Probabilistic Marcinkiewicz-Zygmund inequalities on the rotation group

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Scattered data approximation problems on the rotation group $SO(3)$ naturally arise in various fields like crystallography, chemistry and biology. In order to answer different questions related to polynomial approximation procedures for such problems, Marcinkiewicz-Zygmund inequalities on the rotation group are a very powerful tool. These inequalities provide norm equivalences between polynomials on $SO(3)$ and their sample values at scattered sites. Recently shown equivalences depend on a density parameter of the sampling set and the proven inequalities hold true for polynomials on $SO(3)$ whose degree does not exceed an upper bound which is determined by this density parameter. In this paper, we show that we can enlarge this upper bound for the polynomial degree significantly if we are satisfied by such norm equivalences that hold with a given probability only. Moreover, we show that there are fixed sampling sets for which we get probabilistic Marcinkiewicz-Zygmund inequalities that hold for polynomials on $SO(3)$ of all degrees.

Keywords Marcinkiewicz-Zygmund inequalities, rotation group, scattered data, Wigner-D functions, random polynomials

1 Introduction

Scattered data approximation problems on the rotation group $SO(3)$ are of great importance in many applications in science and engineering. Such problems naturally arise, for example, in crystallographic texture analysis, biochemistry and geometry processing (Butzlaff et al. (1992); Gutzmer (1996); Cramer (2004); Bunge (1982); Kunze et al. (1993); v.d. Boogaart et al. (2007); Funkhouser et al. (2003)). The monograph by Chirikjian and Kyatkin (2001) provides a great collection of problems, where also matrix groups different from $SO(3)$ are involved. Usually the setting is as follows. Given a data set $\mathcal{D} := \{(\boldsymbol{G}_j, y_j) \in SO(3) \times \mathbb{C} : j = 0, \ldots, M-1\}$, we suppose the y_j's to be point evaluations of an unknown function $f : SO(3) \to \mathbb{C}$. Now one tries to recover f from the given data $\mathcal{D}$. There are various methods to handle such scattered data approximation problems on the rotation group. A

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prominent approach uses so-called positive definite basis functions to interpolate the given data $\mathcal{D}$ (Erb and Filbir; Filbir and Schmid (2008); Gutzmer (1996)). Recently, fast algorithms for the calculation of Fourier transforms on the rotation group have become available (Kostelec and Rockmore (2008); Potts et al. (submitted)). This powerful tool at hand, approximation procedures using finite expansions into Wigner-D functions, subsequently called polynomials on $SO(3)$, to approximate the given data $\mathcal{D}$ have attracted more and more attention (Gräf and Kunis; Schmid (2008)). In this respect, so-called Marcinkiewicz-Zygmund inequalities for polynomials on $SO(3)$ are an essential tool to answer different problems that come along with the polynomial approximation on $SO(3)$. Recently, by Schmid (2008) such inequalities have been proven. There it is shown that

$$c_p \|f\|_p \leq \|\boldsymbol{f}\|_{\boldsymbol{w},p} \leq C_p \|f\|_p \tag{1.1}$$

holds true for all polynomials f on $SO(3)$ up to a degree N, where the constants $c_p, C_p > 0$ depend on a density parameter of the underlying sampling set. Here, $\|\cdot\|_p$ is the usual L^p-norm, $\|\cdot\|_{\boldsymbol{w},p}$ denotes a weighted ℓ^p-norm and $\boldsymbol{f}$ is the vector of samples of f at the sampling points $\boldsymbol{G}_j$. We see that (1.1) gives bounds for the norm of the corresponding sampling operator and its inverse. Hence, the inequalities are of importance in connection with stable reconstruction of polynomials on $SO(3)$ from their scattered samples and the design of quadrature formulas. See Mhaskar et al. (2001); Keiner et al. (2007); Filbir and Themistoclakis (2008) for such applications of spherical Marcinkiewicz-Zygmund inequalities.

