\Gamma_n(a_i^1) = \mathit{No}$ for all $n \geq n_1$. Further, Γ_{n_1} only looks at a finite number of entries, lets say entries with index less than i_1.
Now assume that i_k, n_k are already chosen for $k = 1, \ldots, m$. Let (a_i^{m+1}) denote the sequence which has entry 1 exactly at the positions $i_1, \ldots, i_m$. Then there is an n_{m+1} greater than n_m such that $\Gamma_n(a_i^{m+1}) = \mathit{No}$ for all $n \geq n_{m+1}$ and an $i_{m+1} > i_m$ such that $\Gamma_{n_{m+1}}$ only looks at positions less than i_{m+1}.

Now consider the sequence (a_i) which has entry 1 exactly at the positions i_k, $k \in \mathbb{N}$. Then $\Gamma_{n_k}(a_i) = \Gamma_{n_k}(a_i^k) = \mathit{No}$ for every k, hence $\lim_k \Gamma_{n_k}(a_i) = \mathit{No}$.

Thus, there is no one-limit family of algorithms for **Q2**, but with

$$\Gamma_{m,n}(a_i) = \mathit{Yes} \quad \Leftrightarrow \quad \sum_{i=1}^{n} a_i > m,$$

the limit $\lim_m \lim_n \Gamma_{m,n}(a_i)$ always gives the correct answer. Thus

$$\mathrm{SCI}(\mathbf{Q2}) = 2.$$

One step higher: Matrices Now, let $(a_{i,j})_{i,j\in\mathbb{N}}$ be an infinite matrix with entries $a_{i,j} \in \{0,1\}$. (or: $(a_{i,j})$ can be regarded as a sequence of sequences). The following problems

Q3 Does $(a_{i,j})$ have a non-zero entry?

Q4 Does $(a_{i,j})$ have infinitely many non-zero entries?

are easy to translate into the previous problems: just take an enumeration of the elements of $\mathbb{N}^2$, that is a bijection $\mathbb{N} \to \mathbb{N}^2$, $k \mapsto (i(k, j(k))$ in order to regard $(a_{i,j})$ as the sequence $(a_{i(k),j(k)})_k$, which yields that **Q3** (**Q4**) is equivalent to **Q1** (**Q2**, resp.), hence

$$\mathrm{SCI}(\mathbf{Q3}) = 1, \quad \mathrm{SCI}(\mathbf{Q4}) = 2.$$

The key questions The situation becomes less clear with the following questions:

Q5 Does $(a_{i,j})$ have a column containing infinitely many non-zero entries?

Q6 Does $(a_{i,j})$ have infinitely many columns containing infinitely many non-zero entries?

There is no one-limit solution to **Q5**, since this would provide a one-limit solution for **Q2**: Just take a given a sequence as the first column of the matrix whose other columns are zero. However, there is a three-limit solution given by the algorithms

$$\Gamma_{k,m,n}(a_{i,j}) = \mathit{Yes} \quad \Leftrightarrow \quad \sum_{i=1}^{n} a_{i,j} > m \text{ for one } j \in \{1, \dots, k\}.$$

Moreover, it holds that SCI(**Q5**) $\leq$ SCI(**Q6**). To see this, given $(a_{i,j})$, define a new matrix by $(b_{i,j}) := \max\{a_{i,s} : s = 1, \dots, j\}$. Then $(a_{i,j})$ has a column with infinitely many non-zero entries if and only if $(b_{i,j})$ has infinitely many columns with infinitely many non-zero entries. On the other hand, there is a four-limit solution to **Q6** given by

$$\Gamma_{l,k,m,n}(a_{i,j}) = \mathit{Yes} \quad \Leftrightarrow \quad \sum_{i=1}^{n} a_{i,j} > m \text{ for more than } l \text{ numbers } j \in \{1, \dots, k\}.$$

Thus

$$2 \leq \mathrm{SCI}(\mathbf{Q5}) \leq 3, \quad \mathrm{SCI}(\mathbf{Q5}) \leq \mathrm{SCI}(\mathbf{Q6}) \leq 4.$$

