TECHNISCHE UNIVERSITÄT
CHEMNITZ

Faculty of Mathematics

Master thesis

Behaviour of entropy numbers under real interpolation

Robert Nasdala

Born June 11, 1990 in Chemnitz

Chemnitz, July 27, 2017

First advisor: Prof. Dr. Henning Kempka

Second advisor: Prof. Dr. Daniel Potts

Contents

List of Figures

Acknowledgements

First of all I want to express my deepest gratitude towards Professor Henning Kempka whose guidance over the last two years helped me a lot to develop mathematically and personally.

I want to thank Professor Daniel Potts for his overall support and the possibility to recharge over a cup of coffee on a regular basis. I would also like to thank the rest of the working group *Applied Functional Analysis* for their company.

Finally, I have to thank my parents. Not only for their advices over the years, but also for their constant support which granted me certain freedoms for which I am very grateful.

1 Introduction

From classical approximation theory we know interpolation problems in which we are given at least two points in a certain normed vector space and try to construct a continuous function going through them. So in essence we are trying to find a set of points that in a certain sense are lying in between the initially given ones. We want to stick with this idea and generalize it by replacing the points with whole sequence spaces and using the concept of embeddings to describe their position to one another. Of particular interest will be the ℓ_q-spaces of q-summable sequences for values $1 \leq q \leq \infty$. Besides the fact that they are all complete and normed, they possess a lexicographical order

$$\ell_1 \hookrightarrow \ell_q \hookrightarrow \ell_\infty$$

so that all ℓ_q-spaces with $1 < q < \infty$ are lying between ℓ_1 and ℓ_∞ in the sense of embeddings. Clearly, this fits exactly into the desired description of normed space interpolation. A general description is provided through real and complex interpolation space methods, see [1] or [17]. They create a framework in which couples of Banach spaces (X_0, X_1) automatically induce the spaces $X_0 \cap X_1$ and $X_0 + X_1$ in between which we will construct intermediate Banach spaces X so that

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

Since we are only going to deal with real interpolation methods, we will only be able to construct the respective real interpolation spaces $\overline{X}_{\theta,q}$ among all the intermediate spaces. At the same time we will see that the family of real interpolation spaces between ℓ_1 and ℓ_∞ contains much more elements than just the ℓ_q-spaces, so that we will need to introduce a more general class of sequence spaces - the Lorentz spaces $\ell_{p,q}$.

Afterwards we will need two Banach couples (X_0, X_1) and (Y_0, Y_1) to have a look at operators T acting between them. Originally defined as maps from X_0+X_1 to Y_0+Y_1 there is a broad range of results which focus on how properties of the restrictions $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ can influence the operator $T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}$ between the corresponding real interpolation spaces.

In particular we are going to study how the compactness of operators behaves under interpolation. For that we put $B_X = \{x \in X : \|x\|_X \leq 1\}$ as the unit ball of a normed space X and define the n-th (dyadic) entropy numbers [2]

$$e_n(T) = e_n(T : X \to Y) = \inf \left\{ \varepsilon > 0 : \text{ there exist } q \leq 2^{n-1} \text{ points } y_1, y_2, \ldots, y_q \text{ in } Y \right.$$
$$\left. \text{so that } T(B_X) \subset \bigcup_{j=1}^{q} (y_j + \varepsilon B_Y) \right\}.$$

We will see that the compactness of T is equivalent to $(e_n(T))_{n\in\mathbb{N}}$ being a null sequence. Therefore the decay of entropy numbers for increasing n could be interpreted as way to quantify how compact or non-compact T is, which may have been the motivation to actually call $\beta(T) := \lim_{n\to\infty} e_n(T) \geq 0$ the measure of non-compactness. Regarding their behaviour under interpolation it was shown in [4] that there is a constant C depending on θ such that

$$\beta(T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}) \leq C\beta(T : X_0 \to Y_0)^{1-\theta}\beta(T : X_1 \to Y_1)^{\theta}.$$

For quite some time it had been an open problem if there exists a similar inequality for entropy numbers

$$e_k(T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}) \leq Ce_m(T : X_0 \to Y_0)^{1-\theta}e_n(T : X_1 \to Y_1)^{\theta}$$

with finite $k, m, n \in \mathbb{N}$ when no further assumptions about the Banach couple (X_0, X_1) and (Y_0, Y_1) are made. This problem was finally settled in [6] where D. Edmunds and Yu. Netrusov constructed a counterexample for sequence space couples (ℓ_{p_0}, ℓ_{p_1}) and (ℓ_{q_0}, ℓ_{q_1}) with $p_0 < q_0$ and $p_1 < q_1$ such that the entropy number of the interpolated operator $e_n(T : \ell_{p,u} \to \ell_{q,u})$ can not be bounded above by $e_n(T : \ell_{p_0} \to \ell_{q_0})$ and $e_n(T : \ell_{p_1} \to \ell_{q_1})$. After discussing some properties of entropy numbers and a particular technique to estimate them below (see [11]), we will go over the content of their paper [6] in order to grasp the involved ideas and techniques of their given counterexample.

2 Preliminaries

At first we fix some general conventions: Vector spaces are always going to be defined over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Constants are positive real numbers, if not further specified. For any set S we denote by $|S| \in \mathbb{N}_0$ the number of elements in S.

Now we are going to collect some well-known facts about vector spaces, linear operators and sequence spaces. Proofs will be omitted and can be found for example in [9].

2.1 Complete quasi-normed spaces

Definition 2.1 Let X be a vector space. The functional $\|\cdot\| : X \to [0, \infty)$ is called a **norm** if for all $\alpha \in \mathbb{K}$ and $x, y \in X$ it holds

(N1) $\|x\| = 0 \Leftrightarrow x = 0$,

(N2) $\|\alpha x\| = |\alpha|\|x\|$,

(N3) $\|x + y\| \leq \|x\| + \|y\|$.

If we substitute the triangle inequality (N3) with the quasi-triangle inequality, so that there exists a constant $C \geq 1$ such that for all $x, y \in X$

$$\|x + y\| \leq C\left(\|x\| + \|y\|\right),$$

then we call $\|\cdot\|$ a **quasi-norm**. The pair $(X, \|\cdot\|)$ is called a **(quasi-)normed space**.

Since every norm is a quasi-norm with $C = 1$, a lot of definitions and theorems will be stated and proven in their more general quasi-normed version as they naturally include their respective normed space cases.

Definition 2.2 Let $(X, \|\cdot\|)$ be a quasi-normed space.

(a) A sequence $(x_n)_{n\in\mathbb{N}}$ in X **converges to** $x \in X$ if for all $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq n_\varepsilon$, which is denoted by $\lim_{n\to\infty} x_n = x$.

(b) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called a **null sequence** (or $\|\cdot\|$**-null sequence**) if it converges to 0.

(c) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a **Cauchy sequence** if for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for all $m, n \geq N_\varepsilon$.

(d) X is **complete** if every Cauchy sequence of X converges to an element in X.

(e) A complete quasi-normed space is called **quasi-Banach space**.

In order to prove completeness one can simply use the given definition. Yet, sometimes this approach is unnecessarily tedious. Then the following characterization of completeness through the absolute convergence of certain series can be useful:

Lemma 2.3 *Let $(X, \|\cdot\|)$ be a quasi-normed space. The following are equivalent:*

(a) X is complete.

(b) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{n=1}^{\infty} \|x_n\| < \infty$ there exists an $x \in X$ such that

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\| = 0.$$

In some cases it is possible to introduce several norms on a single vector space that yield the same essential properties, for example completeness, which leads us to the concept of norm equivalence.

Definition 2.4 Two quasi-norms $\|\cdot\|$ and $\|\cdot\|'$ on X are **equivalent**, denoted by $\|\cdot\| \sim \|\cdot\|'$, if there are constants C, D such that

$$C\|x\|' \leq \|x\| \leq D\|x\|'$$

for all $x \in X$.

Remark 2.5 *We will use the same notation for the equivalence of scalars $x, y \in \mathbb{R}$. As before we denote by $x \sim y$ that there exist constants C, D such that $Cx \leq y \leq Dx$.*

Equivalent norms are interchangeable as they induce the same notion of convergence:

Theorem 2.6 *Let x be any sequence in X and let $\|\cdot\|$ and $\|\cdot\|'$ be quasi-norms on X. The following are equivalent:*

(a) $\|\cdot\| \sim \|\cdot\|'$

(b) x converges with respect to $\|\cdot\|$ if and only if x converges with respect to $\|\cdot\|'$, and they both converge to the same point.

(c) x is a $\|\cdot\|$-null sequence if and only if x is a $\|\cdot\|'$-null sequence.

2.2 Linear operators and embeddings

Next we will look at *operators* between quasi-normed spaces X and Y which are maps $T : X \to Y, x \mapsto T(x)$. Usually we shorten the notation of images to $Tx := T(x)$. Besides the fundamental notions of linearity, continuity and boundedness of operators we will introduce the concept of continuous embeddings.

Definition 2.7 Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be quasi-normed spaces and $T : X \to Y$ an operator.

(a) T is **linear** if for all $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in X$

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2.$$

We put $\mathrm{L}(X, Y) := \{T : X \to Y, T \text{ linear}\}$.

(b) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. T is **continuous** if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T x_n = T x$.

(c) T is **bounded** if there exists a constant C such that for all $x \in X$

$$\|Tx\|_Y \le C\|x\|_X.$$

The smallest such C is called the **operator norm** and denoted by

$$\|T\| = \|T\|_{X\to Y} := \inf\{C : \|Tx\|_Y \le C\|x\|_X\} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X, x \neq 0\right\}.$$

Lemma 2.8 *Let $T \in \mathrm{L}(X, Y)$. T is continuous if and only if T is bounded.*

Remark 2.9 *The collection of all linear and continuous (or bounded) operators will be denoted by $\mathcal{L}(X, Y) := \{T \in \mathrm{L}(X, Y) : T \text{ continuous}\}$.*

Definition 2.10 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed spaces and $\mathrm{id} : X \to Y, x \mapsto x$ the identity operator. X is **continuously embedded** into Y, denoted by $X \hookrightarrow Y$, if

$$X \subset Y \quad \text{and} \quad \mathrm{id} \in \mathcal{L}(X, Y).$$

With embeddings we can classify vector spaces in terms of their size and relative position to one another which we will use heavily later on in interpolation space theory. Proving an embedding usually comes down to just showing the existence of some constant C for which the respective identity operator is bounded.

2.3 Sequence spaces

A common first example for Banach spaces are the ℓ_q sequence spaces of all q-summable sequences with entries in $\mathbb{K}$. They are not only complete but also well-structured in terms of embeddings which is often described as a *lexicographic order*. Usually defined only for values $q \ge 1$, the concept of quasi-norms enables us to describe these sequence spaces for values $0 < q < 1$ as well.

Definition 2.11 For $0 < q \leq \infty$ the **sequence space** ℓ_q is defined as

$$\ell_q = \left\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{K}, \|x\|_{\ell_q} < \infty \right\}$$

with

$$\|x\|_{\ell_q} = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup_{n\in\mathbb{N}} x_n & \text{for } q = \infty, \end{cases}$$

where $\|\cdot\|_{\ell_q}$ is a quasi-norm for $0 < q < 1$. Additionally, $(c_0, \|\cdot\|_{\ell_\infty})$ denotes the Banach space of all null sequences

$$c_0 = \left\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{K}, \lim_{n\to\infty} x_n = 0 \right\}.$$

Theorem 2.12 *ℓ_q is a quasi-Banach space for $0 < q < 1$ and a Banach space for $1 \leq q \leq \infty$.*

Theorem 2.13 *For $0 < q < r < \infty$ it holds*

$$\ell_q \hookrightarrow \ell_r \hookrightarrow c_0 \hookrightarrow \ell_\infty.$$

Furthermore, the well-known inequalities by Hölder and Minkowski are almost impossible to avoid within the theory of sequence spaces.

Theorem 2.14 (Hölder inequality) *Let $1 \leq q, r \leq \infty$ with $\frac{1}{q} + \frac{1}{r} = 1$. For $x \in \ell_q, y \in \ell_r$ it holds*

$$\|xy\|_{\ell_1} \leq \|x\|_{\ell_q} \|y\|_{\ell_r}.$$

Theorem 2.15 (Minkowski inequality) *For $x, y \in \ell_q$ with $1 \leq q \leq \infty$ it holds*

$$\|x + y\|_{\ell_q} \leq \|x\|_{\ell_q} + \|y\|_{\ell_q}.$$

2.4 Unit balls of sequence spaces

Another important structure in the context of quasi-normed spaces are their unit balls. Later on they will become crucial in the theory of entropy numbers. For now we will just define them and have a look at some examples.

Definition 2.16 Let $(X, \|\cdot\|_X)$ be a quasi-normed space. We call

$$B_X = \{x \in X : \|x\|_X \leq 1\}$$

the (closed) **unit ball of** X.

Going back to sequence spaces we have seen that $\|\cdot\|_{\ell_q}$ is a proper norm for $q \geq 1$ so that it holds the triangle inequality and the respective unit balls are convex. Since we only have a quasi-norm for $q < 1$ it needs to hold a quasi-triangle inequality. In fact, for any $x, y \in \ell_q$ with $0 < q \leq 1$ it holds

$$\|x + y\|_{\ell_q} \leq 2^{\frac{1}{q}-1} \left(\|x\|_{\ell_q} + \|y\|_{\ell_q} \right).$$

This leads to concave unit balls as we can see in Figure 1 for two-dimensional ℓ_q-spaces.

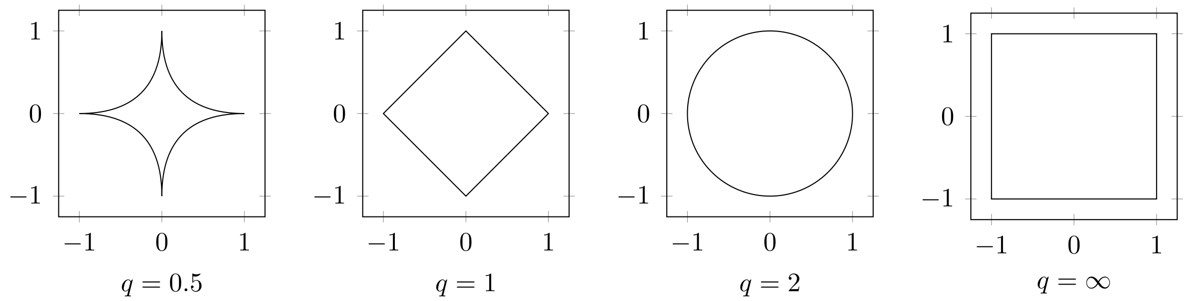

Figure 1: Side-by-side comparison of unit balls B_{ℓ_q}.

3 Real interpolation of Banach spaces

In this chapter we want to unify ideas from interpolation theory with the embedding structure of Banach spaces. Classical interpolation theory is usually concerned with problems where we are given a few points in a certain vector space and want to construct a continuous function going through them. One of the simplest examples for this is to be given any two points $x, y \in \mathbb{R}^2$ and having to draw a line between them. As it turns out, this line is the set

$$\{z \in \mathbb{R}^2 : z = \alpha x + (1-\alpha)y, 0 \leq \alpha \leq 1\}$$

of convex combinations of x and y.

While our interpolation problems will soon become much more complicated, this simple example is sufficient to motivate the upcoming interpolation space theory. For that we have to capture the essentials of the *line through two points* problem: We start with a pair of objects. The goal is to construct a whole family of such objects that are lying in between the initially given ones. Eventually we want these objects to be Banach spaces whose relative positions to one another will be described via embeddings.

With the K-method we will introduce just one particular out of many interpolation methods. We start out by specifying the outer pair of Banach spaces in between which we will interpolate other Banach spaces.

3.1 Banach couples

Definition 3.1 Let X_0, X_1 be Banach spaces and $\mathcal{Z}$ a Hausdorff topological vector space. We call $\overline{X} = (X_0, X_1)$ a **Banach couple** if $X_0 \hookrightarrow \mathcal{Z}$ and $X_1 \hookrightarrow \mathcal{Z}$. Given a Banach couple $\overline{X} = (X_0, X_1)$ we can form the normed spaces $X_0 \cap X_1$ with

$$\|x\|_{X_0 \cap X_1} = \max\left(\|x\|_{X_0}, \|x\|_{X_1}\right)$$

and $X_0 + X_1 = \{x \in \mathcal{Z} : x = x_0 + x_1 \text{ with } x_0 \in X_0, x_1 \in X_1\}$ with the norm

$$\|x\|_{X_0+X_1} = \inf_{x=x_0+x_1} \|x_0\|_{X_0} + \|x_1\|_{X_1}.$$

Theorem 3.2 *Let $\overline{X}$ be a Banach couple. $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces and we have*

$$X_0 \cap X_1 \hookrightarrow X_0, X_1 \hookrightarrow X_0 + X_1.$$

Proof. At first, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $X_0 \cap X_1$. Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X_0 and there exists an $x^{(0)} \in X_0$ so that $\lim_{n\to\infty} x_n = x^{(0)}$. Simultaneously $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X_1 and there is an $x^{(1)} \in X_1$ so that $\lim_{n\to\infty} x_n = x^{(1)}$. Since both X_0 and X_1 are embedded into $\mathcal{Z}$ and its Hausdorffness implies unique limit points, it follows that $x^{(0)} = x^{(1)}$ and $X_0 \cap X_1$ is thus complete.