In this paper, we show that, in the important case $p = 2$, we can relax the condition on the polynomial degree N significantly if we are satisfied by such norm equivalences that hold with a given probability only. More precisely, we assume that the coefficients in the finite expansions into Wigner-D functions are taken randomly from the uniform distribution on a ball of appropriate dimension. Then we can show that inequalities of the form (1.1) hold true with a prescribed probability for polynomials up to a relatively high degree. Moreover, we can prove that there are sampling sets in $SO(3)$ such that we can guarantee (1.1) with a given probability for all polynomial degrees $N \geq 0$. To this end, we follow closely the ideas of Böttcher et al. where probabilistic spherical Marcinkiewicz-Zygmund inequalities have been derived, and we carry over their argumentation to the rotation group $SO(3)$.

The outline of the paper is as follows. In the next section, we review some basic material and necessary notation on the rotation group $SO(3)$ to keep the paper self-contained. We also give a brief introduction about concepts on sampling on $SO(3)$ and present some known results. These results are going to be the main ingredients in order to derive our results in the subsequent section. There we prove probabilistic Marcinkiewicz-Zygmund inequalities on $SO(3)$ and show the existence of universal sampling sets on the rotation group. Finally, we present some numerical examples that confirm our significant relaxations on the conditions for the polynomial degree.

2 Preliminaries

2.1 Analysis on $SO(3)$

Let $SO(3) := \{\boldsymbol{G} \in \mathbb{R}^{3\times 3} : \boldsymbol{G}^T\boldsymbol{G} = \boldsymbol{I}, \det \boldsymbol{G} = 1\}$ denote the non-Abelian compact group of proper rotations in the Euclidean space $\mathbb{R}^3$ and let μ be the normalized Haar measure on $SO(3)$, i.e. we have $\int_{SO(3)} \mathrm{d}\mu(\boldsymbol{G}) = 1$. Using the well-known parameterization of $SO(3)$ via

Euler angles Gelfand et al. (1963), the Haar integral of a measurable function f on $SO(3)$ reads as

$$\int_{SO(3)} f(\boldsymbol{G}) d\mu(\boldsymbol{G}) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(\varphi_1, \theta, \varphi_2) \ \sin\theta \, d\varphi_1 \, d\theta \, d\varphi_2. \tag{2.1}$$

The Hilbert space $L^2(SO(3))$ of all square integrable functions is determined by the scalar product with induced norm

$$\langle f, g \rangle := \int_{SO(3)} f(\boldsymbol{G}) \overline{g(\boldsymbol{G})} \mathrm{d}\mu(\boldsymbol{G}), \quad \|f\|_2^2 := \int_{SO(3)} |f(\boldsymbol{G})|^2 \, \mathrm{d}\mu(\boldsymbol{G}).$$

In order to get an orthogonal basis system for the $L^2(SO(3))$ we will make use of some fundamental results from the representation theory of this non-Abelian compact group.
Let $\{Y_k^l : l \in \mathbb{N}_0, \, k = -l, \ldots, l\}$ denote the canonical orthonormal basis of spherical harmonics on the space of all square integrable functions on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and let $\mathcal{H}_l := \operatorname{span}\{Y_k^l : k = -l, \ldots, l\}$. For given $l \in \mathbb{N}_0$ we assign each element $\boldsymbol{G} \in SO(3)$ the linear transformation $D^l(\boldsymbol{G}) : \mathcal{H}_l \to \mathcal{H}_l$ defined by

$$D^l(\boldsymbol{G}) f(\xi) := f(\boldsymbol{G}^{-1}\xi), \quad f \in \mathcal{H}_l, \ \xi \in \mathbb{S}^2.$$

Each $D^l, l \in \mathbb{N}_0$, can be written as a $(2l+1) \times (2l+1)$- matrix with matrix coefficients defined by the following system of linear equations

$$Y_k^l(\boldsymbol{G}^{-1}\xi) = \sum_{k'=-l}^{l} D_{k,k'}^l(\boldsymbol{G}) Y_{k'}^l(\xi), \quad k = -l, \ldots, l, \ \xi \in \mathbb{S}^2.$$

