A generalization of the Funk–Radon transform

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Radon-type transforms: Basis for Emerging Imaging
100 Years of the Radon Transform
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Funk–Radon transform
Introduction

- **Sphere** \( S^2 = \{ \xi \in \mathbb{R}^3 : \|\xi\| = 1 \} \)
- **Function** \( f : S^2 \to \mathbb{C} \)

- **Circle** is the intersection of \( S^2 \) with a plane:
  \[ \{ \eta \in S^2 : \langle \xi, \eta \rangle = x \}, \]
  \[ \xi \in S^2, \ x \in [-1, 1] \]

**Spherical means**

\[
S : C(S^2) \to C(S^2 \times [-1, 1]),
\]
\[
Sf(\xi, x) = \int_{\langle \xi, \eta \rangle = x} f(\eta) \, d\lambda(\eta)
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- **Function** $f : S^2 \to \mathbb{C}$
- **Circle** is the intersection of $S^2$ with a plane:
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$$Sf(\xi, x) = \int_{\langle \xi, \eta \rangle = x} f(\eta) \, d\lambda(\eta)$$
Funk–Radon transform

- Restriction to great circles
- **Funk–Radon transform** (a.k.a. Funk transform or spherical Radon transform)

\[ \mathcal{F} : C(S^2) \rightarrow C(S^2), \]
\[ \mathcal{F} f(\xi) = \int_{\langle \xi, \eta \rangle = 0} f(\eta) \, d\lambda(\eta) \]

**Questions**

1. **Injectivity**
   (Can we reconstruct \( f \) from its means along all great circles?)
2. **Range of \( \mathcal{F} \)**
Funk–Radon transform

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1. Injectivity
   (Can we reconstruct \( f \) from its means along all great circles?)
2. Range of \( \mathcal{F} \)
Fourier series

Write \( f \in L^2(S^2) \) with respect to spherical harmonics \( Y_n^k \) of degree \( n \)

\[
f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}(n, k) Y_n^k.
\]

Eigenvalue decomposition [Minkowski, 1904]

The Funk–Radon transform

\[
\mathcal{F}Y_n^k(\xi) = P_n(0)Y_n^k(\xi),
\]

\[
P_n(0) = \begin{cases} 
\frac{(n-1)!!}{n!!}, & n \text{ even,} \\
0, & n \text{ odd.}
\end{cases}
\]

\( P_n \) – Legendre polynomial of degree \( n \)

Funk–Radon transform is injective for even functions \( f(\xi) = f(-\xi) \).
Fourier series

Write $f \in L^2(S^2)$ with respect to spherical harmonics $Y^k_n$ of degree $n$

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Sobolev spaces

Let $s \geq 0$. The Sobolev space $H^s(S^2)$ consists of functions $f : S^2 \to \mathbb{C}$ with norm

$$\|f\|_s^2 = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} |\hat{f}(n, k)|^2 \left( n + \frac{1}{2} \right)^{2s}.$$ 

Theorem [Strichartz, 1981]

The Funk–Radon transform is continuous and bijective

$$\mathcal{F} : L^2_{\text{even}}(S^2) \to H^\frac{1}{2}_{\text{even}}(S^2).$$

▶ Inversion of $\mathcal{F}$ is incorrect of degree $\frac{1}{2}$
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Circles with fixed radius

- For $x_0 \in [-1, 1]$ fixed, we define

$$S_{x_0} f(\xi) = \int_{\langle \xi, \eta \rangle = x_0} f(\eta) \, d\eta$$

- Eigenvalue decomposition

$$S_{x_0} Y_n^k = P_n(x_0) Y_n^k$$

“Freak theorem” [Schneider, 1969]

The set of values $x_0$ for which $S_{x_0}$ is not injective is countable and dense in $[-1, 1]$.

$$S_{x_0} : L^2(S^2) \to H^1_2(S^2)$$ is continuous

Explicit algorithm to determine if $S_{x_0}$ is injective for given $x_0$ [Rubin, 2000]
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Vertical slices

\[ Sf(\xi, x) = \int_{\langle \xi, \eta \rangle = x} f(\eta) \, ds(\eta), \quad \xi_3 = 0 \]