Conjecture We conjecture that that the latter indices are larger than 2. When proving this one can maybe learn several things:

- On the one hand this could give an idea or a method or a procedure for the treatment of other problems as well. The point is: How can we construct counter-examples for a two-limit-algorithm? We already met the open problem "Two or Three" for the general spectral problem, and we will see further applications of that type below and in Section 4.
- On the other hand the above constructions probably could be continued in order to obtain problems with SCI = k, $k \in \mathbb{N}$. The idea is just to take analogous questions for sequences of sequences of sequences of ...

Another simple but unsolved problem Let Ω now consist of matrices of the following *very special* type: Each column shall either be zero or contain exactly one nonzero entry.

Q7 Does (a_i) have infinitely many columns being zero?

With the tools of the previous examples it is easy to check $2 \leq$ SCI(**Q7**) ≤ 3, but it is again open which equality holds.

4 Further popular problems

4.1 Computing the norm of the inverse

Given an infinite matrix A which defines a bounded linear operator on l^2 we are now interested in computing the norm of its inverse from the matrix entries. More precisely, we again ask for a family of algorithms $\Gamma_{n_1,\ldots,n_i}$ $(n_1,\ldots,n_i \in \mathbb{N})$ which on the one hand provide approximations to $\|A^{-1}\|$, i.e.

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\cdots\lim_{n_i\to\infty}|\Gamma_{n_1,\ldots,n_i}(A)-\|A^{-1}\||=0, \tag{3}$$

or converge to infinity in case of A being not invertible. On the other hand, each of these algorithms shall take only finitely many entries of A into account and perform finitely many operations.

Proposition 15. *The norm of the inverse of a bounded linear operator cannot be computed in one limit in general.*

Proof. We proceed in a similar way as in the first proof of Proposition 1. For $n \in \mathbb{N}$ let again

$$B_n := \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{n\times n}$$

and $A := \operatorname{diag}\{B_{l_1}, B_{l_2}, B_{l_3}, \ldots\} - I$. Clearly A is invertible and its inverse has norm one. Assume that $\{\Gamma_k\}$ is a one-limit algorithm which in its kth step only reads information contained in the first $N(k)\times N(k)$ entries of A. In order to find a counterexample we again construct an appropriate sequence $(l_n) \subset \mathbb{N}$ by induction: For $C := \operatorname{diag}\{1,0,0,0,\ldots\}$ one obviously has $\|(C-I)^{-1}\| = \infty$. Choose $k_0 := 1$ and $l_1 > N(k_0) = N(1)$.

Assume that $l_1,\ldots,l_n$ are already chosen. Then the operator given by the matrix $\operatorname{diag}\{B_{l_1},\ldots,B_{l_n},C\} - I$ is not invertible, hence there exists a k_n such that $\Gamma_k(\operatorname{diag}\{B_{l_1},\ldots,B_{l_n},C\} - I) > 2$ for every $k \geq k_n$. Now finish the construction by choosing $l_{n+1} > N(k_n) - l_1 - l_2 - \ldots - l_n$.

So, we see that $\Gamma_{k_n}(A) = \Gamma_{k_n}(\operatorname{diag}\{B_{l_1},\ldots,B_{l_n},C\} - I)$ do not converge to $\|A^{-1}\| = 1$, a contradiction. □

Notice that this even proves the lower bound two for the more particular case of self-adjoint operators. Next, we show that the Complexity Index is equal to two and, moreover, we will see that knowing a bound for the dispersion improves the Complexity Index to one.