For the completeness of $X_0 + X_1$ we will follow the arguments in [1, Lemma 2.3.1] and make use of Lemma 2.3. Let $(x_n)_{n\in\mathbb{N}}$ be in $X_0 + X_1$ so that

$$\sum_{n=1}^{\infty} \|x_n\|_{X_0+X_1} < \infty.$$

We can find a decomposition $x_n = x_0^n + x_1^n$ such that

$$\|x_0^n\|_{X_0} + \|x_1^n\|_{X_1} \leq 2\|x_n\|_{X_0+X_1}$$

which implies

$$\sum_{n=1}^{\infty} \|x_0^n\|_{X_0} < \infty, \quad \sum_{n=1}^{\infty} \|x_1^n\|_{X_1} < \infty.$$

With X_0 and X_1 being complete, Lemma 2.3 implies that $\sum_{n=1}^{N} x_0^n$ converges in X_0 and $\sum_{n=1}^{N} x_1^n$ converges in X_1. We put $\sum_{n=1}^{\infty} x_0^n = x_0, \sum_{n=1}^{\infty} x_1^n = x_1$ and $x = x_0 + x_1$. Now we have $x \in X_0 + X_1$. Finally,

$$\left\| x - \sum_{n=1}^{N} x_n \right\|_{X_0+X_1} \leq \left\| x_0 - \sum_{n=1}^{N} x_0^n \right\|_{X_0} + \left\| x_1 - \sum_{n=1}^{N} x_1^n \right\|_{X_1}$$

implies that $\sum_{n=1}^{N} x_n$ converges to x in $X_0 + X_1$. □

Remark 3.3 *The last Theorem shows that $\mathcal{Z} = X_0 + X_1$ is an appropriate choice for our interpolation purposes.*

Besides X_0 and X_1 there are usually a lot more Banach spaces that are embedded between $X_0 \cap X_1$ and $X_0 + X_1$, as Figure 2 suggests. The family of all such spaces is captured in the following term:

Definition 3.4 A Banach space X is called an **intermediate space** (with respect to $\overline{X}$) if

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

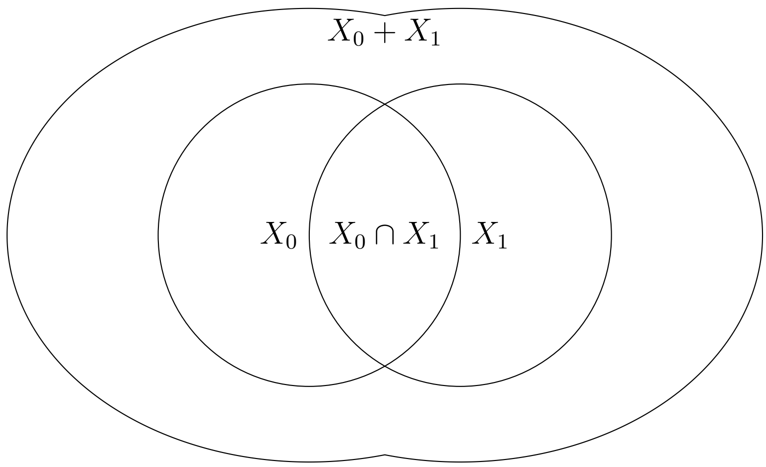

Figure 2: Scheme of the sum and intersection of a Banach couple (X_0, X_1).

3.2 The K-method

The next step is to figure out the intermediate spaces for any given Banach couple. There is a variety of techniques to achieve this, see [17]. With the K-method, which is due to Jack Peetre [13], we will get to know just one of many real interpolation approaches with which we can construct a certain class of intermediate spaces.

Definition 3.5 Let $\overline{X}$ be a Banach couple, $0 < t < \infty$ and $x \in X_0 + X_1$. The **K-functional** is defined as

$$K(x,t) = K(x,t;X_0,X_1) = \inf_{x=x_0+x_1} \|x_0\|_{X_0} + t\|x_1\|_{X_1}.$$

Lemma 3.6 *(a) For fixed x, $K(x,t)$ is a positive, increasing and concave function.*

(b) For $t = 1$ we have $K(x,1) = \|x\|_{X_0+X_1}$.

(c) For fixed $t > 0$, it holds

$$\min(1,t)\|x\|_{X_0+X_1} \leq K(x,t) \leq \max(1,t)\|x\|_{X_0+X_1},$$

hence, $K(x,t)$ is an equivalent norm in $X_0 + X_1$.

Proof. Most of these properties follow immediately from the Definition of the K-functional.

In order to show the concavity we fix $x \in X$ and let $t = \alpha t_0 + (1-\alpha)t_1$ with $t_0, t_1 > 0$ and $0 \leq \alpha \leq 1$. For any decomposition $x = x_0 + x_1$ we have

$$\begin{aligned}\alpha K(x,t_0) + (1-\alpha)K(x,t_1) &\leq \alpha\left(\|x_0\|_{X_0} + t_0\|x_1\|_{X_0}\right) + (1-\alpha)\left(\|x_0\|_{X_0} + t_1\|x_1\|_{X_1}\right)\\ &= \left(\alpha + (1-\alpha)\right)\|x_0\|_{X_0} + \left(\alpha t_0 + (1-\alpha)t_1\right)\|x_1\|_{X_1}\\ &= \|x_0\|_{X_0} + t\|x_1\|_{X_1}.\end{aligned}$$

Taking the Infimum over all decompositions of x yields

$$\alpha K(x,t_0) + (1-\alpha)K(x,t_1) \le K(x, \alpha t_0 + (1-\alpha)t_1)$$

and the concavity of $K(x,\cdot)$. □

Definition 3.7 Let $\overline{X}$ be a Banach couple, $0 < \theta < 1$ and $1 \le q \le \infty$. We call

$$\overline{X}_{\theta,q} = (X_0, X_1)_{\theta,q} = \{x \in X_0 + X_1 : \|x\|_{\theta,q} < \infty\}$$

the **real interpolation space** with the norm

$$\|x\|_{\theta,q} = \|x\|_{(X_0,X_1)_{\theta,q}} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(x,t)\right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} & \text{for } q < \infty, \\ \sup\limits_{0<t<\infty} t^{-\theta} K(x,t) & \text{for } q = \infty. \end{cases}$$

Now we will discuss certain properties of interpolation spaces. At first we show that these spaces inherit the completeness of the respective Banach couple, see [17, page 25].

Theorem 3.8 *Let $\overline{X}$ be a Banach couple, $0 < \theta < 1$ and $1 \le q \le \infty$.*

(a) For $x \in \overline{X}_{\theta,q}$ it holds with some constant C (depending only on θ, q)

$$K(x,t) \le C t^\theta \|x\|_{\theta,q}.$$

(b) $\overline{X}_{\theta,q}$ is an intermediate space.

Proof. (a) Let $q < \infty$ and $t > 0$. Since the K-functional is increasing, we have $K(x,t) \le K(x,s)$ for $t \le s$ which implies

$$\begin{aligned} K(x,t) &= K(x,t) \left(\theta q t^{\theta q}\right)^{\frac{1}{q}} \left(\int_t^\infty s^{-\theta q} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}} \\ &\le \left(\theta q t^{\theta q}\right)^{\frac{1}{q}} \left(\int_t^\infty K(x,s)^q s^{-\theta q} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}} = (\theta q)^{\frac{1}{q}} t^\theta \|x\|_{\theta,q}. \end{aligned}$$

The case $q = \infty$ follows immediately from the definition of $\|\cdot\|_{\theta,\infty}$.

(b) At first we have to show that $\overline{X}_{\theta,q}$ is a Banach space. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\overline{X}_{\theta,q}$. $(K(x_n,t))_{n\in\mathbb{N}}$ is also a Cauchy sequence because of $K(x,t) \le C t^\theta \|x\|_{\theta,q}$. In particular we have $\|x\|_{X_0+X_1} = K(x,1) \le C\|x\|_{\theta,q}$ so that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X_0 + X_1$ as well. Thus, there exists an $x \in X_0 + X_1$ such that $\lim_{n\to\infty} x_n = x$. Now it has to be shown that $(x_n)_{n\in\mathbb{N}}$ converges to x in $\overline{X}_{\theta,q}$.

Let $q < \infty$ ($q = \infty$ is shown analogously) and $0 < \varepsilon < N < \infty$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, for all $\delta > 0$ there exists an $n_0(\delta) \in \mathbb{N}$ such that $\|x_m - x_n\|_{\theta,q} < \delta$ for $m > n \ge n_0(\delta)$ and in particular we have

$$\left(\int_\varepsilon^N \left[t^{-\theta} K(x_m - x_n, t)\right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \le \|x_m - x_n\|_{\theta,q} < \delta.$$

The triangle inequality, the last estimate, the K-functional being increasing and the fact that $K(x,t) \le \max(1,t)\|x\|_{X_0+X_1}$ yield

$$\begin{aligned}
&\left(\int_\varepsilon^N \left[t^{-\theta} K(x-x_n,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\le \left(\int_\varepsilon^N \left[t^{-\theta} K(x_m-x_n,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + \left(\int_\varepsilon^N \left[t^{-\theta} K(x-x_m,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\le \delta + \left(\int_\varepsilon^N \left[t^{-\theta} K(x-x_m,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\le \delta + \left(\int_\varepsilon^N \left[t^{-\theta} K(x-x_m,N)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\le \delta + N\|x-x_m\|_{X_0+X_1} \left(\int_\varepsilon^N t^{-\theta q} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&= \delta + N\|x-x_m\|_{X_0+X_1} \left(\frac{1}{\theta q}\right)^{\frac{1}{q}} \left(\varepsilon^{-\theta q} - N^{-\theta q}\right)^{\frac{1}{q}} \\
&\le \delta + N\|x-x_m\|_{X_0+X_1} \left(\frac{1}{\theta q}\right)^{\frac{1}{q}} \varepsilon^{-\theta}.
\end{aligned}$$

For all $\varepsilon, N, \delta > 0$ there exists an $m_0(\varepsilon, N, \delta) \in \mathbb{N}$ such that $N\|x-x_m\|_{X_0+X_1} \left(\frac{1}{\theta q}\right)^{\frac{1}{q}} \varepsilon^{-\theta} \le \delta$ for all $m \ge m_0(\varepsilon, N, \delta)$ so that

$$\left(\int_\varepsilon^N \left[t^{-\theta} K(x-x_n,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \le 2\delta.$$

With $N \to \infty$ and $\varepsilon \to 0$ we finally have

$$\|x-x_n\|_{\theta,q} \le 2\delta$$

for $n \ge n_0(\delta)$ and therefore $x_n \to x \in \overline{X}_{\theta,q}$ for $n \to \infty$ so that $\overline{X}_{\theta,q}$ is complete.

In order to show that $\overline{X}_{\theta,q}$ is an intermediate space we have to show the embeddings

$$X_0 \cap X_1 \hookrightarrow \overline{X}_{\theta,q} \hookrightarrow X_0 + X_1.$$

$\overline{X}_{\theta,q} \hookrightarrow X_0 + X_1$ is an immediate consequence of (a) because for $t = 1$ and all $x \in \overline{X}_{\theta,q}$ we have $\|x\|_{X_0+X_1} = K(x,1) \le C\|x\|_{\theta,q}$.

Now let $x \in X_0 \cap X_1$. Since $K(x,t) \le \min(1,t)\|x\|_{X_0 \cap X_1}$, we have

$$\|x\|_{\theta,q} = \left(\int_0^\infty \left[t^{-\theta} K(x,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \le \|x\|_{X_0\cap X_1} \left(\int_0^\infty \left[t^{-\theta} \min(1,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

$$= \|x\|_{X_0\cap X_1}\left(\int_0^1 t^{(-\theta+1)q-1}\mathrm{d}t + \int_1^\infty t^{-\theta q-1}\mathrm{d}t\right)^{\frac{1}{q}}$$
$$= \|x\|_{X_0\cap X_1}\left(\frac{1}{(1-\theta)q} + \frac{1}{\theta q}\right)^{\frac{1}{q}}$$

which implies $X_0 \cap X_1 \hookrightarrow \overline{X}_{\theta,q}$. □

After showing that the real interpolation spaces are indeed complete and a family of intermediate spaces, we will now take a closer look at the individual spaces within this family and how they position to one another. At first we will observe that we have an ordered pair of Banach spaces in the notation of real interpolation spaces and switching their roles in the K-functional leads to a different class of interpolation spaces. More importantly, we will then observe that with two parameter there are also two ways to make a real interpolation space bigger, either by increasing q for fixed θ or by simply increasing θ.

Lemma 3.9 *Let $\overline{X}$ be a Banach couple.*

(a) If $0 < \theta < 1$ and $1 \leq q \leq \infty$ then

$$(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}.$$

(b) If $0 < \theta < 1$ and $1 \leq q \leq r \leq \infty$ then

$$\overline{X}_{\theta,q} \hookrightarrow \overline{X}_{\theta,r}.$$

(c) If $X_0 \hookrightarrow X_1, 0 < \theta < \eta < 1$ and $1 \leq r, \tilde{r} \leq \infty$ then

$$\overline{X}_{\theta,r} \hookrightarrow \overline{X}_{\eta,\tilde{r}}.$$

Proof. The original proof can be found in [17, pages 25-26].

(*a*) Let $q < \infty$. Substituting $t^{-1} = s, \mathrm{d}t = s^{-2}\mathrm{d}s$ in the definition of the K-functional yields

$$\|x\|_{(X_0,X_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta}K(x,t;X_0,X_1)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty \left[t^{-\theta}\inf_{x=x_0+x_1}(\|x_0\|_{X_0} + t\|x_1\|_{X_1})\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty \left[t^{-\theta+1}\inf_{x=x_0+x_1}(t^{-1}\|x_0\|_{X_0} + \|x_1\|_{X_1})\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty \left[s^{\theta-1}\inf_{x=x_0+x_1}(s\|x_0\|_{X_0} + \|x_1\|_{X_1})\right]^q \frac{s^{-2}\mathrm{d}s}{s^{-1}}\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \left(\int_0^\infty \left[s^{-(1-\theta)} K(x,s;X_1,X_0) \right]^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}} \\
&= \|x\|_{(X_1,X_0)_{1-\theta,q}}.
\end{aligned}
$$

(b) Let $1 \leq q < r < \infty$. We have to show the existence of a constant C so that $\|x\|_{\theta,r} \leq C\|x\|_{\theta,q}$ for all $x \in \overline{X}_{\theta,q}$. By increasing $t^{-\theta}K(x,t)$ to its supremum over all t and using the fact that there is a constant such that $K(x,t) \leq Ct^\theta \|x\|_{\theta,q}$ for all $x \in \overline{X}_{\theta,q}$ (see Lemma 3.8) we get

$$
\begin{aligned}
\|x\|_{\theta,r} &= \left(\int_0^\infty \left[t^{-\theta} K(x,t) \right]^q \left[t^{-\theta} K(x,t) \right]^{r-q} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \\
&\leq \left[\sup_{0<t<\infty} t^{-\theta} K(x,t) \right]^{\frac{r-q}{r}} \left(\int_0^\infty \left[t^{-\theta} K(x,t) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \\
&\leq (C\|x\|_{\theta,q})^{1-\frac{q}{r}} \left(\int_0^\infty \left[t^{-\theta} K(x,t) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \\
&= C^{1-\frac{q}{r}} \|x\|_{\theta,q}^{1-\frac{q}{r}} \|x\|_{\theta,q}^{\frac{q}{r}} \\
&= C^{1-\frac{q}{r}} \|x\|_{\theta,q}.
\end{aligned}
$$

The arguments for $r = \infty$ with the respective supremum norm are essentially the same.