- **Circles perpendicular to the equator**
  - Injective for symmetric functions
  \[ f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3) \]
  - Proof 1: Orthogonal projection onto the equatorial plane
    [Gindikin, Reeds & Shepp, 1994]
  - Proof 2: Spherical harmonics
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**Known results**

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Local two-radii problem

- $B_R = \{ \xi \in S^2 : \xi_3 > \cos R \}$
  - spherical cap north of the circle of latitude $R \in (0, \pi]$
- $r_1, r_2 < R$ ... latitudes of centers

Theorem

[Volchkov & Volchkov, 2014]

Then

$$S f(\xi, t) = 0, \quad \xi_{d+1} = \cos r_j, \quad t > \cos(R - r_j), \quad j \in \{1, 2\}$$

implies $f = 0$ for all $f \in L^1_{\text{loc}}(B_R)$ if and only if $R \geq r_1 + r_2$ and

$$P^k_\nu(\cos r_1)^2 + P^k_\nu(\cos r_2)^2 > 0 \quad \forall k \in \mathbb{N}_0, \quad \nu > k.$$
Higher dimensions

- $S^d$ ... $d$-dimensional sphere
- $\mathbb{R}^{d+1}$ ... $(d + 1)$-dimensional ambient space
- Great circle becomes $(d - 1)$-dimensional subsphere
- For $f \in C(S^d)$, we set

$$\mathcal{F} f (\xi) = \int_{S^d \cap \{ \langle \xi, \eta \rangle = 0 \}} f(\eta) \, d\mu$$

**Theorem** [Strichartz, 1981]

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   - Introduction
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   - Definition
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Slices through a fixed point

Replace 0 by

\[ ze^{d+1}, \quad 0 \leq z < 1 \]

inside the sphere.

Intersection with hyperplane through \( ze^{d+1} \) is given by

\[ \{ \eta \in S^d : \langle \xi, \eta \rangle = z \xi_{d+1} \}. \]

**Definition**

\[ U_z : C(S^d) \rightarrow C(S^d), \]

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\[ \mathcal{U}_z f(\xi) = \int_{\langle \xi, \eta \rangle = z \xi_{d+1}} f(\eta) \, d\lambda(\eta) \]
Theorem

Let \( z \in [0, 1) \), \( \sigma = \sqrt{\frac{1+z}{1-z}} \) and \( f \in C^1(S^d) \) be supported in the interior of \( \{ \xi \in S^d : -1 \leq \xi_{d+1} \leq z \} \). Then \( f \) can be reconstructed by

\[
(f \circ \pi^{-1}) \left( \frac{2\sigma x}{1 + \sqrt{1 + 4|x|^2}} \right) =
\begin{cases}
(1) \frac{d-2}{2} (1 - z) \sqrt{1 + 4|x|^2} \left( \frac{1 + \sqrt{1 + 4|x|^2}}{1 + \sqrt{1 + 4|x|^2}} \right)^{d-1} \\
\triangle x^2 \int_{S^{d-1}} \int_0^{\pi/2} (\mathcal{U}_z f)(\cos \theta \phi + (\sin \theta) e^{d+1}) \log \left| x \cdot \phi - \frac{1}{2} \sqrt{1 - z^2} \tan \theta \right| \frac{d\theta d\phi}{\sqrt{1 - z^2 \sin^2 \theta \cos \theta}}, \text{ if } d \geq 2 \text{ even}
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\end{cases}
\]
Let $z \in [0, 1)$. The generalized Radon transform $U_z$ can be represented through

$$U_z = N_z F \mathcal{M}_z.$$ 

- $\mathcal{M}_z f(\xi) = \left( \frac{\sqrt{1 - z^2}}{1 + z \xi_{d+1}} \right)^{d-1} f \circ h(\xi), \quad \xi \in S^d$

  $$h(\xi) = \sum_{i=1}^{d} \frac{\sqrt{1 - z^2}}{1 + z \xi_{d+1}} \xi_i e^i + \frac{z + \xi_{d+1}}{1 + z \xi_{d+1}} e^{d+1}$$