Theorem 16. *The norm of the inverse of a bounded linear operator can be computed in two limits.*

Proof. Introduce the numbers

$$\begin{aligned}
\gamma &:= \|A^{-1}\|^{-1} = \min\{\sigma_1(A), \sigma_1(A^*)\} \\
\gamma_m &:= \min\{\sigma_1(AP_m), \sigma_1(A^*P_m)\} \\
\gamma_{m,n} &:= \min\{\sigma_1(P_nAP_m), \sigma_1(P_nA^*P_m)\} \\
\delta_{m,n} &:= \min\{k/m : k \in \mathbb{N}, k/m \geq \sigma_1(P_nAP_m) \text{ or } k/m \geq \sigma_1(P_nA^*P_m)\}
\end{aligned}$$

and note that $\gamma_m \downarrow_m \gamma$, and $\gamma_{m,n} \uparrow_n \gamma_m$ for every fixed m. Moreover, $(\delta_{m,n})_n$ is bounded and monotone, and $\gamma_{m,n} \leq \delta_{m,n} \leq \gamma_{m,n} + 1/m$. Thus, $(\delta_{m,n})_n$ converges for every m, and for $\epsilon > 0$ there is an m_0, and for every $m \geq m_0$ there is an $n_0 = n_0(m)$ such that

$$|\gamma - \delta_{m,n}| \leq |\gamma - \gamma_m| + |\gamma_m - \gamma_{m,n}| + |\gamma_{m,n} - \delta_{m,n}| \leq \epsilon/3 + \epsilon/3 + 1/m \leq \epsilon$$

whenever $m \geq m_0$ and $n \geq n_0(m)$. Since $\delta_{m,n}$ and hence $\Gamma_{m,n}(A) := \delta_{m,n}^{-1}$ can again be computed with finitely many operations using the Cholesky decomposition this completes the proof. □

Corollary 17. *If a bound $f : \mathbb{N} \to \mathbb{N}$ for the dispersion of the bounded linear operator A is known then the norm of its inverse can be computed in one limit, using the above algorithm with $n = f(m)$.*

4.2 Solving linear systems

Here we consider the following problem: Given an invertible bounded linear operator A on l^2 and $b \in l^2$ we want to compute $x \in l^2$ which solves $Ax = b$. The algorithms we want to allow shall again read only finitely many entries of A and b, and shall generate an approximation for x from these entries using finitely many arithmetic operations. That is the algorithms Γ operate from $\mathcal{L}(l^2) \times l^2$ to l^2, i.e. map the pair (A, b) to an approximation $\Gamma(A, b) \in l^2$ for x.

Proposition 18. *The solution of a linear problem with an invertible bounded linear operator cannot be computed in one limit in general.*

Proof. For $n, m \in \mathbb{N} \setminus \{1\}$ let

$$B_{n,m} := \begin{pmatrix} 1/m & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 1 & & & & 1/m \end{pmatrix} \in \mathbb{C}^{n \times n}$$

and for $(l_n) \subset \mathbb{N} \setminus \{1\}$ set $A := \operatorname{diag}\{B_{l_1,2}, B_{l_2,3}, \ldots, B_{l_n,n+1}, \ldots\}$. Clearly, A is invertible. Furthermore, we define b as the element of l^2 having the entry $1/(n+1)$ at the l_nth position and zeros elsewhere. Note that the inverse of the operator $C_m := \operatorname{diag}\{1/m, 1, 1, \ldots\}$ is $\operatorname{diag}\{m, 1, 1, \ldots\}$.

As usual, one now assumes that there is a one-limit-family of algorithms Γ_n and one constructs sequences (l_n), (k_n) in such a way that $\Gamma_{k_n}(A, b)$ equals $\Gamma_{k_n}(\operatorname{diag}\{B_{l_1,2}, B_{l_2,3}, \ldots, B_{l_{n-1},n}, C_{n+1}\}, b)$, hence has entry close to one at its $(\sum_{i=1}^{n-1} l_i)$th position, which leads to a contradiction. □

Theorem 19. *Linear systems (with an invertible bounded linear operator) can be solved in two limits.*