(c) Let $X_0 \hookrightarrow X_1$, $0 < \theta < \eta < 1$ and $0 < q \leq \infty$. By the definition of real interpolation spaces we know that

$$
X_0 \cap X_1 = X_0 \hookrightarrow \overline{X}_{\theta,q} \hookrightarrow X_1 = X_0 + X_1.
$$

In particular we have $\overline{X}_{\theta,\infty} \hookrightarrow X_1$. Thus, there is a constant C such that for all $x \in \overline{X}_{\theta,\infty}$ it holds

$$
\|x\|_{X_1} \leq C\|x\|_{\theta,\infty}
$$

and the definition of the K-functional yields for $x \in X_1$ that

$$
K(x,t) \leq t\|x\|_{X_1}.
$$

With property (b) we have in particular the embeddings

$$
\overline{X}_{\theta,q} \hookrightarrow \overline{X}_{\theta,\infty} \quad \text{and} \quad \overline{X}_{\eta,1} \hookrightarrow \overline{X}_{\eta,r}.
$$

Thus, all we need to show is that $\overline{X}_{\theta,\infty} \hookrightarrow \overline{X}_{\eta,1}$ for $\theta < \eta$. Let $x \in \overline{X}_{\theta,\infty}$. By the previous observations we have

$$
\|x\|_{\eta,1} = \int_0^1 t^{-\eta} K(x,t) \frac{\mathrm{d}t}{t} + \int_1^\infty t^{-\eta} K(x,t) \frac{\mathrm{d}t}{t}
$$

$$
\begin{aligned}
&\leq \|x\|_{X_1} \int_0^1 t^{-\eta+1} \frac{\mathrm{d}t}{t} + \sup_{0<t<\infty} t^{-\theta} K(x,t) \int_1^\infty t^{-\eta+\theta} \frac{\mathrm{d}t}{t} \\
&\leq C\|x\|_{X_1} + C'\|x\|_{\theta,\infty} \\
&\leq C''\|x\|_{\theta,\infty}
\end{aligned}
$$

with constants C, C', C'' depending only on θ and η. □

4 The interpolation of sequence spaces

So far we have seen two major concepts. On one hand, among the classical ℓ_q sequence space with $q \geq 1$, which are all Banach spaces, we have ℓ_1 as the smallest and ℓ_∞ as the biggest one. On the other hand, within the interpolation space theory we have built a framework where we have a smallest and a biggest Banach space in between which we can construct a family of intermediate Banach spaces.

We will combine these two particular ideas by asking which spaces lie in between ℓ_1 and ℓ_∞ and utilizing real interpolation theory to provide an answer. Since $\ell_1 \hookrightarrow \ell_\infty$, we have

$$\ell_1 = \ell_1 \cap \ell_\infty \hookrightarrow (\ell_1, \ell_\infty)_{\theta,q} \hookrightarrow \ell_1 + \ell_\infty = \ell_\infty.$$

Our next task is to find a precise description for the real interpolation spaces $(\ell_1, \ell_\infty)_{\theta,q}$. Once we have defined the Lorentz spaces $\ell_{p,q}$, it will turn out that

$$(\ell_1, \ell_\infty)_{\theta,q} = \ell_{\frac{1}{1-\theta},q}$$

for $1 \leq q \leq \infty$ and $0 < \theta < 1$. The proof for this is rather technical and while the major steps can be found in [17, pages 125-126], we will work it out more detailed to showcase the difficulties in figuring out the actual interpolation spaces.

4.1 Lorentz spaces

Despite their lexicographical order in the form of $\ell_q \hookrightarrow \ell_r$ for $0 < q < r$, the embedding structure of the ℓ_q-spaces is still pretty coarse. A massive refinement is caused by keeping the general structure of the quasi-norms $\|\cdot\|_{\ell_q}$ but additionally rearranging and weighting the sequences before evaluation.

Definition 4.1 Let $x = (x_n)_{n\in\mathbb{N}}$ be a sequence. If there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$|x_{\pi(1)}| \geq |x_{\pi(2)}| \geq |x_{\pi(3)}| \geq \ldots \geq 0,$$

we call $x^* = (x_n^*)_{n\in\mathbb{N}} = \big(|x_{\pi(n)}|\big)_{n\in\mathbb{N}}$ the **decreasing** (or **non-increasing) rearrangement of** x.

Remark 4.2 *As suggested in [8, page 10], the decreasing rearrangement could also be defined as the sequence* $(x_n^*)_{n\in\mathbb{N}}$ *with the elements*

$$x_n^* := \inf\{m > 0 : |\{k \in \mathbb{N} : x_k \geq m\}| < n\}$$

which is the discrete version (using the counting measure) of the usual definition of the non-increasing rearrangement of a function. In both cases the rearrangement is unique if it exists.

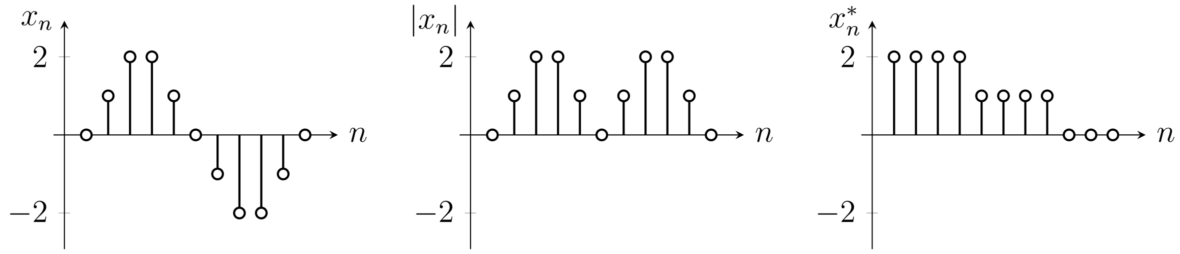

Figure 3: The major steps in forming the decreasing rearrangement of a sequence.

Example 4.3 *We want to observe the effect of a decreasing rearrangement for a single oscillation of a discretized sinusoidal function which is filled up zeros at the end:*

$$(x_n)_{n\in\mathbb{N}} = (0, 1, 2, 2, 1, 0, -1, -2, -2, -1, 0, \ldots)$$

We eliminate negative elements by only considering their absolute values

$$(|x_n|)_{n\in\mathbb{N}} = (0, 1, 2, 2, 1, 0, 1, 2, 2, 1, 0, \ldots)$$

and rearrange the entries of the resulting sequence in descending order

$$(x_n^*)_{n\in\mathbb{N}} = (2, 2, 2, 2, 1, 1, 1, 1, 0, 0, 0, \ldots).$$

In Figure 3 we have a side-by-side comparison of these three states.

Definition 4.4 For $0 < p, q \leq \infty$ the **Lorentz (sequence) space** $\ell_{p,q}$ is defined as

$$\ell_{p,q} = \left\{ x \in c_0 : \|x\|_{\ell_{p,q}} < \infty \right\}$$

with the quasi-norm

$$\|x\|_{\ell_{p,q}} = \left\| \left(n^{\frac{1}{p}-\frac{1}{q}} x_n^* \right)_{n\in\mathbb{N}} \right\|_{\ell_q} = \begin{cases} \left(\sum_{n=1}^{\infty} \left(n^{\frac{1}{p}-\frac{1}{q}} x_n^* \right)^q \right)^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup_{n\in\mathbb{N}} n^{\frac{1}{p}} x_n^* & \text{for } q = \infty. \end{cases}$$

We observe that $\|\cdot\|_{\ell_{q,q}} = \|\cdot\|_{\ell_q}$ and $\ell_{q,q} = \ell_q$ for $0 < q \leq \infty$ because the weights $n^{\frac{1}{p}-\frac{1}{q}}$ disappear for $p = q$ and the rearrangement becomes irrelevant for the value of the ℓ_q-norm.

4.2 Lorentz spaces as interpolation spaces

Now we are ready to show that

$$(\ell_1, \ell_\infty)_{\theta,q} = \ell_{\frac{1}{1-\theta},q}$$

with equivalent norms. For this we need two major steps:

1. Find an explicit expression for the K-functional $K(x, t; \ell_1, \ell_\infty)$.
2. Show the existence of all constants for the equivalences

$$\|x\|^q_{(\ell_1,\ell_\infty)_{\theta,q}} \sim \sum_{k=1}^{\infty} k^{-1-\theta q} \left(\sum_{n=1}^{k} x_n^* \right)^q \sim \|x\|^q_{\ell_{\frac{1}{1-\theta},q}}.$$

The first step will be summarized in the following Lemma, whose proof utilizes only some properties of the K-functional and decreasing rearrangements.

Lemma 4.5 *Let $N \in \mathbb{N}$, $x = (x_k)_{k\in\mathbb{N}} \in \ell_\infty$ with $x = y + z$ and $y \in \ell_1, z \in \ell_\infty$. Then*

$$K(x, t; \ell_1, \ell_\infty) = t x_1^* \quad (0 < t \leq 1)$$

and

$$K(x, N; \ell_1, \ell_\infty) = \sum_{k=1}^{N} x_k^* \quad (N \in \mathbb{N}).$$

Proof. Let $0 < t \leq 1$. We know that $\ell_1 \hookrightarrow \ell_\infty$ with $\|x\|_{\ell_\infty} \leq \|x\|_{\ell_1}$ for all $x \in \ell_\infty$. It follows that

$$\begin{aligned} K(x, t; \ell_1, \ell_\infty) = \inf_{x=y+z} \|y\|_{\ell_1} + t\|z\|_{\ell_\infty} &\geq \inf_{x=y+z} \|y\|_{\ell_\infty} + t\|z\|_{\ell_\infty} \\ &\geq \inf_{x=y+z} \|y + tz\|_{\ell_\infty} \\ &= t \inf_{x=y+z} \left\| \frac{y}{t} + z \right\|_{\ell_\infty} \\ &\geq t \inf_{x=y+z} \|y + z\|_{\ell_\infty} = t\|x\|_{\ell_\infty} = t x_1^*. \end{aligned}$$

The converse inequality is obtained by decomposing x into $y = 0, z = x$. Then we get

$$K(x, t; \ell_1, \ell_\infty) \leq \|y\|_{\ell_1} + t\|z\|_{\ell_\infty} = t\|x\|_{\ell_\infty} = t x_1^*.$$

Now let $N \in \mathbb{N}$ and again we have to show both inequalities. Without loss of generality we can assume that x is a decreasing rearrangement, so that $x_k = x_k^*$ for all $k \in \mathbb{N}$. At first let $x_k^* = y_k + z_k$ be any decomposition. Then we have

$$\sum_{k=1}^{N} x_k^* \leq \sum_{k=1}^{N} |y_k + z_k| \leq \sum_{k=1}^{N} |y_k| + \sum_{k=1}^{N} |z_k| \leq \|y\|_{\ell_1} + N \sup_k |z_k|$$

for all decompositions of x_k^*, so in particular it holds

$$\sum_{k=1}^{N} x_k^* \leq \inf_{x=y+z} \|y\|_{\ell_1} + N\|z\|_{\ell_\infty} = K(x, N; \ell_1, \ell_\infty).$$

For the converse inequality we choose specific y and z in the following way

$$y_k = \begin{cases} x_k^* - x_N^* & \text{for } k = 1, \dots, N, \\ 0 & \text{for } k \geq N+1, \end{cases}$$

$$z_k = x_k^* - y_k = \begin{cases} x_k^* - (x_k^* - x_N^*) = x_N^* & \text{for } k = 1, \dots, N, \\ x_k^* & \text{for } k \geq N+1. \end{cases}$$

For the ℓ_1-norm we use the fact that y is a decreasing and positive sequence and get

$$\|y\|_{\ell_1} = \sum_{k=1}^{N} |x_k^* - x_N^*| = \sum_{k=1}^{N} (x_k^* - x_N^*) = \sum_{k=1}^{N} x_k^* - N x_N^*$$

and for the ℓ_∞-norm we have

$$\|z\|_{\ell_\infty} = \max\left(\sup_{k=1,\dots,N} x_N^*, \sup_{k \geq N+1} x_k^* \right) = x_N^*,$$

yielding in total

$$K(x, N; \ell_1, \ell_\infty) \leq \|y\|_{\ell_1} + N\|z\|_{\ell_\infty} = \left(\sum_{k=1}^{N} x_k^* - N x_N^* \right) + N x_N^* = \sum_{k=1}^{N} x_k^*.$$

□

Remark 4.6 *If x is not decreasing then it is not guaranteed that the first N entries of x are also its N absolute largest entries. Therefore we introduce an index set $I \subset \mathbb{N}$ with $|I| = N$ associated with N entries of largest absolute value of x so that I is a set of indices such that for all $i \in I, j \in \mathbb{N} \setminus I$ it holds $|x_i| \geq |x_j|$. By $x_N, N \in I$ we still want to denote the smallest element (of absolute value) among those identified with I, hence, for all $i \in I$ it holds $|x_N| \leq |x_i|$. Now we can adjust the decomposition of x into y and z in the following way:*

$$y_k = \begin{cases} x_k - \frac{x_k}{|x_k|}|x_N| & \text{for } k \in I, \\ 0 & \text{else}, \end{cases}$$

$$z_k = x_k - y_k = \begin{cases} \frac{x_k}{|x_k|}|x_N| & \text{for } k \in I, \\ x_k & \text{else}. \end{cases}$$

The second major step of showing the equivalence of the interpolation space norm and the Lorentz norm will be divided into two parts. The first one is the following Lemma.

Lemma 4.7 *Let $1 \leq q < \infty, 0 < t < \infty$ and $x = (x_k)_{k \in \mathbb{N}} \in \ell_\infty$. Then*

$$\|x\|^q_{(\ell_1, \ell_\infty)_{\theta,q}} \sim \sum_{k=1}^{\infty} k^{-1-\theta q} \left(\sum_{n=1}^{k} x_n^* \right)^q$$

with some constants depending on q and θ.

Proof. At first we recall that the K-functional is increasing so that

$$K(x,t) \leq K(x,u)$$

for all $0 < t \leq u$. Now we split the interpolation norm in the following way

$$\begin{aligned}
\|x\|^q_{(\ell_1,\ell_\infty)_{\theta,q}} &= \int_0^\infty \left(t^{-\theta}K(x,t)\right)^q \frac{\mathrm{d}t}{t} \\
&= \int_0^1 t^{-1-\theta q}K(x,t)^q \,\mathrm{d}t + \int_1^\infty t^{-1-\theta q}K(x,t)^q \,\mathrm{d}t \\
&= \int_0^1 t^{-1-\theta q}K(x,t)^q \,\mathrm{d}t + \sum_{k=1}^\infty \int_k^{k+1} t^{-1-\theta q}K(x,t)^q \,\mathrm{d}t.
\end{aligned}$$

With the previously shown expression of $K(x,t,\ell_1,\ell_\infty)$ for $0 < t \leq 1$ we can write

$$\int_0^1 t^{-1-\theta q}K(x,t)^q \,\mathrm{d}t = \int_0^1 t^{-1-\theta q}(tx_1^*)^q \,\mathrm{d}t = \int_0^1 t^{q(1-\theta)-1}(x_1^*)^q \,\mathrm{d}t = \frac{(x_1^*)^q}{q(1-\theta)}$$

and end up with

$$\|x\|^q_{(\ell_1,\ell_\infty)_{\theta,q}} = \left(\frac{1}{q(1-\theta)}\right)(x_1^*)^q + \sum_{k=1}^\infty \int_k^{k+1} t^{-1-\theta q}K(x,t)^q \,\mathrm{d}t. \tag{1}$$

Now we have to find upper and lower bounds for the series in (1). In order to estimate above we observe that it holds $K(x,t) \leq K(x,k+1)$ and $t^{-1-\theta q} \leq k^{-1-\theta q}$ for $k \leq t < k+1$. Plugging this in yields

$$\begin{aligned}
\sum_{k=1}^\infty \int_k^{k+1} t^{-1-\theta q}K(x,t)^q \mathrm{d}t &\leq \sum_{k=1}^\infty k^{-1-\theta q}K(x,k+1)^q \\
&= \sum_{k=2}^\infty (k-1)^{-1-\theta q}K(x,k)^q \\
&= \sum_{k=2}^\infty \left(\frac{k-1}{k}\right)^{-1-\theta q} k^{-1-\theta q}K(x,k)^q \\
&\leq \sum_{k=2}^\infty k^{-1-\theta q}K(x,k)^q.
\end{aligned} \tag{2}$$

The lower bound is obtained analogously by observing that for $k \leq t < k+1$ it holds $K(x,t) \geq K(x,k)$ and $t^{-1-\theta q} \geq (k+1)^{-1-\theta q}$, thus

$$\sum_{k=1}^\infty \int_k^{k+1} t^{-1-\theta q}K(x,t)^q \mathrm{d}t \geq \sum_{k=1}^\infty (k+1)^{-1-\theta q}K(x,k)^q$$

$$= \sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^{-1-\theta q} k^{-1-\theta q} K(x,k)^q$$
$$\geq \sum_{k=2}^{\infty} k^{-1-\theta q} K(x,k)^q. \qquad (3)$$

By combining (1) and (2) we have

$$\|x\|^q_{(\ell_1,\ell_\infty)_{\theta,q}} \leq \max\left(\frac{1}{q(1-\theta)}, 1\right) \sum_{k=1}^{\infty} k^{-1-\theta q} \left(\sum_{n=1}^{k} x_n^*\right)^q ,$$

and combining (1) and (3) yields

$$\|x\|^q_{(\ell_1,\ell_\infty)_{\theta,q}} \geq \min\left(\frac{1}{q(1-\theta)}, 1\right) \sum_{k=1}^{\infty} k^{-1-\theta q} \left(\sum_{n=1}^{k} x_n^*\right)^q .$$

□

Before we continue with the second part we have to establish two specific estimates for partial harmonic series.

Lemma 4.8 *Let $n \in \mathbb{N}, 0 < \lambda < 1$ and $1 < \gamma < \infty$. It holds*

(a) $\displaystyle\sum_{k=1}^{n} k^{-\lambda} \leq \left(\frac{1}{1-\lambda}\right) n^{1-\lambda}$,

(b) $\displaystyle\sum_{k=n}^{\infty} k^{-\gamma} \leq \left(1 + \frac{1}{\gamma-1}\right) n^{1-\gamma}$.