- $\mathcal{F} ...$ Funk–Radon transform

  $$\mathcal{N}_z f(\xi) = (1 - z^2 \xi_{d+1}^2)^{-\frac{d-1}{2}} f \circ g(\xi), \quad \xi \in S^d$$

  $$g(\xi) = \frac{1}{\sqrt{1 - z^2 \xi_{d+1}^2}} \left( \sum_{i=1}^{d} \xi_i e^i + \sqrt{1 - z^2 \xi_{d+1} e^{d+1}} \right)$$
Geometric interpretation of \( h \) (for \( S^2 \))

**Theorem**

The map

\[
    h(\xi) = \pi^{-1} \left( \sqrt{\frac{1+z}{1-z}} \pi(\xi) \right)
\]

is conformal. It consists of

1. Stereographic projection \( \pi : S^d \to \mathbb{R}^d \)
2. Uniform scaling \( \mathbb{R}^d \to \mathbb{R}^d, \ x \mapsto \sqrt{\frac{1+z}{1-z}} \ x \)
3. Inverse stereographic projection \( \pi^{-1} : \mathbb{R}^d \to S^d \)

We are going to see that

\( h \) maps great circles to small circles through \( ze^{d+1} \).
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- $G$ ... great circle of $S^2$
- $E$ ... equator of $S^2$

- $G$ intersects $E$ in two antipodal points (or is identical to $E$)

- $\pi(E) = E$
- $\pi(G)$ is circle or line in $\mathbb{R}^2$ and intersects $\pi(E)$ in two antipodal points
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2) Uniform scaling

- Scaling with factor $\sigma = \sqrt{\frac{1+z}{1-z}}$

- Unit circle $E$ becomes circle $\sigma(\pi(E))$ with radius $\sigma$

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3) Inverse stereographic projection $\pi^{-1}$

- Circle with radius $s$
  becomes circle of latitude $z$;
  $h(E)$

- $h(G) = \pi^{-1}(\sigma(\pi(G)))$
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Nullspace of $\mathcal{U}_z$

**Theorem**  \[ Q., 2017 \]

For $\xi \in \mathbb{S}^d$ we define $\xi^* \in \mathbb{S}^d$ as the point reflection of the sphere about the point $ze^{d+1}$.

Let $f \in L^2(\mathbb{S}^d)$. Then

$$\mathcal{U}_z f = 0$$

if and only if for almost every $\xi \in \mathbb{S}^d$

$$f(\xi) = - \left( \frac{1 - z^2}{1 + z^2 - 2z\eta_{d+1}} \right)^{d-1} f(\xi^*).$$
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Range of $\mathcal{U}_z$

**Theorem** [Q., 2017]

The generalized Radon transform

$$\mathcal{U}_z : \tilde{H}^s_e(S^d) \to H^s_e + \frac{d-1}{2} (S^d)$$

is bijective and continuous.

- $\tilde{H}^s_e(S^d) = \left\{ f \in H^s(S^d) \mid f(\xi) = \left( \frac{1 - z^2}{1 + z^2 - 2z\eta_{d+1}} \right)^{d-1} f(\xi) \right\}$

- $H^s_e + \frac{d-1}{2} (S^d)$ ... Sobolev space of even functions
Sketch of proof

\[ \mathcal{U} : H^s(\mathbb{S}^d) \xrightarrow{M_z} H^s(\mathbb{S}^d) \xrightarrow{\mathcal{F}} H^{s+\frac{d-1}{2}}(\mathbb{S}^d) \xrightarrow{N_z} H^{s+\frac{d-1}{2}}(\mathbb{S}^d) \]
Sketch of proof

\[ \mathcal{U} : \mathcal{H}^s(S^d) \xrightarrow{\mathcal{M}_z} \mathcal{H}^s(S^d) \xrightarrow{\mathcal{F}} \mathcal{H}^{s + \frac{d-1}{2}}(S^d) \xrightarrow{\mathcal{N}_z} \mathcal{H}^{s + \frac{d-1}{2}}(S^d) \]