The functions $D_{k,k'}^l$ are often called Wigner-D functions of degree l and orders k and k'. It is well-known that the $D^l, l \in \mathbb{N}_0$, form a complete set of unitary irreducible representations of the rotation group. Thus the dual object $SO(3)\widehat{\ }$ can be identified with $\mathbb{N}_0$ and due to the Peter-Weyl Theorem the matrix coefficients $D_{k,k'}^l$ form an orthogonal basis of the $L^2(SO(3))$. Hence, every $f \in L^2(SO(3))$ can be expanded in a $SO(3)$ Fourier series

$$f = \sum_{l \in \mathbb{N}_0} \sum_{k,k'=-l}^{l} \sqrt{2l+1} \, \hat{f}_{k,k'}^l D_{k,k'}^l$$

with $SO(3)$ Fourier coefficients $\hat{f}_{k,k'}^l = \sqrt{2l+1} \langle f, D_{k,k'}^l \rangle = \sqrt{2l+1} \int_{SO(3)} f(\boldsymbol{G}) \overline{D_{k,k'}^l(\boldsymbol{G})} \mathrm{d}\mu(\boldsymbol{G})$.
Furthermore, for $f \in L^2(SO(3))$, Parseval's equality reads as

$$\|f\|_2^2 = \sum_{l \in \mathbb{N}_0} \sum_{k,k'=-l}^{l} \left| \hat{f}_{k,k'}^l \right|^2. \tag{2.2}$$

A remarkable and useful property of the Wigner-D functions is the addition theorem

$$\sum_{k,k'=-l}^{l} D_{k,k'}^l(\boldsymbol{G}) \, \overline{D_{k,k'}^l(\boldsymbol{H})} = U_{2l}\left(\cos \frac{d(\boldsymbol{G}, \boldsymbol{H})}{2}\right), \tag{2.3}$$

where $U_l(\cos\omega) = \sin((l+1)\omega) / \sin(\omega)$ denotes the l-th Chebyshev polynomial of second kind.

We call functions with finite Fourier expansion polynomials on $SO(3)$. Hence, we define the space of polynomials on $SO(3)$ with degree at most N by

$$\mathbf{\Pi}_N := \operatorname{span}\left\{D^l_{k,k'} : l = 0, \ldots, N;\ k, k' = -l, \ldots, l\right\}.$$

The spaces $\mathbf{\Pi}_N$, indeed, admit a polynomial behavior, i.e. for $f \in \mathbf{\Pi}_{N_1}$ and $g \in \mathbf{\Pi}_{N_2}$ we have, cf. (Schmid, 2009, Eq. (3.54)),

$$f \cdot g \in \mathbf{\Pi}_{N_1+N_2}. \tag{2.4}$$

We have $d_N := \dim(\mathbf{\Pi}_N) = \frac{1}{6}(2N+1)(2N+2)(2N+3)$ for the dimension of the space $\mathbf{\Pi}_N$. For the $SO(3)$ Fourier coefficients of polynomials $f \in \mathbf{\Pi}_N$ we introduce the index set

$$J_N := \{(l, k, k') \ :\ l = 0, \ldots, N;\ k, k' = -l, \ldots, l\}.$$

2.2 Sampling data

Besides the parameterization via Euler angles, the parameterization of the rotation group via the projective space is of some importance to us. This parameterization yields a translation invariant metric on $SO(3)$ which enables us to quantify different sampling sets on the rotation group.

Let $\mathcal{K}_\pi$ be the closed ball in $\mathbb{R}^3$ of radius π centered at the origin and identify antipodal points on its surface. This is the three dimensional projective space. An element $\boldsymbol{G} \in SO(3)$ is identified with a point in the projective space $\mathcal{K}_\pi$ by $\boldsymbol{G} \to \omega \cdot r$ where r, satisfying $\boldsymbol{G}r = r$ and $\|r\| = 1$, is the rotation axis and ω, which can be chosen in $[0, \pi]$, is the rotation angle of $\boldsymbol{G}$. A function that only depends on the rotation angle $\omega = \omega(\boldsymbol{G})$ is called conjugate invariant or central. For a central function f on $SO(3)$ the Haar integral (2.1) simplifies to

$$\int_{SO(3)} f(\boldsymbol{G}) \mathrm{d}\mu(\boldsymbol{G}) = \frac{2}{\pi} \int_0^\pi f(\omega) \sin^2\left(\frac{\omega}{2}\right) \mathrm{d}\omega. \tag{2.5}$$

Furthermore, it is easy to see that

$$\mathrm{d}(\boldsymbol{G}, \boldsymbol{H}) := \omega(\boldsymbol{H}^{-1}\boldsymbol{G})$$

defines a translation invariant metric on $SO(3)$.