Proof. If $n \in \mathbb{N}$ and $0 < \lambda < 1$ then

$$\sum_{k=1}^{n} k^{-\lambda} \leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} x^{-\lambda}\, \mathrm{d}x = 1 + \int_{1}^{n} x^{-\lambda}\, \mathrm{d}x = 1 + \frac{n^{1-\lambda}-1}{1-\lambda} = \frac{n^{1-\lambda}-\lambda}{1-\lambda} \leq \frac{n^{1-\lambda}}{1-\lambda}.$$

For $n \in \mathbb{N}$ and $1 < \gamma < \infty$ we have with the same kind of argument that

$$\sum_{k=n}^{\infty} k^{-\gamma} \leq n^{-\gamma} + \int_{n}^{\infty} x^{-\gamma}\, \mathrm{d}x = n^{-\gamma} - \frac{n^{1-\gamma}}{1-\gamma} = \left(\frac{1}{n} + \frac{1}{\gamma-1}\right) n^{1-\gamma} \leq \left(1 + \frac{1}{\gamma-1}\right) n^{1-\gamma}.$$

□

Lemma 4.9 *Let $1 \leq q < \infty$ and $x = (x_k)_{k\in\mathbb{N}} \in \ell_\infty$. Then*

$$\sum_{k=1}^{\infty} k^{-\theta q-1} \left(\sum_{n=1}^{k} x_n^*\right)^q \sim \sum_{k=1}^{\infty} k^{q(1-\theta)-1} (x_k^*)^q = \|x\|^q_{\ell_{p,q}}$$

with $1-\theta = \frac{1}{p}$ and some constants depending on q and θ.

Proof. The lower bound is obtained by

$$\sum_{k=1}^{\infty} k^{-\theta q-1}\left(\sum_{n=1}^{k} x_n^*\right)^q \geq \sum_{k=1}^{\infty} k^{-\theta q-1}\left(k x_k^*\right)^q = \sum_{k=1}^{\infty} k^{q(1-\theta)-1}(x_k^*)^q.$$

On the other hand, let $1 > \theta > \varepsilon > 0$ and $\frac{1}{q}+\frac{1}{q'}=1$. By the Hölder inequality and changing the order of summation it holds that

$$\begin{aligned}\sum_{k=1}^{\infty} k^{-\theta q-1}\left(\sum_{n=1}^{k} x_n^*\right)^q &= \sum_{k=1}^{\infty} k^{-\theta q-1}\left(\sum_{n=1}^{k}\left(x_n^* n^{(1-\theta)-\frac{1}{q}+\varepsilon}\right) n^{\theta-\frac{1}{q'}-\varepsilon}\right)^q \\ &\leq \sum_{k=1}^{\infty} k^{-\theta q-1}\sum_{n=1}^{k}\left(x_n^*\right)^q n^{(1-\theta)q-1+\varepsilon q}\left(\sum_{n=1}^{k} n^{(\theta-\varepsilon)q'-1}\right)^{\frac{q}{q'}} \\ &= \sum_{n=1}^{\infty}\left(x_n^*\right)^q n^{(1-\theta)q-1+\varepsilon q}\sum_{k=n}^{\infty} k^{-\theta q-1}\left(\sum_{l=1}^{k} l^{(\theta-\varepsilon)q'-1}\right)^{\frac{q}{q'}}.\end{aligned}$$

Now we use the fact that

$$\sum_{l=1}^{k} l^{(\theta-\varepsilon)q'-1} \leq \begin{cases} k\cdot k^{(\theta-\varepsilon)q'-1} = k^{(\theta-\varepsilon)q'} & \text{for } (\theta-\varepsilon)q'-1\geq 0, \\ \left(\frac{1}{(\theta-\varepsilon)q'}\right) k^{(\theta-\varepsilon)q'} & \text{for } (\theta-\varepsilon)q'-1<0,\end{cases}$$

where the second case follows by Lemma 4.8 (a). We put $C_1 := \max\left(1, \frac{1}{(\theta-\varepsilon)q'}\right)$ and obtain

$$\sum_{k=1}^{\infty} k^{-\theta q-1}\left(\sum_{n=1}^{k} x_n^*\right)^q \leq C_1\sum_{n=1}^{\infty}\left(x_n^*\right)^q n^{(1-\theta)q-1+\varepsilon q}\sum_{k=n}^{\infty} k^{-\theta q-1+\theta q-\varepsilon q}.$$

By Lemma 4.8 (b) we have

$$n^{\varepsilon q}\sum_{k=n}^{\infty} k^{-(1+\varepsilon q)} \leq n^{\varepsilon q}\left(1+\frac{1}{\varepsilon q}\right) n^{-\varepsilon q} = \left(1+\frac{1}{\varepsilon q}\right) =: C_2,$$

thus, we have finally found some constant $C' = C_1\cdot C_2$ such that

$$\sum_{k=1}^{\infty} k^{-\theta q-1}\left(\sum_{n=1}^{k} x_n^*\right)^q \leq C'\sum_{n=1}^{\infty}\left(x_n^*\right)^q n^{(1-\theta)q-1}.$$

□

Theorem 4.10 *Let $1 \leq q \leq \infty$ and $p = \frac{1}{1-\theta}$ with $0 < \theta < 1$.*
Then we have (with equivalent norms)

$$(\ell_1, \ell_\infty)_{\theta,q} = \ell_{p,q}.$$

Proof. Let $q < \infty$. With Lemma 4.7 and Lemma 4.9 we have already shown that

$$\|x\|_{(\ell_1,\ell_\infty)_{\theta,q}} \sim \|x\|_{\ell_{p,q}}.$$

For the case $q = \infty$ we only need to slightly adjust the proofs of these Lemmas. □

As these kind of proofs are quite lengthy we will leave it at the one above, but it should be mentioned that the last result can be considered as a special case of even more general Theorems. Since real interpolation spaces $\overline{X}_{\theta,q}$ are always Banach spaces, they themselves can form feasible Banach couples $\left(\overline{X}_{\theta_0,q_0}, \overline{X}_{\theta_1,q_1}\right)$ that yield certain families of interpolation spaces. This idea of interpolating between interpolation spaces is captured in the so called *reiteration theorem*, see [1, pages 50-51]. The next result (see [17, page 127, Theorem 2]) shows the application of this theorem for Banach couples of Lorentz spaces:

Theorem 4.11 *Let $0 < \theta < 1$ and $1 < p_0, p_1 < \infty$ with $p_0 \neq p_1$ such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

and let $1 \leq q, q_0, q_1 \leq \infty$. It holds (with equivalent norms)

$$(\ell_{p_0,q_0}, \ell_{p_1,q_1})_{\theta,q} = \ell_{p,q}.$$

If we had shown this statement first, then everything concerning the real interpolation of ℓ_q-spaces would have been just a special case. For example, if we put $p_0 = q_0, p_1 = q_1$, then

$$(\ell_{q_0}, \ell_{q_1})_{\theta,q} = \ell_{p,q} \quad \text{and} \quad \ell_{q_0} \hookrightarrow \ell_{p,q} \hookrightarrow \ell_{q_1}$$

and additionally putting $q_0 = 1$ and $q_1 = \infty$ yields the content of Theorem 4.10.

4.3 Properties of Lorentz spaces

It certainly is possible to study Lorentz spaces without ever mentioning interpolation space theory. For $1 \leq p, q \leq \infty$ proofs of their completeness and embeddings can be found for example in [10], which can be adapted for values $0 < p, q < 1$. But because we have proven that for $1 < p < \infty$ and $1 \leq q \leq \infty$ Lorentz spaces are interpolation spaces, we can go back to the general interpolation theory to collect a couple of immediate consequences caused by choosing the particular Banach couple (ℓ_1, ℓ_∞). About completeness we can say the following, with (b) and (c) being implied by Lemma 3.8:

Theorem 4.12 *(a) $(\ell_{p,q}, \|\cdot\|_{\ell_{p,q}})$ is a quasi-Banach space for $0 < p < \infty, 0 < q \leq \infty$.*

(b) $(\ell_{p,q}, \|\cdot\|_{\ell_{p,q}})$ *is a Banach space for* $1 < p < \infty, 1 \le q \le p$.

(c) For $1 < p < \infty, 1 \le q \le \infty$ *there is an equivalent norm* $\|\cdot\|'_{\ell_{p,q}}$ *in* $\ell_{p,q}$ *such that* $(\ell_{p,q}, \|\cdot\|'_{\ell_{p,q}})$ *is a Banach space.*

With two parameters the embedding structure of $\ell_{p,q}$-spaces also becomes a little bit more complicated, as shown in Figure 4. Now there are two ways to make them bigger - either by increasing the value of p regardless of q or by fixing p and increasing q.

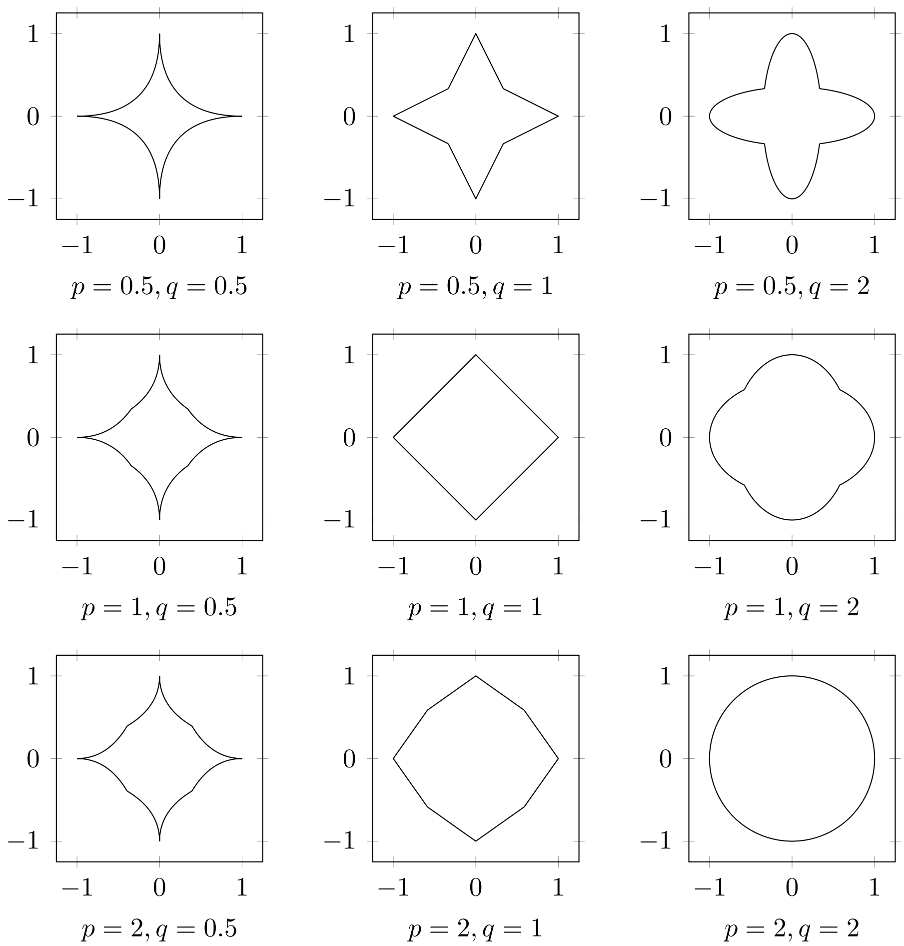

Figure 4: Side-by-side comparison of two-dimensional unit balls $B_{\ell_{p,q}}$.

Theorem 4.13 *Let* $0 < p_1, p_2 < \infty, 0 < q_1, q_2 \le \infty$. *It holds*

(a) $\ell_{p_1,q_1} \hookrightarrow \ell_{p_2,q_2}$ *for* $p_1 < p_2$,

(b) $\ell_{p,q_1} \hookrightarrow \ell_{p,q_2}$ *for* $p = p_1 = p_2$ *and* $q_1 \le q_2$.

Once again, the cases in which the Lorentz spaces are normed have turned into a Corollary of Lemma 3.9.

4.4 Remark on the interpolation of quasi-normed spaces

Originally the definition of Lorentz spaces included the values $0 < p, q < 1$ for which they turned out to be quasi-Banach spaces. This naturally leads to the question whether these particular spaces can be interpolated as well.

In [16] we find an approach to extend the real interpolation methods and in particular the K-method for couples of quasi-Banach spaces. For this we need another kind of norm: Let X be a vector space over $\mathbb{K}$ and $0 < r \leq 1$. The functional $\|\cdot\| : X \to [0, \infty)$ is called an r**-norm**, if for all $\alpha \in \mathbb{K}$ and $x, y \in X$ it holds

(N1) $\|x\| = 0 \Leftrightarrow x = 0$,

(N2r) $\|\alpha x\| = |\alpha|^r \|x\|$,

(N3) $\|x + y\| \leq \|x\| + \|y\|$.

$(X, \|\cdot\|)$ is therefore called r**-normed space** or r**-Banach space**, if it is complete.

Y. Sagher proceeds in [16] by introducing and proving all majors notions and properties of interpolation space theory for couples of r-Banach spaces, that we have already seen for Banach couples. Since we know by the *Aoki-Rolewicz theorem* [15] that every quasi-norm is also an r-norm, he had established the real interpolation methods for quasi-Banach couples as well. Furthermore he proved several results, for example the *reiteration theorem*, in the context of *Lorentz function spaces* $L_{p,q}$. While these proofs are adjustable for their discrete counterparts in form of the Lorentz sequence spaces $\ell_{p,q}$, we would need in both cases quite a few more advanced techniques and therefore omit the details.

The sole purpose of this remark is to indicate that the reiteration of Lorentz spaces as seen in Theorem 4.11 is certainly possible for values $0 < p_0, p_1, q_0, q_1 \leq 1$ as well and that there are ways to describe the quasi-normed Lorentz spaces as interpolation spaces, too. However, this is only meant to be an outlook on even more general theory and since the previously mentioned interpolation theory was only discussed thoroughly for Banach couples, we will from now on restrict statements concerning $\ell_{p,q}$-spaces to values $p, q \geq 1$.

5 Entropy numbers

We have to introduce a couple more terms and properties before we start investigating the behavior of operators and their properties under interpolation. We will start out by recalling the necessary definitions to describe compact sets and operators and introduce an equivalent description of relative compactness (see again [9]).

Definition 5.1 Let X be a quasi-Banach space and $M \subset X$.

(a) Let I be any index set and $U_i \subset X$ open for $i \in I$. The system $\{U_i\}_{i \in I}$ is an **open cover** of M if $M \subset \bigcup_{i \in I} U_i$.

(b) Let $\{U_i\}_{i \in I}$ be an open cover of M and $J \subset I$ with $|J| \in \mathbb{N}$. If $M \subset \bigcup_{j \in J} U_j$ then we call $\{U_j\}_{j \in J}$ is a **finite cover** of M.

(c) We put $U(x, \varepsilon) := \{y \in X : \|x - y\|_X < \varepsilon\}$.

(d) Let $\varepsilon > 0$. $K \subset X$ is an ε**-net** for M, if $M \subset \bigcup_{x \in K} U(x, \varepsilon)$.

(e) M is **compact**, if every open cover of M contains a finite cover of M.

(f) M is **relative compact**, if the closure of M is compact.

As we can see, an ε-net is just an open cover with a specific system of open sets $U(x, \varepsilon)$.

Theorem 5.2 *Let X be a quasi-Banach space and $M \subset X$. M is relative compact if and only if for all $\varepsilon > 0$ there exists a finite ε-net for M.*

Definition 5.3 Let X, Y be quasi-Banach spaces and B_X the unit ball of X.

(a) A linear map $T : X \to Y$ is **compact**, if $T(B_X)$ is relative compact.

(b) $\mathcal{K}(X, Y) := \{T \in \mathcal{L}(X, Y) : T \text{ compact}\}$.

Apparently, the compactness of an operator is characterized by the possibility to find a finite ε-net for $T(B_X)$ for all $\varepsilon > 0$. Following the example of [2] we want to go a step further by restricting the number of sets involved in such nets. More precisely, after fixing a certain number of open sets we want to know for which values $\varepsilon > 0$ we can still form an ε-net for $T(B_X)$ and in particular what the smallest one amongst them is. This problem motivates the following definitions:

Definition 5.4 Let X, Y be quasi-Banach spaces and $T \in \mathcal{L}(X, Y)$.