Proof. Let A be invertible and $Ax = b$ with the unknown x. Since P_m are compact projections converging strongly to the identity, we get that the ranks $\operatorname{rk} P_m = \operatorname{rk}(AP_m) = \operatorname{rk}(P_n A P_m)$ for every m and all $n \geq n_0$ with an n_0 depending on m. Then, obviously, $P_m A^* P_n A P_m$ is invertible, and we can define

$$\Gamma_{m,n}(A, b) := (P_m A^* P_n A P_m)^{-1} P_m A^* P_n b.$$

Note that this can be computed by finitely many operations again using e.g. Choleskys decomposition, and it converges to $y_m := (P_m A^* A P_m)^{-1} P_m A^* b$ as $n \to \infty$. It is well known that y_m is also a (least squares) solution of the optimization problem $\|AP_m y - b\| \to \min$, that is

$$\|AP_m y_m - b\| \leq \|AP_m x - b\| \leq \|A\| \|P_m x - A^{-1} b\| = \|A\| \|P_m x - x\| \to 0$$

as $m \to \infty$. Therefore $\|y_m - x\| = \|P_m y_m - x\|$ is not greater than

$$\|A^{-1}\| \|A(P_m y_m - x)\| = \|A^{-1}\| \|AP_m y_m - b\| \leq \|A^{-1}\| \|A\| \|P_m x - x\| \to 0,$$

which yields the convergence $y_m \to x$ and finishes the proof. □

Corollary 20. *Linear systems with an invertible bounded linear operator for which a bound f on the dispersion is known can be solved in one limit, using the above algorithm with $n = f(m)$.*

Proof. The smallest singular values of the operators AP_m are uniformly bounded below by $\|A^{-1}\|^{-1}$ which, together with $\|P_{f(m)} AP_m - AP_m\| \to 0$, yields that the limes inferior of the smallest singular values of $P_{f(m)} AP_m$ is positive, hence the inverses of the operators $B_m := P_m A^* A P_m$ and $C_m := P_m A^* P_{f(m)} A P_m$ on the range of P_m exist for sufficiently large m and have uniformly bounded norm. Moreover, $\|B_m^{-1} - C_m^{-1}\| \leq \|B_m^{-1}\| \|C_m - B_m\| \|C_m^{-1}\|$ tend to zero as $m \to \infty$.

This particularly implies that the norms $\|y_m - (P_m A^* P_{f(m)} A P_m)^{-1} P_m A^* b\|$ with y_m as above tend to zero as $m \to \infty$, and we easily conclude that the norms $\|y_m - \Gamma_{m,f(m)}(A, b)\|$ tends to zero as well. With the convergence $\|y_m - x\| \to 0$ from the previous proof, now also $\|x - \Gamma_{m,f(m)}(A, b)\| \to 0$ holds as $m \to \infty$, the assertion. □

4.3 Convergence of real sequences

We consider the function

$$\limsup : l^\infty(\mathbb{N}, \mathbb{R}) \to \mathbb{R}, \quad (a_i) \mapsto \limsup_i a_i.$$

Each of the algorithms $\Gamma_{m,n}(a_i) := \max\{a_i : i \in \{m, \ldots, n\} \cup \{m\}\}$ obviously takes into account only finitely many entries of (a_i). Moreover, we see that $\Gamma_{m,n}(a_i) \uparrow_n \sup\{a_i : i \geq m\} \downarrow_m \limsup(a_i)$, hence $\mathrm{SCI}(\limsup) \leq 2$. On the other hand a recursive construction as it was done several times in the above sections even yields $\mathrm{SCI}(\limsup) = 2$: More precisely, we consider sequences (a_i) whose entries a_i are 0 or 1, and note that the $\limsup a_i$ is 1 if infinitely many entries are 1 and is 0 otherwise. As in the second proof of Proposition 1 or the proof of Proposition 13 one then deduces that every one-limit-algorithm is condemned to fail.