We can use the metric d to quantify a sampling set

$$\mathcal{X} := \{\boldsymbol{G}_j \in SO(3) : j = 0, \ldots, M-1\}.$$

To do so we define two parameters. The first one is given by the separation distance

$$q_{\mathcal{X}} := \min_{0 \le j < k \le M-1} \mathrm{d}(\boldsymbol{G}_j, \boldsymbol{G}_k),$$

which measures in some respect the "nonuniformity" of $\mathcal{X} \subset SO(3)$. We call a sampling set $\mathcal{X}$ q-separated for some $q \in (0, \pi]$ if $q_{\mathcal{X}} \ge q$. From (Filbir and Schmid, 2008, Lemma 5.1) we know that every q-separated sampling set $\mathcal{X}$ has cardinality

$$M \le \frac{109\pi}{2q^3}. \tag{2.6}$$

For further considerations we decompose a q-separated sampling set $\mathcal{X}$ into shells

$$R_{\mathcal{X},q,m} := \{\boldsymbol{G} \in \mathcal{X} : mq \leq \mathrm{d}(\boldsymbol{G}, \boldsymbol{I}) < (m+1)q\}, \quad m \in \mathbb{N}_0,$$

where the cardinality can be estimated by (Gräf and Kunis, Lemma 3.1)

$$|R_{\mathcal{X},q,m}| \leq 48m^2 + 48m + 28. \tag{2.7}$$

Note that $R_{\mathcal{X},q,m} = \emptyset$ whenever $mq > \pi$. On the other hand, the mesh norm

$$\delta_{\mathcal{X}} := 2 \max_{\boldsymbol{H} \in SO(3)} \min_{j=0,\ldots,M-1} \mathrm{d}(\boldsymbol{G}_j, \boldsymbol{H})$$

describes the "density" of $\mathcal{X} \subset SO(3)$. We call a sampling set $\mathcal{X}$ δ-dense for some $\delta \in (0, \pi]$ if $\delta_{\mathcal{X}} \leq \delta$.

In order to compensate for clusters in the sampling set $\mathcal{X}$, it is reasonable to weight the sampling nodes $\boldsymbol{G}_j \in \mathcal{X}$. To this end, we introduce for a given sampling set $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \ldots, M-1\}$ an associated partition

$$\mathcal{R} := \{\Omega_j \subset SO(3) : j = 0, \ldots, M-1\}$$

of $SO(3)$, i.e. $\mathcal{R}$ is a collection of M closed regions $\Omega_j \subset SO(3)$, having no common interior points and covering the whole rotation group, i.e $\bigcup_{j=0}^{M-1} \Omega_j = SO(3)$. Moreover, we require that $\boldsymbol{G}_j$ is an interior point of Ω_j for all $j = 0, \ldots, M-1$. With respect to the partition $\mathcal{R}$ we define the corresponding weights by

$$\boldsymbol{w} := (w_0, \ldots, w_{M-1})^T \in \mathbb{R}^M, \quad w_j := \int_{\Omega_j} \mathrm{d}\mu(\boldsymbol{G}) = \mu(\Omega_j).$$

Finally, the partition norm R of the partition $\mathcal{R}$ is given by

$$R := \max_{j=0,\ldots,M-1} \operatorname{diam} \Omega_j := \max_{j=0,\ldots,M-1} \max_{\boldsymbol{G},\boldsymbol{H} \in \Omega_j} \mathrm{d}(\boldsymbol{G}, \boldsymbol{H}).$$