(a) The **n-th (dyadic) inner entropy number of T** is defined by

$$\begin{aligned}\varphi_n(T) = \sup\{\delta > 0 : &\text{ there exist } x_1, x_2, \ldots, x_p \text{ in } B_X, p > 2^{n-1}, \\ &\text{such that } \|Tx_i - Tx_j\|_Y > 2\delta \text{ for } i \neq j, 1 \leq i, j \leq p\}.\end{aligned}$$

(b) The n**-th (dyadic) entropy number of T** is defined by

$$e_n(T) = \inf \Bigg\{ \varepsilon > 0 : \text{ there exist } y_1, y_2, \ldots, y_q \text{ in } Y, q \leq 2^{n-1}, \\ \text{such that } T(B_X) \subset \bigcup_{j=1}^{q} (y_j + \varepsilon B_Y) \Bigg\}.$$

(c) The **measure of non-compactness of T** is defined by

$$\beta(T) = \inf \Bigg\{ \eta > 0 : \text{ there is a finite number of } y_1, y_2, \ldots, y_q \text{ in } Y \\ \text{such that } T(B_X) \subset \bigcup_{j=1}^{q} (y_j + \eta B_Y) \Bigg\}.$$

Clearly, $\beta(T) = \lim_{n\to\infty} e_n(T)$.

Remark 5.5 *For the rest of this section we keep on using the introduced short notation $\beta(T), e_n(T)$ and $\varphi_n(T)$. Later on we repeatedly have to specify the domain and co-domain of T in order to avoid ambiguities and write*

$$\beta(T : X \to Y), \quad e_n(T : X \to Y) \quad \textit{and} \quad \varphi_n(T : X \to Y).$$

Overall we will discuss only a small selection of the properties of these numbers. For more details see [2]. At first we will observe that the entropy numbers are always decreasing and bounded above by the operator norm. Afterwards we will prove the equivalence of entropy and inner entropy numbers.

Theorem 5.6 *Let X, Y be quasi-Banach spaces and $T \in \mathcal{L}(X, Y)$. Then*

(a) $\|T\|_{X\to Y} \geq e_1(T) \geq e_2(T) \geq \ldots \geq 0$,

(b) $\|T\|_{X\to Y} = e_1(T)$ *if Y is a Banach space.*

Proof. We will only show (b) and slightly modify the original proof in [8, page 8]. First of all, we know $\|T\|_{X\to Y} = \inf\{\lambda \geq 0 : T(B_X) \subset \lambda B_Y\}$ and thus $\|T\|_{X\to Y} \geq e_1(T)$.

Now let Y be a Banach space. If $T(B_X) \subset y_0 + \lambda B_Y$ for $y_0 \in Y$ and $\lambda \geq 0$ then for all $x \in B_X$ there are $y_1, y_2 \in B_Y$ such that $Tx = y_0 + \lambda y_1$ and $-Tx = y_0 + \lambda y_2$ yielding $2Tx = \lambda(y_1 - y_2)$. Since $\|\cdot\|_Y$ is considered to be a norm and $\|y_1\|_Y \leq 1, \|y_2\|_Y \leq 1$, we get

$$\|Tx\|_Y = \frac{\lambda}{2} \|y_1 - y_2\|_Y \leq \frac{\lambda}{2} (\|y_1\|_Y + \|y_2\|_Y) \leq \lambda$$

with the right hand side being independent of x. Taking the supremum over all x implies $\|T\|_{X\to Y} \leq \lambda$ and taking the infimum over all λ for which $e_1(T) < \lambda$ finally yields $\|T\|_{X\to Y} \leq e_1(T)$. □

Theorem 5.7 *Let X, Y be quasi-Banach spaces and $T \in \mathcal{L}(X, Y)$. Then*

$$\varphi_n(T) \leq e_n(T) \leq 2\varphi_n(T)$$

for all $n \in \mathbb{N}$.

Proof. We present the proof found in [2, pages 7-8] which was adjusted accordingly to the dyadic version of the (inner) entropy number of an operator.

In order to show the left inequality, let $\varepsilon > e_n(T)$ and $0 < \delta < \varphi_n(T)$. Then there are $y_1, y_2, \ldots, y_q$ in Y with $q \leq 2^{n-1}$ such that $T(B_X) \subset \bigcup_{j=1}^{q}(y_j + \varepsilon B_Y)$, and there exist $x_1, x_2, \ldots, x_p$ in B_X with $p > 2^{n-1}$ such that $\|Tx_i - Tx_j\|_Y > 2\delta$ for $i \neq j$. Since $q \leq 2^{n-1} < p$, there are more such x_i than sets $y_j + \varepsilon B_Y$. Hence, there must be a $y_j + \varepsilon B_Y$ containing two Tx_i. Suppose $Tx_i \in y_j + \varepsilon B_Y$ and $Tx_k \in y_j + \varepsilon B_Y$. Then we have

$$\|Tx_i - Tx_k\|_Y \leq \|Tx_i - y_j\|_Y + \|y_j - Tx_k\|_Y \leq 2\varepsilon$$

and therefore $2\delta < \|Tx_i - Tx_k\|_Y \leq 2\varepsilon$ which implies $\varphi_n(T) \leq e_n(T)$.

Conversely, for any δ with $\varphi_n(T) < \delta$ there exists a maximal subset $\{x_1, x_2, \ldots, x_p\}$ of B_X with $p > 2^{n-1}$ such that $\|Tx_i - Tx_j\| > 2\delta$ for $i \neq j$. The maximality is due to the fact that for any $x \in B_X$ there is at least one element x_i in this subset such that

$$\|Tx - Tx_i\| \leq 2\delta.$$

Hence,

$$T(B_X) \subset \bigcup_{j=1}^{q}(y_j + 2\delta B_Y).$$

This implies $e_n(T) \leq 2\delta$ and thus, $e_n(T) \leq 2\varphi_n(T)$. □

At last we will prove the equivalence of an operator being compact and the respective entropy numbers being a null sequence. This characterization will be crucial for the next chapter in which we want to prove the non-compactness of certain operators.

Theorem 5.8 *Let X, Y be quasi-Banach spaces and $T \in \mathcal{L}(X, Y)$. Then*

$$T \in \mathcal{K}(X, Y) \quad \Leftrightarrow \quad \lim_{n\to\infty} e_n(T) = 0.$$

Proof. Let $T \in \mathcal{K}(X, Y)$. Therefore $T(B_X)$ is relatively compact and thus, for every $\varepsilon > 0$ there is a finite ε-net for $T(B_X)$. Without loss of generality we can assume that these finite ε-nets consist of a dyadic number of open sets, hence, $\lim_{n\to\infty} e_n(T) = 0$.

Conversely, let $\lim_{n\to\infty} e_n(T) = 0$. For all $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n > n_\varepsilon$ it holds $e_n(T) < \varepsilon$ and the involved elements $y_1, y_2, \ldots, y_{2^{n-1}}$ form a finite ε-net for $T(B_X)$. Since this holds for all $\varepsilon > 0$, it follows that $T(B_X)$ is relative compact and $T \in \mathcal{K}(X, Y)$. □

Because $0 \neq \beta(T)(= \lim_{n\to\infty} e_n(T))$ is sufficient to determine that T is not compact, the value of $\beta(T)$ might be interpreted as a way to quantify how non-compact T is. We might go so far as to say that it is a measure of non-compactness - hence its name.

6 Operators between interpolation spaces

Finally we want to look at the behavior of operators and their respective entropy numbers under interpolation. Previously in the theory of interpolation spaces we started out with a Banach couple $\overline{X} = (X_0, X_1)$ yielding a pair of outer Banach spaces $X_0 \cap X_1$ and $X_0 + X_1$ in between which the spaces X_0, X_1 and $\overline{X}_{\theta,q}$ were embedded. Taking another Banach couple $\overline{Y} = (Y_0, Y_1)$ yields the same embedding structure with Y_0, Y_1 and $\overline{Y}_{\theta,q}$ lying in between $Y_0 \cap Y_1$ and $Y_0 + Y_1$.

In order to study operators between interpolation spaces we initially define them in terms of the biggest appearing spaces so that they map from $X_0 + X_1$ to $Y_0 + Y_1$ and restrict them afterwards to X_0, X_1 and $\overline{X}_{\theta,q}$. In the end we want to describe relationships between these restrictions. We will see that certain properties of the operators over X_0 and X_1 are handed down to the operator over $\overline{X}_{\theta,q}$. Furthermore we will have a close look at how the entropy numbers of these three restrictions relate.

6.1 Compactness under interpolation

At first we introduce the notation of operators between Banach couples:

Definition 6.1 (a) For Banach couples $\overline{X}$ and $\overline{Y}$ we denote by $T \in \mathcal{L}\left(\overline{X}, \overline{Y}\right)$ that

$$T : X_0 + X_1 \to Y_0 + Y_1$$

is linear and that the restrictions to X_0 and Y_0

$$T|_{X_0} : X_0 \to Y_0 \quad \text{and} \quad T|_{X_1} : X_1 \to Y_1$$

are linear and bounded.

(b) If the Banach couple $\overline{X}$ or $\overline{Y}$ is reduced to a single space, so that $X_0 = X_1 = X$ or $Y_0 = Y_1 = Y$, we just write $T \in \mathcal{L}\left(X, \overline{Y}\right)$ or $T \in \mathcal{L}\left(\overline{X}, Y\right)$.

In Figure 5 we have summarized all the previously mentioned restrictions in a single scheme. In the sequel we will denote those by just T as well and additionally state the domain and co-domain, when we need to avoid ambiguity.

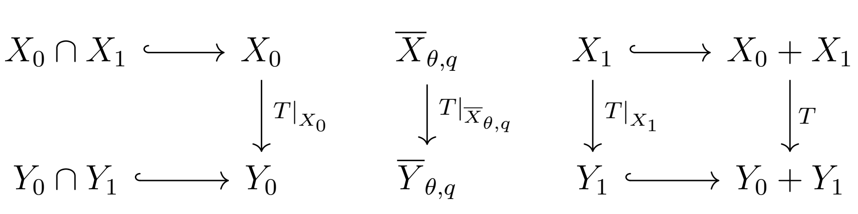

Figure 5: Scheme of the definition of an operator $T \in \mathcal{L}\left(\overline{X}, \overline{Y}\right)$.

The next result is the first one of the kind that the restriction of T to $\overline{X}_{\theta,q}$ inherits a property of $T : X_0 \to Y_0$ or $T : X_1 \to Y_1$ or both, which in this case will be continuity (or boundedness).

Theorem 6.2 (Interpolation property, [3, Theorem 1.3]) *Let $\overline{X}$ and $\overline{Y}$ be Banach couples and $T \in \mathcal{L}\left(\overline{X}, \overline{Y}\right)$. For all $0 < \theta < 1$ and $1 \leq q \leq \infty$*

$$T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}$$

is a bounded operator and it holds

$$\|T\|_{\overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}} \leq \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}. \tag{4}$$

Proof. Let $\overline{X}$ and $\overline{Y}$ be Banach couples, $T \in \mathcal{L}\left(\overline{X}, \overline{Y}\right)$ and $x_0 + x_1 = x \in X_0 + X_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. Then $Tx_0 + Tx_1 = Tx \in Y_0 + Y_1$ with $Tx_0 \in Y_0$ and $Tx_1 \in Y_1$. We put

$$M_0 := \|T\|_{X_0 \to Y_0} \quad \text{and} \quad M_1 := \|T\|_{X_1 \to Y_1}.$$

The definition of the K-functional as well as $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ being bounded yield

$$\begin{aligned}
K(Tx, t; Y_0, Y_1) &= \inf \left\{ \|Tx_0\|_{Y_0} + t\|Tx_1\|_{Y_1} : Tx = Tx_0 + Tx_1 \right\} \\
&\leq \|Tx_0\|_{Y_0} + t\|Tx_1\|_{Y_1} \\
&\leq M_0 \|x_0\|_{X_0} + tM_1 \|x_1\|_{X_1} \\
&= M_0 \left(\|x_0\|_{X_0} + t\frac{M_1}{M_0} \|x_1\|_{X_1} \right),
\end{aligned}$$

so that by taking the infimum over all decompositions of x we get

$$K(Tx, t; Y_0, Y_1) \leq M_0 K\left(x, t\frac{M_1}{M_0}; X_0, X_1\right).$$

With this estimate and the substitution $s := t\frac{M_1}{M_0}, \mathrm{d}t = \frac{M_0}{M_1}\mathrm{d}s$ we have

$$\begin{aligned}
\|Tx\|_{\overline{Y}_{\theta,q}} &= \left(\int_0^\infty \left[t^{-\theta} K(Tx, t; Y_0, Y_1) \right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\
&\leq \left(\int_0^\infty \left[t^{-\theta} M_0 K\left(x, t\frac{M_1}{M_0}; X_0, X_1\right) \right]^q \frac{1}{t}\mathrm{d}t \right)^{\frac{1}{q}} \\
&= M_0 \left(\int_0^\infty \left[\left(\frac{M_0}{M_1} s \right)^{-\theta} K\left(x, s; X_0, X_1\right) \right]^q \left(\frac{M_1}{M_0} \frac{1}{s} \right) \frac{M_0}{M_1}\mathrm{d}s \right)^{\frac{1}{q}} \\
&= M_0 \left(\frac{M_1}{M_0} \right)^{\theta} \left(\int_0^\infty \left[s^{-\theta} K\left(x, s; X_0, X_1\right) \right]^q \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q}}
\end{aligned}$$

$$= M_0^{1-\theta} M_1^{\theta} \|x\|_{\overline{X}_{\theta,q}}$$

and thus, it holds

$$\|T\|_{\overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}} \leq \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}.$$

□

Remark 6.3 *In general,* (4) *contains a constant* $C \geq 1$ *in the upper bound, so that*

$$\|T\|_{\overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}} \leq C \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}, \tag{5}$$

see for example [1, page 27]. For all the interpolation spaces we are dealing with it holds $C = 1$*, yet we mention this constant to emphasize the similarities to other inequalities that we will see later on in this section.*

Since continuous operators have a finite norm, the interpolation property shows in particular that the continuity of $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ is handed down to the interpolated operator $T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}$. A similar effect is known for compactness:

Theorem 6.4 ([3, Theorem 3.2]) *Let* $\overline{X}$ *and* $\overline{Y}$ *be Banach couples,* $T \in \mathcal{L}(\overline{X}, \overline{Y})$*,* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$*. If*

$$T \in \mathcal{K}(X_0, Y_0) \quad \textit{or} \quad T \in \mathcal{K}(X_1, Y_1)$$

then

$$T \in \mathcal{K}(\overline{X}_{\theta,q}, \overline{Y}_{\theta,q}).$$

We notice that this time the compactness of just one of the operators $T : X_0 \to Y_0$ or $T : X_1 \to Y_1$ is sufficient for $T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}$ to inherit it. This behaviour of compact operators under real interpolation can also be expressed in an inequality of the form (5) that yields the previous result by recalling that $\beta(T) = 0$ is equivalent to $T \in \mathcal{K}(X, Y)$.

Theorem 6.5 ([4]) *Let* $\overline{X}$ *and* $\overline{Y}$ *be Banach couples and* $T \in \mathcal{L}\left(\overline{X}, \overline{Y}\right)$*. For all* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$ *there is a constant* C *depending on* θ *such that*

$$\beta(T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}) \leq C \beta(T : X_0 \to Y_0)^{1-\theta} \beta(T : X_1 \to Y_1)^{\theta}.$$

Naturally the question arises if there are such an inequalities for entropy numbers $e_n(T)$ with $n < \infty$, too. Indeed, those can be proven if we one of the end spaces is fixed, which is achieved by replacing $\overline{X}$ or $\overline{Y}$ in $T \in \mathcal{L}(\overline{X}, \overline{Y})$ with a single Banach space X or Y. The resulting notation of $T \in \mathcal{L}\left(X, \overline{Y}\right)$ or $T \in \mathcal{L}\left(\overline{X}, Y\right)$ was already introduced in Definition 6.1.

Theorem 6.6 *Let $m, n \in \mathbb{N}$, $0 < \theta < 1$, $1 \le q \le \infty$ and let $\overline{X}, \overline{Y}$ be Banach couples.*

(a) If Y is a Banach space and $T \in \mathcal{L}(\overline{X}, Y)$, then

$$e_{n+m-1}(T : \overline{X}_{\theta,q} \to Y) \le 2Ce_n(T : X_0 \to Y)^{1-\theta} e_m(T : X_1 \to Y)^{\theta}.$$

(b) If X is a Banach space and $T \in \mathcal{L}(X, \overline{Y})$, then

$$e_{n+m-1}(T : X \to \overline{Y}_{\theta,q}) \le 2Ce_n(T : X \to Y_0)^{1-\theta} e_m(T : X \to Y_1)^{\theta}.$$

A proof for this Theorem can be found in [14, page 170, Prop. 12.1.11. and 12.1.12].

6.2 Entropy numbers under real interpolation

The final question is if there exists an inequality of the form

$$e_k(T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}) \le Ce_m(T : X_0 \to Y_0)^{1-\theta} e_n(T : X_1 \to Y_1)^{\theta},$$

for finite $k, m, n \in \mathbb{N}$ when no further assumptions about the Banach couples are made. This problem was finally settled in [6] where D. Edmunds and Yu. Netrusov constructed a counterexample for a pair of Banach couples (ℓ_{p_0}, ℓ_{q_0}) and (ℓ_{p_1}, ℓ_{q_1}) such that the entropy number of the interpolated operator $T : \overline{X}_{\theta,q} \to \overline{Y}_{\theta,q}$ can not be bounded above by the product of the entropy numbers of the original two operators.