We have

\[ \mathcal{M}_z f(\xi) = \left( \frac{\sqrt{1 - z^2}}{1 + z \xi_{d+1}} \right)^{d-1} f \circ h(\xi), \quad \xi \in S^d \]

where

\[ h(\xi) = \sum_{i=1}^{d} \frac{\sqrt{1 - z^2}}{1 + z \xi_{d+1}} \xi_i e^i + \frac{z + \xi_{d+1}}{1 + z \xi_{d+1}} e^{d+1}. \]
Sketch of proof

\[ \mathcal{U} : H^s(S^d) \xrightarrow{\mathcal{M}_z} H^s(S^d) \xrightarrow{\mathcal{F}} H^{s + \frac{d-1}{2}}(S^d) \xrightarrow{\mathcal{N}_z} H^{s + \frac{d-1}{2}}(S^d) \]

Lemma 1

Let \( s \in \mathbb{N}_0 \) and \( g \in C^s(S^d) \). Then there exists \( c_s \) such that

\[ \| fg \|_{H^s(S^d)} \leq c_s \| f \|_{H^s(S^d)} \| g \|_{C^s(S^d)} \quad \forall f \in H^s(S^d), \forall g \in C^s(S^d). \]

Lemma 2

Let \( s \in \mathbb{N}_0 \) and \( g \in C^s(S^d \to S^d) \) be a diffeomorphism. Then there exists \( C_{g,s} \) such that

\[ \| f \circ g \|_{H^s(S^d)} \leq C_{g,s} \| f \|_{H^s(S^d)} \quad \forall f \in H^s(S^d). \]
Sketch of proof

\[ \mathcal{U} : H^s(S^d) \xrightarrow{\mathcal{M}_z} H^s(S^d) \xrightarrow{\mathcal{F}} H^{s+\frac{d-1}{2}}(S^d) \xrightarrow{\mathcal{N}_z} H^{s+\frac{d-1}{2}}(S^d) \]

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Theorem [Strichartz, 1981]

The Funk–Radon transform

\[ \mathcal{F} : L^2_{\text{even}}(\mathbb{S}^d) \to H^{\frac{d-1}{2}}_{\text{even}}(\mathbb{S}^d) \]

is continuous and bijective.
Sketch of proof

\[ \mathcal{U} : H^s(S^d) \xrightarrow{\mathcal{M}_z} H^s(S^d) \xrightarrow{\mathcal{F}} H^{s+\frac{d-1}{2}}(S^d) \xrightarrow{\mathcal{N}_z} H^{s+\frac{d-1}{2}}(S^d) \]

We have

\[ \mathcal{N}_z f(\xi) = (1 - z^2 \xi_{d+1}^2)^{-\frac{d-1}{2}} \text{ multiplication } \]
\[ \underbrace{f \circ g}_\text{composition} (\xi), \quad \xi \in S^d \]

where

\[ g : S^d \to S^d, \quad g(\xi) = \frac{1}{\sqrt{1 - z^2 \xi_{d+1}^2}} \left( \sum_{i=1}^{d} \xi_i e^i + \sqrt{1 - z^2 \xi_{d+1} e^{d+1}} \right). \]
Sketch of proof

\[ \mathcal{U} : H^s(\mathbb{S}^d) \xrightarrow{\mathcal{M}_z} H^s(\mathbb{S}^d) \xrightarrow{\mathcal{F}} H^{s+\frac{d-1}{2}}(\mathbb{S}^d) \xrightarrow{\mathcal{N}_z} H^{s+\frac{d-1}{2}}(\mathbb{S}^d) \]

- The Lemmas imply that \( \mathcal{N}_z : H^s \rightarrow H^s \) is continuous for \( s \in \mathbb{N}_0 \)
- The same holds for \( \mathcal{N}^{-1}_z \)
- Continuity for arbitrary \( s > 0 \) follows with theory of interpolation spaces [Triebel, 1995]
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