For the decision problem

$$C : l^\infty(\mathbb{N}, \mathbb{R}) \to M := \{\mathit{Yes}, \mathit{No}\}, \quad (a_i) \mapsto ((a_i) \text{ converges})$$

we get a three-limit-algorithm by

$$\Gamma_{k,m,n}(a_i) :=$$
$$(\max\{a_i : i \in \{m, \ldots, n\} \cup \{m\}\} - \min\{a_i : i \in \{m, \ldots, n\} \cup \{m\}\} < 1/k).$$

The above idea for a construction easily yields that also this problem cannot be solved in one limit in general. Thus, one obtains the lower bound two and ends up with $2 \leq \mathrm{SCI}(C) \leq 3$. Unfortunately, it is again an open problem what this Complexity Index precisely is.

4.4 Deciding whether $0 \in \mathrm{sp}(A)$

In our final example we return to the spectrum and ask: How difficult is it to decide whether a given point $\lambda \in \mathbb{C}$ belongs to the spectrum of a given bounded linear operator A or not? Obviously this question is equivalent to asking whether $0 \in \mathrm{sp}(A - \lambda I)$. Thus, it suffices to study the question for $\lambda = 0$. To make this more precise, we consider the functions

$$\mathrm{sp}^0 : \mathcal{L}(l^2) \to \{\textit{Yes}, \textit{No}\}, \quad A \mapsto (0 \in \mathrm{sp}(A))$$

and we recall from the proof of Theorem 16 the algorithms $\Gamma_{m,n}(A) := \delta_{m,n}^{-1}$, which approximate the norm of the inverse of A in case A is invertible, or tend to infinity otherwise. If we define $\Gamma_{k,m,n}(A) := (\delta_{m,n}^{-1} < 1/k)$ we arrive at a three-limit-algorithm

$$\mathrm{sp}^0(A) = \lim_{k\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} \Gamma_{k,m,n}(A)$$

for our problem, hence $\mathrm{SCI}(\mathrm{sp}^0) \leq 3$. Due to Corollary 17 this upper bound decreases by 1 if one considers operators for which a bound on their dispersion is known.

On the other hand, if we assume that there is a one-limit-algorithm given by a family $\{\Gamma_n\}$ then we can again construct counterexamples very easily: For a decreasing sequence (a_i) of positive numbers we consider the diagonal operator $A := \mathrm{diag}\{a_i\}$. Clearly, 0 belongs to the spectrum of A if and only if the a_is tend to zero. As a start, set $(a_i^1) := (1, 1, \ldots)$, choose n_1 such that $\Gamma_n(a_i^1) = \textit{No}$ for all $n \geq n_1$, and i_1 such that Γ_{n_1} doesn't see the diagonal entries a_i with indices $i \geq i_1$. Then set $(a_i^2) := (1, 1, \ldots, 1, 1/2, 1/2, \ldots)$ with 1/2s starting at the i_1th position. If $n_1, \ldots, n_{k-1}$ and $i_1, \ldots, i_{k-1}$ are already choosen then pick n_k such that $\Gamma_n(a_i^k) = \textit{No}$ for all $n \geq n_k$, and i_k such that Γ_{n_k} doesn't see the diagonal entries a_i with indices $i \geq i_k$, and set $(a_i^{k+1}) := (1, \ldots, 2^{-k}, 2^{-k}, \ldots)$ with 2^{-k}s starting at the i_kth position. Now, the contradiction is as usual and we see

Theorem 21. *In order to decide whether $\lambda \in \mathbb{C}$ belongs to $\mathrm{sp}(A)$ one needs at least two and at most three limits. If a bound on the dispersion of A is known then the Solvability Complexity Index equals two.*

Remark 22. Notice that the above counterexample actually provides self-adjoint diagonal operators for which there don't exist any one-limit-algorithms that can decide sp^0, although there is a one-limit algorithm which computes the whole spectrum as is shown in Theorem 12. This seems to be a bit surprising at a first glance. Actually, the present question is really stronger in a sense: From Theorem 12 we only get approximations for the spectrum which converge with respect to the Hausdorff distance, but which can still have even an empty intersection with $\mathrm{sp}(A)$, whereas Theorem 21 addresses the inclusion $\lambda \in \mathrm{sp}(A)$.

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