Besides going over the content of [6] we will discuss some additional aspects of bounding entropy numbers in order to be able to grasp the involved ideas and techniques of the final counterexample.

6.2.1 Combinatorial Lemmas

Estimates of entropy numbers in any direction are most commonly based on either a couple of volume arguments or a combination of some combinatorial arguments. Since proving those is sometimes quite technical, we will prepare some combinatorial Lemmas right away so that they are ready to be used afterwards when we want to actually show certain upper and lower bounds of entropy numbers.

Lemma 6.7 *Let $v, d \in \mathbb{N}, v \le d$. Then*

$$\left(\frac{d}{v}\right)^v \le \binom{d}{v} \le \left(\frac{ed}{v}\right)^v \quad \text{and} \quad \binom{2d}{d} \le 2^{2d}.$$

Proof. For the lower bound in the first inequality we have to show

$$\left(\frac{d}{v}\right)^v \le \prod_{k=0}^{v-1} \frac{d-k}{v-k} = \frac{d(d-1)\dots(d-v+1)}{v(v-1)\dots 1} = \frac{d!}{v!(d-v)!} = \binom{d}{v},$$

which is implied by the fact that $\frac{d}{v} \leq \frac{d-k}{v-k}$ is equivalent to $v \leq d$ for all $k \in \{0, 1, \ldots, v-1\}$.

For the upper bound we will use the fact that

$$e^v = \sum_{k=0}^{\infty} \frac{v^k}{k!} \geq \frac{v^v}{v!} \quad \Leftrightarrow \quad \frac{1}{v!} \leq \frac{e^v}{v^v}$$

and conclude that

$$\binom{d}{v} = \frac{d!}{v!(d-v)!} = \frac{d(d-1)\ldots(d-v+1)}{v!} \leq \frac{d^v}{v!} \leq \left(\frac{ed}{v}\right)^v.$$

The second inequality is implied by the Binomial theorem, yielding

$$\binom{2d}{d} \leq \sum_{k=0}^{2d} \binom{2d}{k} = (1+1)^{2d} = 2^{2d}.$$

□

For the next result we introduce a notation for a set of subsets with a fixed number of elements. Given any set E and any $v \in \mathbb{N}$ with $v \leq |E|$, we put

$$\mathcal{L}(E, v) := \{D \subset E : |D| = v\}.$$

Lemma 6.8 *Let E be a set and $v \in \mathbb{N}$ such that $|E| \geq 64e^3 v$. Then there exists a subset $\mathcal{S}(E, v) \subset \mathcal{L}(E, v)$ with the following properties:*

(a) for any distinct $D_1, D_2 \in \mathcal{S}(E, v)$

$$|D_1 \cap D_2| \leq \frac{v}{2},$$

(b)

$$|\mathcal{S}(E, v)|^4 \geq |\mathcal{L}(E, v)| = \binom{|E|}{v}.$$

Proof. We will only discuss the case of n being even. The odd cases can be handled analogously. Let $\frac{v}{2} \in \mathbb{N}$, $n = |E| \in \mathbb{N}$ so that $n \geq 64e^3 v$, and let $D_1 \in \mathcal{L}(E, v)$. For the sake of readability we will shorten the notation of certain sets by putting

$$\mathcal{L}_{\geq k} := \{D_2 \in \mathcal{L}(E, v) : |D_1 \cap D_2| \geq k\},$$
$$\mathcal{L}_{\leq k} := \{D_2 \in \mathcal{L}(E, v) : |D_1 \cap D_2| \leq k\}$$

with $k \in \{0, 1, \ldots, v\}$. At first we observe that

$$|\mathcal{L}_{\leq v}| = \sum_{k=0}^{v} \binom{v}{k}\binom{n-v}{v-k} = \binom{n}{v} = \binom{|E|}{v},$$

which is the well-known Vandermonde identity. Additionally it holds

$$|\mathcal{L}_{\leq v}| \leq |\mathcal{L}_{\leq \frac{v}{2}}| + |\mathcal{L}_{\geq \frac{v}{2}}| \leq |\mathcal{L}_{\leq \frac{v}{2}}| \cdot |\mathcal{L}_{\geq \frac{v}{2}}| \tag{6}$$

for whenever $|\mathcal{L}_{\leq \frac{v}{2}}| \geq 2$ and $|\mathcal{L}_{\geq \frac{v}{2}}| \geq 2$, which is obviously the case for all $\frac{v}{2} \in \mathbb{N}$.

The next step is to show

$$|\mathcal{L}_{\geq \frac{v}{2}}| \leq \binom{v}{\frac{v}{2}}\binom{n}{\frac{v}{2}}. \tag{7}$$

By using $\binom{v}{k} \leq \binom{v}{v/2}$ for $k = 0, 1, \ldots, v/2$ we get

$$|\mathcal{L}_{\geq \frac{v}{2}}| = \sum_{k=\frac{v}{2}}^{v} \binom{v}{k}\binom{n-v}{v-k} \leq \binom{v}{\frac{v}{2}} \sum_{k=\frac{v}{2}}^{v} \binom{n-v}{v-k} = \binom{v}{\frac{v}{2}} \sum_{k=0}^{\frac{v}{2}} \binom{n-v}{k}.$$

Hence, it remains to show that

$$\sum_{k=0}^{\frac{v}{2}} \binom{n-v}{k} \leq \binom{n}{\frac{v}{2}}.$$

By using the Vandermonde identity, $\binom{v/2}{k} \geq 1$ for all $k \in \{0, \ldots, v/2\}$ and inverting the order of summation, we have

$$\binom{n}{\frac{v}{2}} = \sum_{k=0}^{\frac{v}{2}} \binom{\frac{v}{2}}{k}\binom{n-\frac{v}{2}}{\frac{v}{2}-k} \geq \sum_{k=0}^{\frac{v}{2}} \binom{n-\frac{v}{2}}{\frac{v}{2}-k} = \sum_{k=0}^{\frac{v}{2}} \binom{n-\frac{v}{2}}{k} \geq \sum_{k=0}^{\frac{v}{2}} \binom{n-v}{k},$$

where the last estimate holds for $n - v \geq v/2$, which fits our initial assumption on the size of E. Therefore (7) holds. By combining (6) and (7) we conclude that

$$\frac{\binom{n}{v}}{\binom{v}{v/2}\binom{n}{v/2}} \leq \frac{|\mathcal{L}_{\leq v}|}{|\mathcal{L}_{\geq \frac{v}{2}}|} \leq |\mathcal{L}_{\leq \frac{v}{2}}|$$

and thus we have found a subset $\mathcal{L}_{\leq \frac{v}{2}} =: \mathcal{S}(E, v) \subset \mathcal{L}(E, v)$ having property (a) and containing at least $\frac{\binom{n}{v}}{\binom{v}{v/2}\binom{n}{v/2}}$ elements.

In order to show property (b) we apply Lemma 6.7 which yields

$$|\mathcal{S}(E, v)| \geq \frac{\binom{n}{v}}{\binom{v}{v/2}\binom{n}{v/2}} \geq \frac{\left(\frac{n}{v}\right)^v}{2^v\left(\frac{2en}{v}\right)^{v/2}} = \frac{\left(\frac{n}{v}\right)^v}{2^v 2^{v/2}\left(\frac{en}{v}\right)^{v/2}} \cdot \frac{e^v}{e^v} = \frac{\left(\frac{ne}{v}\right)^{v/2}}{2^v 2^{v/2} e^v} = \frac{\left(\frac{ne}{v}\right)^{2\cdot v/4}}{(64e^4)^{v/4}}$$

and by the initial assumption of $n \geq 64e^3 v$ and again Lemma 6.7 we finally get

$$\frac{\left(\frac{ne}{v}\right)^{v/4}\left(\frac{ne}{v}\right)^{v/4}}{(64e^4)^{v/4}} \geq \frac{\left(\frac{ne}{v} \cdot \frac{64e^3 ve}{v}\right)^{v/4}}{(64e^4)^{v/4}} = \left(\frac{ne}{v}\right)^{v/4} \geq \binom{n}{v}^{1/4}.$$

□

The next statement is quite similar to the previous one and its proof is based on pretty much the same core ideas.

Lemma 6.9 *Let $a \in \mathbb{N}, a \geq 8$ and suppose that $(F_i)_{i=1}^n$ is a family of sets with $|F_i| \geq a$ for all i. Then there exists a subset $\mathcal{A} \subset \prod_{i=1}^n F_i$ such that*

$$|\mathcal{A}| \geq a^{\frac{n}{6}}$$

and for any $f = (f_i)_{i=1}^n, g = (g_i)_{i=1}^n \in \mathcal{A}$ with $f \neq g$, it holds the estimate

$$|\{i \in \{1, \ldots, n\} : f_i = g_i\}| \leq \frac{n}{2}.$$

Proof. Without loss of generality let $\frac{n}{2} \in \mathbb{N}$, $|F_i| = a$ for all i and $g \in \prod_{i=1}^n F_i$. Again we will shorten the notation by putting

$$\mathcal{F}_{\geq k} := \left\{ f \in \prod_{i=1}^n F_i : |\{i \in \{1, \ldots, n\} : f_i = g_i\}| \geq k \right\},$$
$$\mathcal{F}_{\leq k} := \left\{ f \in \prod_{i=1}^n F_i : |\{i \in \{1, \ldots, n\} : f_i = g_i\}| \leq k \right\}.$$

Already we can make some observations. First of all we have

$$a^n = \left| \prod_{i=1}^n F_i \right| = \sum_{k=0}^n \left| \left\{ f \in \prod_{i=1}^n F_i : |\{i \in \{1, \ldots, n\} : f_i = g_i\}| = k \right\} \right| \leq |\mathcal{F}_{\leq \frac{n}{2}}| + |\mathcal{F}_{\geq \frac{n}{2}}|$$

and it holds

$$|\mathcal{F}_{\leq \frac{n}{2}}| + |\mathcal{F}_{\geq \frac{n}{2}}| \leq |\mathcal{F}_{\leq \frac{n}{2}}| \cdot |\mathcal{F}_{\geq \frac{n}{2}}| \tag{8}$$

for $|\mathcal{F}_{\leq \frac{n}{2}}| \leq 2, |\mathcal{F}_{\geq \frac{n}{2}}| \geq 2$ which indeed holds for all $\frac{n}{2} \in \mathbb{N}$. Additionally we have

$$\left| \left\{ f \in \prod_{i=1}^n F_i : |\{i \in \{1, \ldots, n\} : f_i = g_i\}| = k \right\} \right| = \binom{n}{k} (a-1)^{n-k},$$

because after choosing k of the n coordinates that will coincide, there are $n-k$ coordinates with each $(a-1)$ elements left to be chosen.

Now we show that

$$|\mathcal{F}_{\geq \frac{n}{2}}| \leq a^{\frac{5}{6}n}. \tag{9}$$

By using our above observations we have

$$|\mathcal{F}_{\geq \frac{n}{2}}| = \sum_{k=\frac{n}{2}}^n \left| \left\{ f \in \prod_{i=1}^n F_i : |\{i \in \{1, \ldots, n\} : f_i = g_i\}| = k \right\} \right|$$

$$= \sum_{k=\frac{n}{2}}^{n} \binom{n}{k} (a-1)^{n-k} \leq \binom{n}{n/2} \sum_{k=\frac{n}{2}}^{n} (a-1)^{n-k}$$

$$= \binom{n}{n/2} \sum_{k=0}^{\frac{n}{2}} (a-1)^k \leq \binom{n}{n/2} a^{\frac{n}{2}},$$

where the last estimate $\sum_{k=0}^{\frac{n}{2}} (a-1)^k \leq a^{\frac{n}{2}}$ can be shown via induction ($n \to n+2$) such that

$$\sum_{k=0}^{\frac{n}{2}+1} (a-1)^k \leq a^{\frac{n}{2}} + (a-1)^{\frac{n}{2}+1} = a^{\frac{n}{2}+1} \left(\frac{1}{a} + \left(\frac{a-1}{a} \right)^{\frac{n}{2}+1} \right)$$

$$= a^{\frac{n}{2}+1} \left(\frac{1}{a} + \left(1 - \frac{1}{a} \right)^{\frac{n}{2}+1} \right) \leq a^{\frac{n}{2}+1},$$

where we used for the last estimate that

$$\frac{1}{a} + \left(1 - \frac{1}{a} \right)^{\frac{n}{2}+1} \leq 1 \quad \Leftrightarrow \quad \left(1 - \frac{1}{a} \right)^{\frac{n}{2}} \leq 1,$$

which obviously holds for all n.

Furthermore, because we assumed $a \geq 8$, it holds

$$\binom{n}{n/2} a^{\frac{n}{2}} \leq 2^n a^{\frac{n}{2}} = 8^{\frac{n}{3}} a^{\frac{n}{2}} \left(\frac{a^{n/3}}{a^{n/3}} \right) = \left(\frac{8}{a} \right)^{n/3} a^{\frac{5}{6}n} \leq a^{\frac{5}{6}n}.$$

By combining (8) and (9) we finally arrive at

$$a^{\frac{n}{6}} = \frac{a^n}{a^{\frac{5}{6}n}} \leq \frac{|\mathcal{F}_{\leq \frac{n}{2}}| + |\mathcal{F}_{\geq \frac{n}{2}}|}{|\mathcal{F}_{\geq \frac{n}{2}}|} \leq |\mathcal{F}_{\leq \frac{n}{2}}|$$

so that choosing $\mathcal{F}_{\leq \frac{n}{2}} = \mathcal{A}$ yields the claim. □

6.2.2 Bounds of entropy numbers

In the sequel we will encounter finite-dimensional ℓ_q-spaces and sequences spaces whose norms sum over index sets different from the natural numbers. For that we need more precise notation: Let E be any non-empty countable set. We put

$$\|x\|_{\ell_q(E)} = \left(\sum_{n \in E} |x_n|^q \right)^{\frac{1}{q}}$$

for finite q (with the usual adjustment for $q = \infty$) and $\ell_q(E)$ denotes the space of all sequences with $\|x\|_{\ell_q(E)} < \infty$.

In particular, for $E = \{1, 2, \dots, n\}$ we end up with the finite dimensional ℓ_q-spaces, denoted by ℓ_q^n, which are basically the $\mathbb{R}^n$ equipped with the norm $\|x\|_{\ell_q^n} = (\sum_{k=1}^n |x_k|^q)^{\frac{1}{q}}$. And choosing $E = \mathbb{N}$ yields the usual ℓ_q-spaces. Additionally, all the logarithms will have base 2, even though we will keep writing $\log(x)$ instead of $\log_2(x)$.

Now we are going to be concerned with the actual bounds for entropy numbers of the identity operator acting between ℓ_q^n-spaces. The upper bounds can be found in [8, Proposition 3.2.2] and the lower bounds in [5, Theorem 2]. We will go over the proof of one of the lower bounds as it showcases a particular combinatorial argument that we will encounter again in the main result.

Theorem 6.10 *Let $n \in \mathbb{N}$ and $0 < p < q \leq \infty$ and $\mathrm{id} : \ell_p^n \to \ell_q^n$. There are constants C_1, C_2 independent of k and n such that*

$$C_1 A(k,n) \leq e_k(\mathrm{id}) \leq C_2 A(k,n)$$

with

$$A(k,n) = \begin{cases} 1 & \text{if } k \leq \log n, \\ \left(\frac{\log(1+\frac{n}{k})}{k}\right)^{\frac{1}{p}-\frac{1}{q}} & \text{if } \log n \leq k \leq n, \\ 2^{-\frac{k}{n}} n^{-\left(\frac{1}{p}-\frac{1}{q}\right)} & \text{if } k \geq n. \end{cases}$$

Proof. We will follow the arguments in [11] and show the lower bound of $e_k(\mathrm{id} : \ell_p^n \to \ell_q^n)$ for $\log n \leq k \leq n$:

Let $m, n \in \mathbb{N}$ with $n \geq 4$ and $1 \leq m \leq \frac{n}{4}$. We put

$$S := \left\{ x = (x_k)_{k=1}^n \in \{-1, 0, 1\}^n : \sum_{k=1}^n |x_k| = 2m \right\}$$

and observe that $|S| = \binom{n}{2m} 2^{2m}$. Additionally we have $(2m)^{-\frac{1}{p}} S \subset B_{\ell_p^n}$. We define the *Hamming distance* h on S by

$$h(x,y) = |\{k \in \{1, 2, \dots, n\} : x_k \neq y_k\}|.$$

Obviously, for fixed $x \in S$ we have

$$|\{y \in S : h(x,y) \leq m\}| \leq \binom{n}{m} 3^m.$$

Now take any subset $A \subset S$ with $|A| \leq N := \binom{n}{2m} / \binom{n}{m}$. The estimate

$$|\{y \in S : \text{there exists an } x \in A \text{ with } h(x,y) \leq m\}| \leq |A| \binom{n}{m} 3^m \leq \binom{n}{2m} 3^m < |S|$$

implies the existence of an element $y \in S$ with $h(x,y) > m$ for all $x \in A$. Hence, it is possible to construct a set $A' \subset S$ with $|A'| > N$ and $h(x,y) > m$ and thus $\|x-y\|_{\ell_q^n} > m^{\frac{1}{q}}$ for all distinct $x, y \in A'$. So, we have found with $(2m)^{-\frac{1}{p}}A'$ a subset of $B_{\ell_p^n}$ containing more than N elements that have a mutual ℓ_q^n-distance

$$\|x-y\|_{\ell_q^n} > \varepsilon := (2m)^{-\frac{1}{p}} m^{\frac{1}{q}}. \tag{10}$$

Recalling the definition of inner entropy numbers $\varphi_k(\cdot)$, see Definition 5.4, and the fact that $e_k(\cdot) \geq \varphi_k(\cdot)$, it becomes apparent that by putting $k := \lfloor \log N \rfloor$ we can write (10) as

$$e_k(\mathrm{id} : \ell_p^n \to \ell_q^n) \geq \frac{\varepsilon}{2} = c_1 m^{\frac{1}{q}-\frac{1}{p}}$$

with $c_1 = 2^{-(1+\frac{1}{p})}$.

With a couple more arguments regarding the relationship of N, k and n it can be shown that

$$e_k(\mathrm{id} : \ell_p^n \to \ell_q^n) \geq c \left(\frac{\log(\frac{n}{k}+1)}{k} \right)^{\frac{1}{p}-\frac{1}{q}}$$

for $\log n \leq k \leq n$, but we omit them. □

The main reason to mention this proof was to highlight the particular chain of arguments revolving around the relationship $\varphi_k(\cdot) \leq e_k(\cdot) \leq 2\varphi_k(\cdot)$ that we had seen in Theorem 5.7. With this approach the difficult task of precisely evaluating an entropy number $e_k(T : X \to Y)$ changes into the construction of an appropriate subset of B_X containing a sufficiently large number of elements, so that their images under T have a minimal non-zero distance to one another. But as useful as this approach might be, finding such subsets can still be quite difficult depending on the respective operator T as well as its domain and co-domain spaces.

Next we state another Theorem regarding bounds of entropy numbers and this time not for the identity map but a *diagonal operator* that weights the elements of a sequences.

Theorem 6.11 ([12]) *Let $0 < p < q \leq \infty$. We put $\lambda := \frac{1}{p} - \frac{1}{q} > 0$ and define the diagonal map D by*

$$D : \ell_p \to \ell_q, \quad (x_i)_{i \in \mathbb{N}} \mapsto \left(\frac{x_i}{(\log(i+1))^{\lambda}} \right)_{i \in \mathbb{N}}.$$

There exist constants C_1, C_2 independent of n such that

$$C_1 n^{-\lambda} \leq e_n(D) \leq C_2 n^{-\lambda}.$$

After establishing these general entropy bounds we will have to introduce just one more tool in the form of bounds for the Lorentz norm of certain sequences, which will be crucial for deriving a lower bound for certain (inner) entropy numbers afterwards.

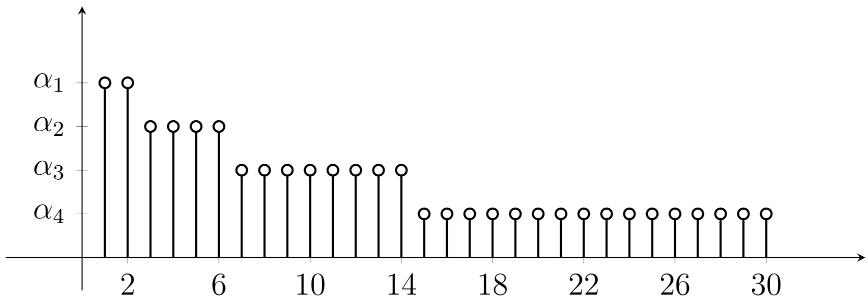

Figure 6: The first entries of the unweighted decreasing rearrangement of $\sum_{i=1}^{n} \alpha_i \chi_{E_i}$ with $|E_i| = 2^i$ and decreasing α_i.

Lemma 6.12 *Let $0 < u, p \leq \infty$ and let E be any set and $(E_i)_{i=1}^{n}$ any family of disjoint subsets of E with $|E_i| = 2^i$ for all $i = 1, \ldots, n$. Then there are positive constants C_1, C_2, depending on u and p, such that for any scalars $\alpha_1, \ldots, \alpha_n$ it holds*

$$C_1 \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}} \leq \left\| \sum_{i=1}^{n} \alpha_i \chi_{E_i} \right\|_{\ell_{p,u}} \leq C_2 \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}}.$$

Proof. We will only discuss the cases where u and p are finite. Without loss of generality we assume that $(\alpha_i)_{i=1,\ldots,n}$ is a non-increasing rearrangement. We can interpret $\alpha_i \chi_{E_i}$ to be a sequence of length $\sum_{i=1}^{n} 2^i = 2(2^n - 1)$, where the first 2 entries are equal to α_1, the next 2^2 entries are equal to α_2, and so forth, and the last 2^n entries are equal to α_n, see Figure 6. This leads to the expression of the $\ell_{p,u}$-norm as

$$\left\| \sum_{i=1}^{n} \alpha_i \chi_{E_i} \right\|_{\ell_{p,u}} = \left(\sum_{i=1}^{n} \left(\sum_{j=2^i-1}^{2(2^i-1)} \left(j^{\frac{1}{p}-\frac{1}{u}} |\alpha_i| \right)^u \right) \right)^{\frac{1}{u}}.$$

The estimates of this norm are based on the following idea: Above every sector of length 2^i we have the same α_i that is multiplied with different weights of the form $j^{\frac{1}{p}-\frac{1}{u}}$ which are either increasing or decreasing, depending on whether $p \leq u$ or $p > u$. In both cases there is a biggest (and a smallest) element $j^{\frac{1}{p}-\frac{1}{u}} \alpha_i$ on the left or right end of a sector with which all the other elements of each sector are estimated above (or below). Additionally we will repeatedly use the fact that $1 \leq \frac{2^i}{2^i-1} \leq 2$ for $i \in \mathbb{N}$.

At first, let $p \leq u$ (or equivalently $\frac{1}{p} - \frac{1}{u} \geq 0$) so that $(j^{\frac{1}{p}-\frac{1}{u}})_{j=1,\ldots,2(2^i-1)}$ is an increasing sequence. We get

$$\begin{aligned}
\left(\sum_{i=1}^{n} \left(\sum_{j=2^i-1}^{2(2^i-1)} \left(j^{\frac{1}{p}-\frac{1}{u}} |\alpha_i| \right)^u \right) \right)^{\frac{1}{u}} &\leq \left(\sum_{i=1}^{n} 2^i \left([2(2^i-1)]^{\frac{1}{p}-\frac{1}{u}} |\alpha_i| \right)^u \right)^{\frac{1}{u}} \\
&= \left(\sum_{i=1}^{n} \frac{2^i}{2(2^i-1)} [2(2^i-1)]^{\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}}
\end{aligned}$$

$$\leq \left(\sum_{i=1}^{n}(2^{i+1})^{\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

$$= 2^{\frac{1}{p}}\left(\sum_{i=1}^{n}2^{i\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

and

$$\left(\sum_{i=1}^{n}\left(\sum_{j=2^i-1}^{2(2^i-1)}\left(j^{\frac{1}{p}-\frac{1}{u}}|\alpha_i|\right)^u\right)\right)^{\frac{1}{u}} \geq \left(\sum_{i=1}^{n}2^i\left([2^i-1]^{\frac{1}{p}-\frac{1}{u}}|\alpha_i|\right)^u\right)^{\frac{1}{u}}$$

$$= \left(\sum_{i=1}^{n}\frac{2^i}{2^i-1}[2^i-1]^{\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

$$\geq \left(\sum_{i=1}^{n}[2^{i-1}]^{\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

$$= 2^{-\frac{1}{p}}\left(\sum_{i=1}^{n}2^{i\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}.$$

Conversely, if $p > u$ so that $(j^{\frac{1}{p}-\frac{1}{u}})_{j=1,\ldots,2(2^i-1)}$ is a decreasing sequence, then we have

$$\left(\sum_{i=1}^{n}\left(\sum_{j=2^i-1}^{2(2^i-1)}\left(j^{\frac{1}{p}-\frac{1}{u}}|\alpha_i|\right)^u\right)\right)^{\frac{1}{u}} \leq \left(\sum_{i=1}^{n}2^i\left([2^i-1]^{\frac{1}{p}-\frac{1}{u}}|\alpha_i|\right)^u\right)^{\frac{1}{u}}$$

$$= \left(\sum_{i=1}^{n}\frac{2^i}{2^i-1}[2^i-1]^{\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

$$\leq 2^{\frac{1}{u}}\left(\sum_{i=1}^{n}2^{i\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

and

$$\left(\sum_{i=1}^{n}\left(\sum_{j=2^i-1}^{2(2^i-1)}\left(j^{\frac{1}{p}-\frac{1}{u}}|\alpha_i|\right)^u\right)\right)^{\frac{1}{u}} \geq \left(\sum_{i=1}^{n}2^i\left([2(2^i-1)]^{\frac{1}{p}-\frac{1}{u}}|\alpha_i|\right)^u\right)^{\frac{1}{u}}$$

$$= \left(\sum_{i=1}^{n}\frac{2^i}{2(2^i-1)}[2(2^i-1)]^{\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

$$\geq \left(\sum_{i=1}^{n}\frac{1}{2}[2(2^{i-1})]^{\frac{u}{p}}|\alpha_i|^u\right)^{\frac{1}{u}}$$

$$= 2^{-\frac{1}{u}} \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}}.$$

In total we have shown that

$$\min\left(2^{-\frac{1}{u}}, 2^{-\frac{1}{p}}\right) \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}} \leq \left\| \sum_{i=1}^{n} \alpha_i \chi_{E_i} \right\|_{\ell_{p,u}} \leq \max\left(2^{\frac{1}{u}}, 2^{\frac{1}{p}}\right) \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}}.$$

□

Remark 6.13 *We will need two particular modifications of the previous Lemma in which the α_i still ought to be decreasing scalars while the number of elements in the sets E_i are changed into different powers of 2. The arguments used for the estimates stay exactly the same and lead to similar looking constants:*

(a) For any family $(E_i)_{i=1}^n$ of disjoint subsets of E with $|E_i| = 2^{i-1}$ for each i it holds

$$2^{-\frac{1}{u}-\frac{1}{p}} \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}} \leq \left\| \sum_{i=1}^{n} \alpha_i \chi_{E_i} \right\|_{\ell_{p,u}} \leq \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}}. \tag{11}$$

(b) For any family $(E_i)_{i=1}^n$ of disjoint subsets of E with $|E_1| = 2$ and $|E_i| = 2^{i-1}$ for each $i \in \{2, 3, \ldots, n\}$ it holds once again

$$2^{-\frac{1}{u}-\frac{1}{p}} \left(\sum_{i=1}^{n} 2^{i\frac{u}{p}} |\alpha_i|^u \right)^{\frac{1}{u}} \leq \left\| \sum_{i=1}^{n} \alpha_i \chi_{E_i} \right\|_{\ell_{p,u}}. \tag{12}$$

Remark 6.14 *For the next result let E be any countably infinite set and $(E_i)_{i\in\mathbb{N}}$ a family of disjoint subsets of E with $|E_i| = 2^{2^i}$ for all i. Let λ be a positive constant and denote by T the linear operator defined by*

$$T((x_i)_{i\in\mathbb{N}}) = \left(2^{-i\lambda} x_i\right)_{i\in\mathbb{N}}, \quad x_i \in E_i,$$

where the x_i are allowed to be vectors, so that all elements from E_i are weighted with $2^{-i\lambda}$.

Theorem 6.15 *Let $1 \leq p, q, u, v \leq \infty$ be such that $\lambda := \frac{1}{p} - \frac{1}{q} > 0$ and $q \geq v$ and we put $\sigma := \frac{1}{u} - \frac{1}{v}$. Let $T : \ell_{p,u}(E) \to \ell_{q,v}(E)$ and E be as in the previous Remark. Then there is a constant $C = C(p, q, u, v)$ such that*

$$e_m(T) \geq C m^{-\lambda} (\log m)^{-\sigma+\lambda}$$

for all $m \in \mathbb{N}$.

$$\begin{array}{ccccccccccc} & & \mathcal{L}(E_{k_0+1}, 2^{s-k_0-1}) & & \mathcal{L}(E_{k_0+2}, 2^{s-k_0-2}) & & & & \mathcal{L}(E_{s-1}, 2) & & \mathcal{L}(E_s, 1) \\ & & \cup & & \cup & & & & \cup & & \cup \\ \mathcal{A} & \subset & A_{k_0+1} & \times & A_{k_0+2} & \times & \ldots & \times & A_{s-1} & \times & A_s \end{array}$$

Figure 7: Visualization of the set $\mathcal{A}$ constructed in the proof of Theorem 6.15.

Proof. By substituting $m = 2^s s$ we get the estimate

$$\begin{aligned} e_{2^s s}(T) &\geq C(2^s s)^{-\lambda}(\log(2^s s))^{-\sigma+\lambda} \\ &= C2^{-s\lambda}s^{-\lambda}(s+\log s)^{\lambda}(s+\log s)^{-\sigma} \\ &\geq C2^{-s\lambda}s^{-\lambda}s^{\lambda}\min(1, 2^{-\sigma})s^{-\sigma} \\ &= C2^{-s\lambda}s^{-\sigma} \end{aligned}$$

for all $s \in \mathbb{N}$. In order to derive such a lower bound for $e_{2^s s}(T)$ we recall that $e_{2^s s}(T) \geq \varphi_{2^s s}(T)$ and

$$\begin{aligned} \varphi_{2^s s}(T) = \sup\{\delta > 0 : &\text{ there exist } x_1, x_2, \ldots, x_p \text{ in } B_{\ell_{p,u}}, p > 2^{2^s s-1}, \\ &\text{such that } \|Tx_i - Tx_j\|_{\ell_{q,v}} > 2\delta \text{ for } i \neq j, 1 \leq i, j \leq p\}, \end{aligned}$$

revealing two main tasks. At first we need to construct a subset $\mathcal{A} \subset B_{\ell_{p,u}}$ containing more than $2^{2^s s-1}$ elements. Afterwards we have to verify that the images of the elements in $\mathcal{A}$ under T have a certain minimal distance to one another.

Before we take on the first task, we need to make the following consideration: Let $s \in \mathbb{N}$ be sufficiently large and put $k_0 := \min\left\{k \in \mathbb{N} : s \leq 2^k\right\}$. Then obviously $2^{k_0-1}+1 \leq s \leq 2^{k_0}$ and furthermore it holds

$$s - k_0 \geq \frac{1}{3}s. \tag{13}$$

Since $1 - \frac{k_0}{s} \geq 1 - \frac{k_0}{2^{k_0-1}+1}$, (13) is already implied by observing that the sequence $\left(\frac{k_0}{2^{k_0-1}+1}\right)_{k_0 \in \mathbb{N}} = (\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{4}{9}, \ldots)$ is strictly decreasing for all $2 \leq k \in \mathbb{N}$ so that $\frac{2}{3}$ is the largest possible entry.

Now we recall that the sets E_i contain 2^{2^i} elements and $\mathcal{L}(E_i, 2^{s-i})$ is the collection of all subsets of E_i containing exactly 2^{s-i} elements. We claim that there exists a set $\mathcal{A} \subset A_{k_0+1} \times A_{k_0+2} \times \ldots \times A_{s-1} \times A_s$ and $A_i \subset \mathcal{L}(E_i, 2^{s-i})$ for $i \in \{k_0+1, \ldots, s\}$, see Figure 7, possessing the following properties:

(a) For all distinct $F_0, F_1 \in A_i, i \in \{k_0+1, \ldots, s\}$ it holds

$$|F_0 \cap F_1| \leq 2^{s-i-1}.$$

(b) For all $f = (f_i)_{i=k_0+1}^{s}, g = (g_i)_{i=k_0+1}^{s} \in \mathcal{A}$ with $f \neq g$ it holds

$$|\{i \in \{k_0+1, \ldots, s\} : f_i = g_i\}| \leq \frac{s-k_0}{2}.$$

(c) There is a constant C such that

$$\log(|A_i|) \geq C2^s.$$

(d) There is a constant C_1 such that

$$\log(|\mathcal{A}|) \geq C_1 2^s s.$$

(a) and (c) are consequences of Lemma 6.8. Putting $v = 2^{s-i}$ directly yields the existence of sets A_i which take the role of the subsets $\mathcal{S}(E, 2^{s-i})$. For these we had shown the inequality $|A_i|^4 = |\mathcal{S}(E, 2^{s-i})|^4 \geq |\mathcal{L}(E, 2^{s-i})|$. This implies (c) when we apply Lemma 6.7, because

$$\log|A_i| \geq \log|\mathcal{L}(E, 2^{s-i})| \geq \log\binom{2^{2^i}}{2^{s-i}} \geq 2^{s-i}\log\left(\frac{2^{2^i}}{2^{s-i}}\right) = 2^{s-i}\log\left(2^{2^i-s+1}\right)$$

and by recalling that we are using base 2 logarithms we get

$$2^{s-i}\log\left(2^{2^i-s+1}\right) = 2^{s-i}\left(2^i - s + 1\right) \geq 2^s\left(1 - \frac{s}{2^i}\right) \geq 2^s\left(1 - \frac{1}{2}\right),$$

where we used for the last estimate that $s \leq 2^{k_0}$ and $2^{-i} \leq 2^{-(k_0+1)}$ for $i \in \{k_0+1, \ldots, s\}$.

Similarly, properties (b) and (d) are implied by Lemma 6.9 with $n = s - k_0$. By putting $a := \min(|A_{k_0+1}|, |A_{k_0+2}|, \ldots, |A_s|)$ and applying the just shown property (c) we get

$$\log|\mathcal{A}| \geq \left(\frac{s-k_0}{6}\right)\log a \geq \left(\frac{s-k_0}{6}\right)C2^s = \frac{1}{6}C\left(\frac{s-k_0}{s}\right)2^s s \geq \frac{1}{18}C2^s s,$$

where we used the fact that $s - k_0 \geq \frac{1}{3}s$.

Now that we are sure of the existence of $\mathcal{A}$ we take any $f = (f_i)_{i=k_0+1}^{s} \in \mathcal{A}$ and put

$$x_f := s^{-\frac{1}{u}}\left(\sum_{i=k_0+1}^{s} 2^{-\frac{s-i}{p}}\chi_{f_i}\right) = s^{-\frac{1}{u}}\left(\sum_{i=1}^{s-k_0} 2^{-\frac{i-1}{p}}\chi_{f_{s+1-i}}\right)$$

with inverted order of summation in the second series. Since $|\chi_{f_{s+1-i}}| = 2^{i-1}$ for all $i \in \{1, \ldots, s-k_0\}$, we can apply inequality (11) with the coefficients $\alpha_i = 2^{-\frac{i-1}{p}}$. By additionally using the fact that $s^{-\frac{1}{u}}(s-k_0)^{\frac{1}{u}} \leq 1$ we get

$$\|x_f\|_{p,u} = s^{-\frac{1}{u}}\left\|\sum_{i=1}^{s-k_0} 2^{-\frac{i-1}{p}}\chi_{f_{s+1-i}}\right\|_{\ell_{p,u}} \leq s^{-\frac{1}{u}}\left(\sum_{i=1}^{s-k_0} 2^{i\frac{u}{p}}|2^{-\frac{i-1}{p}}|^u\right)^{\frac{1}{u}}$$

$$= s^{-\frac{1}{u}} \left(\sum_{i=1}^{s-k_0} 2^{\frac{u}{p}} \right)^{\frac{1}{u}}$$
$$= 2^{\frac{1}{p}} s^{-\frac{1}{u}} (s-k_0)^{\frac{1}{u}} \leq 2^{\frac{1}{p}}.$$

Hence, $\mathcal{A}$ can be scaled down such that it is a subset of the unit ball of $\ell_{p,u}$. With the previous estimate and property (d) we have already shown that the set $\mathcal{A}$ contains enough elements and can be scaled accordingly to fit into the definition of $\varphi_{2^s s}(T)$. Hence, it only remains to show that there exists some constant C_2 so that

$$\|Tx_f - Tx_g\|_{\ell_{q,v}} \geq C_2 2^{-s\lambda} s^{\sigma}$$

for distinct $f, g \in \mathcal{A}$. At first we simply use the definition of T and once again reverse the order of summation so that

$$\begin{aligned}
\|Tx_f - Tx_g\|_{\ell_{q,v}} &= \left\| T\left(s^{-\frac{1}{u}} \sum_{i=k_0+1}^{s} 2^{-\frac{s-i}{p}} (\chi_{f_i} - \chi_{g_i}) \right) \right\|_{\ell_{q,v}} \\
&= \left\| s^{-\frac{1}{u}} \sum_{i=k_0+1}^{s} 2^{-\frac{s-i}{p} - i\lambda} (\chi_{f_i} - \chi_{g_i}) \right\|_{\ell_{q,v}} \\
&= \left\| s^{-\frac{1}{u}} \sum_{i=1}^{s-k_0} 2^{-\frac{i-1}{p} - (s+1-i)\lambda} (\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}) \right\|_{\ell_{q,v}} \\
&= s^{-\frac{1}{u}} 2^{-s\lambda + \frac{1}{q}} \left\| \sum_{i=1}^{s-k_0} 2^{-\frac{i}{q}} (\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}) \right\|_{\ell_{q,v}}
\end{aligned}$$

and notice the decreasing coefficients $2^{-\frac{i}{q}}$.

For further estimates of the $\ell_{q,v}$-norm we have to look back at the definition of $\mathcal{A}$ and the properties of distinct elements $f = (f_{s+1-i})_{i=1}^{s-k_0}, g = (g_{s+1-i})_{i=1}^{s-k_0}$ from this set. Property (b) tells us that f and g can coincide in at most half their coordinates and by (a) we know that if $f_{s+1-i} \neq g_{s+1-i}$, then these two sets share at most half of their elements. Thus, for the widths of the difference of the indicator functions it holds $|\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}| \leq 2^i$ for all $i \in \{1, \ldots, s-k_0\}$ and

$$|\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}| \begin{cases} = 0 & \text{if } f_{s+1-i} = g_{s+1-i}, \\ = 2 & \text{if } f_{s+1-i} \neq g_{s+1-i} \text{ and } i = 1, \\ \geq 2^{i-1} & \text{if } f_{s+1-i} \neq g_{s+1-i} \text{ and } i \geq 2. \end{cases}$$

So as long as the respective coordinates do not coincide, we will have exactly 2 elements above E_s and the number of elements above every other component E_{s+1-i} with $i \geq 2$ is a multiple of 2 between 2^{i-1} and 2^i. In particular, if f and g are disjoint then the resulting sequences have the same form as $\sum_{i=1}^{n} \alpha_i \chi_{E_i}$ that we had seen in Lemma 6.12, see Figure 6.

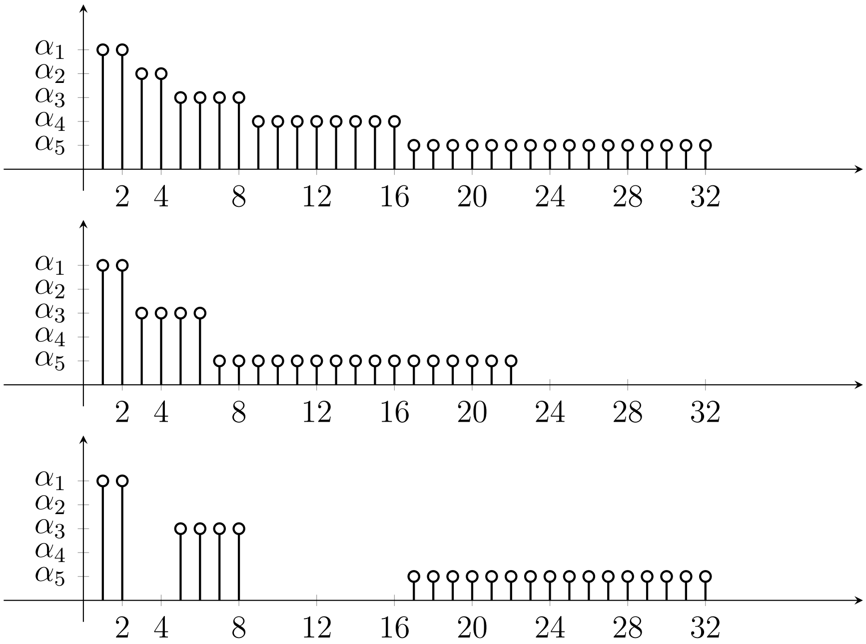

Figure 8: The first entries of the decreasing rearrangement of $\sum_{i=1}^{s-k_0} 2^{-\frac{i}{q}}(\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}})$ after applying property (a) of $\mathcal{A}$; after additionally applying property (b) of $\mathcal{A}$; and how it ought be weighted and estimated in the proof of Theorem 6.15.

Finally we need to combine the assumption of $q \geq v$ with the properties (a) and (b) of $\mathcal{A}$ and the lower bound in (12) in order to derive the desired lower bound for

$$\|Tx_f - Tx_g\|_{\ell_{q,v}} = s^{-\frac{1}{u}} 2^{-s\lambda + \frac{1}{q}} \left\| \sum_{i=1}^{s-k_0} 2^{-\frac{i}{q}} (\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}) \right\|_{\ell_{q,v}} .$$

The first step to minimize the $\ell_{q,v}$-norm is to suppose that f and g are so that there is maximal overlay above every E_i in the sense of (a), such that $|\chi_{f_s} - \chi_{g_s}| = 2$ and $|\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}| = 2^{i-1}$ for all $i \in \{2, 3, \ldots, s-k_0\}$. Hence, (12) is applicable with $E_i = \chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}$ and $\alpha_i = 2^{-\frac{i}{q}}$, yielding

$$\left\| \sum_{i=1}^{s-k_0} 2^{-\frac{i}{q}} (\chi_{f_{s+1-i}} - \chi_{g_{s+1-i}}) \right\|_{\ell_{q,v}} \geq 2^{-\frac{1}{v} - \frac{1}{q}} \left(\sum_{i=1}^{s-k_0} 2^{i\frac{v}{q}} |2^{-\frac{i}{q}}|^v \right)^{\frac{1}{v}} \geq 2^{-\frac{1}{v} - \frac{1}{q}} \left(\frac{s-k_0}{2} \right)^{\frac{1}{v}}, \quad (14)$$

where we supposed for the second estimate that in the sense of (b) the maximum possible number of coordinates coincide. Since we assumed $q \geq v$ and thus have decreasing weights $n^{\frac{1}{q} - \frac{1}{v}}$ in the Lorentz norm, we would get the same lower bound even if we applied (b) first. As shown in Figure 8, if some coordinates are eliminated by property (b) then the rest are pushed to the left as the Lorentz norm decreasingly rearranges the sequence. But because of the decreasing weights, the norm can only get smaller if we shift the remaining coordinates to where they would have ended up before applying (b). With the same argument inequality (12) remains valid even if we allow some $\alpha_i = 0$. By using $s - k_0 \geq \frac{1}{3}s$ in (14) we finally have

$$\|Tx_f - Tx_g\|_{\ell_{q,v}} \geq s^{-\frac{1}{u}} 2^{-s\lambda + \frac{1}{q}} 2^{-\frac{1}{v} - \frac{1}{q}} \left(\frac{s}{6}\right)^{\frac{1}{v}} = 12^{-\frac{1}{v}} 2^{-s\lambda} s^{-\left(\frac{1}{u} - \frac{1}{v}\right)}$$

for all distinct $f, g \in \mathcal{A}$.

□

This finally enables us to formulate the result in which entropy numbers do not behave well under real interpolation:

Theorem 6.16 *For all $\lambda \in (0,1)$ there exist Banach spaces X_0, X_1, Y_0, Y_1 and a linear map $T : X_0 + X_1 \to Y_0 + Y_1$ such that for all $1 \leq u \leq \infty, 0 < \theta < 1$ with $1 - \theta \geq \frac{1}{u}$ and $n \in \mathbb{N}$*

$$e_n(T : X_0 \to Y_0) \leq c_0 n^{-\lambda},$$
$$e_n(T : X_1 \to Y_1) \leq c_1 n^{-\lambda}$$

and

$$e_n(T : (X_0, X_1)_{\theta,u} \to (Y_0, Y_1)_{\theta,u}) \geq cn^{-\lambda}(\log n)^{\lambda},$$

where the constants c, c_0, c_1 are positive and independent on n.

Proof. We put $X_0 = \ell_{p_0}, X_1 = \ell_{p_1}, Y_0 = \ell_{q_0}, Y_0 = \ell_{q_1}$ with $1 < p_0, p_1, q_0, q_0 < \infty$, $p_0 \neq q_0$ and

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \lambda > 0$$

With the same operator T as in Theorem 6.15 it follows by Theorem 6.11 that

$$e_n(T : \ell_{p_0} \to \ell_{q_0}) \leq C_0 n^{-\lambda} \quad \text{and} \quad e_n(T : \ell_{p_1} \to \ell_{q_1}) \leq C_1 n^{-\lambda},$$

which was applicable since T was specifically defined as a diagonal operator acting over some sets E_i with lengths $|E_i| = 2^{2^i}$ and the respective weights $2^{-i\lambda}$.

Additionally we had seen in the Theorems 4.10 and 4.11 that

$$(\ell_{p_0}, \ell_{p_1})_{\theta,u} = \ell_{p,u} \quad \text{and} \quad (\ell_{q_0}, \ell_{q_1})_{\theta,u} = \ell_{q,u}$$

with $0 < \theta < 1$, $1 \leq u \leq \infty$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Hence, Theorem 6.15 implies

$$e_n(T : (\ell_{p_0}, \ell_{p_1})_{\theta,u} \to (\ell_{q_0}, \ell_{q_1})_{\theta,u}) \geq C_2 n^{-\lambda}(\log n)^{\lambda}$$

when $q \geq u$.

□

6.3 Summary

A lot of preparations had to be done, but we finally have an example for an operator acting between real interpolation spaces with an ill-behaving entropy number. Yet, it should be noted that Theorem 6.15 and Theorem 6.16 were originally stated in [6] without the restriction $q \geq v$. Since there are strong similarities with certain arguments in the earlier work [5] by Edmunds and Netrusov, there might be a way to derive the lower bound just with the help of Lemma 6.12 for $q < v$ as well. But so far we could not figure out such an argument. Additionally their main result includes the quasi-normed Lorentz spaces. Even though we excluded these cases previously, our presented argument in the proof of Theorem 6.15 works for those as well.

As a general note we want to highlight that among several follow-up papers, Edmunds and Netrusov themselves quickly provided another counterexample in [7] with an even worse behaviour of the entropy numbers for vector-valued sequence spaces and in particular for Besov spaces.

While we can consider the problem of the behaviour of entropy numbers under real interpolation to be settled, the exact same problem is still open for complex interpolation methods. Similarly it can already be shown that they behave well under specific assumptions. Yet, there is missing an answer when no further assumptions are made on the involved spaces. Hence, we can be excited when and how this problem will be solved, too.

References

[1] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[2] B. Carl and I. Stephani. *Entropy, compactness and the approximation of operators*, volume 98 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1990.

[3] F. Cobos. Interpolation theory and compactness. *In: Spring School in Analysis, Function Spaces, Inequalities and Interpolation*, pages 31–75, 2009.

[4] F. Cobos, P. Fernández-Martínez, and A. Martínez. Interpolation of the measure of non-compactness by the real method. *Studia Math.*, 135(1):25–38, 1999.

[5] D. E. Edmunds and Yu. Netrusov. Entropy numbers of embeddings of Sobolev spaces in Zygmund spaces. *Studia Math.*, 128(1):71–102, 1998.

[6] D. E. Edmunds and Yu. Netrusov. Entropy numbers and interpolation. *Mathematische Annalen*, 351(4):963–977, 2011.

[7] D. E. Edmunds and Yu. Netrusov. Entropy numbers of operators acting between vector-valued sequence spaces. *Math. Nachr.*, 286(5-6):614–630, 2013.

[8] D. E. Edmunds and H. Triebel. *Function spaces, entropy numbers, differential operators*, volume 120 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1996.

[9] D. D. Haroske. *Höhere Analysis I.* lecture notes. Friedrich-Schiller-Universität Jena, winter term 2011/2012.

[10] M. Kato. On Lorentz spaces $l_{p,q}\{E\}$. *Hiroshima Math. J.*, 6(1):73–94, 1976.

[11] T. Kühn. A lower estimate for entropy numbers. *J. Approx. Theory*, 110(1):120–124, 2001.

[12] T. Kühn. Entropy numbers of general diagonal operators. *Rev. Mat. Complut.*, 18(2):479–491, 2005.

[13] J. Peetre. *A theory of interpolation of normed spaces.* Notas de Matemática, No. 39. Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas, Rio de Janeiro, 1968.

[14] A. Pietsch. *Operator ideals*, volume 16 of *Mathematische Monographien [Mathematical Monographs].* VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[15] S. Rolewicz. On a certain class of linear metric spaces. *Bull. Acad. Polon. Sci. Cl. III.*, 5:471–473, XL, 1957.

[16] Y. Sagher. Interpolation of r-Banach spaces. *Studia Math.*, 41:45–70, 1972.

[17] H. Triebel. *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, second edition, 1995.

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig angefertigt, nicht anderweitig zu Prüfungszwecken vorgelegt und keine anderen als die angegebenen Hilfsmittel verwendet habe. Sämtliche wissentlich verwendete Textausschnitte, Zitate oder Inhalte anderer Verfasser wurden ausdrücklich als solche gekennzeichnet.

Chemnitz, den 27. Juli 2017

Robert Nasdala