Remark 2.1
Given a set $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \ldots, M-1\}$ of sampling points on $SO(3)$, there are many possibilities to construct an associated partition $\mathcal{R}$. In many situations, the so-called Voronoi partition $\mathcal{R}^V$, which is determined by

$$\Omega_j^V := \left\{\boldsymbol{H} \in SO(3) : \ \mathrm{d}(\boldsymbol{G}_j, \boldsymbol{H}) = \min_{k=0,\ldots,M-1} \mathrm{d}(\boldsymbol{G}_k, \boldsymbol{H})\right\}, \quad j = 0, \ldots, M-1,$$

is a reasonable choice. Using the Voronoi partition $\mathcal{R}^V$ corresponding to a given sampling set $\mathcal{X}$ we obtain

$$\frac{1}{2}\delta_{\mathcal{X}} \leq R^V \leq \delta_{\mathcal{X}},$$

where R^V denotes the partition norm of $\mathcal{R}^V$. This relation, in turn, makes it possible by using the Voronoi partition to express all of the following results that depend on the partition norm of the underlying partition in terms of the mesh norm $\delta_{\mathcal{X}}$ of the given sampling set $\mathcal{X} \subset SO(3)$. $\square$

In the next step, we would like to get a uniform upper bound on the weights w_j by means of the partition norm R. For this, let $B_r(\boldsymbol{G}) := \{\boldsymbol{H} \in SO(3) : \mathrm{d}(\boldsymbol{G}, \boldsymbol{H}) \le r\}$ be the ball of radius $r \ge 0$ with center $\boldsymbol{G} \in SO(3)$. Then we can find for every $j = 0, \dots, M-1$ an element $\boldsymbol{H}_j \in \Omega_j$ such that $\Omega_j \subset B_R(\boldsymbol{H}_j)$. With (2.5) we get for arbitrary $j = 0, \dots, M-1$

$$w_j \le \mu\left(B_R(\boldsymbol{H}_j)\right) = \mu\left(B_R(\boldsymbol{I})\right) = \frac{2}{\pi}\int_0^R \sin^2\left(\frac{t}{2}\right) \mathrm{d}t \le \frac{1}{2\pi}\int_0^R t^2 \mathrm{d}t = \frac{R^3}{6\pi}. \qquad (2.8)$$

We define the discrete spaces $\ell^p_{\boldsymbol{w}}(SO(3))$, $1 \le p \le \infty$, corresponding to the sampling set $\mathcal{X}$ with associated partition $\mathcal{R}$ in the usual manner with norm

$$\|f\|_{\boldsymbol{w},p} := \begin{cases} \left(\sum\limits_{j=0}^{M-1} w_j |f(\boldsymbol{G}_j)|^p\right)^{1/p} & 1 \le p < \infty, \\ \sup\limits_{j=0,\dots,M-1} |f(\boldsymbol{G}_j)| & p = \infty. \end{cases}$$

To use the compact matrix-vector notation we introduce the non-equispaced $SO(3)$ Fourier matrix

$$\boldsymbol{D} := \left(D^l_{k,k'}(\boldsymbol{G}_j)\right)_{j=0,\dots,M-1;(l,k,k')\in J_N} \in \mathbb{C}^{M\times d_N}$$

for the sampling set $\mathcal{X}$ and polynomial degree N, as well as the sampling vector

$$\boldsymbol{f} := (f(\boldsymbol{G}_0), \dots, f(\boldsymbol{G}_{M-1}))^T \in \mathbb{C}^M$$

of a given polynomial $f \in \boldsymbol{\Pi}_N$. Furthermore, we identify the polynomial f with its $SO(3)$ Fourier coefficients vector

$$\hat{\boldsymbol{f}} := (\hat{f}^l_{k,k'})_{(l,k,k')\in J_N} \in \mathbb{C}^{d_N}.$$

A major role in our further considerations is played by the weighted $SO(3)$ Fourier matrix

$$\boldsymbol{A} := \boldsymbol{W}^{\frac{1}{2}} \boldsymbol{D} \hat{\boldsymbol{W}}^{\frac{1}{2}} \in \mathbb{C}^{M\times d_N}, \qquad (2.9)$$

where

$$\hat{\boldsymbol{W}} := \mathrm{diag}(\hat{\boldsymbol{w}}) \in \mathbb{R}^{d_N\times d_N},\ \hat{\boldsymbol{w}} := (2l+1)_{(l,k,k')\in J_N} \in \mathbb{R}^{d_N}$$

is a scale matrix and

$$\boldsymbol{W} = \mathrm{diag}(\boldsymbol{w}) \in \mathbb{R}^{M\times M}$$

contains the weights w_j of the associated partition $\mathcal{R}$ on the diagonal.

With help of the weighted $SO(3)$ Fourier matrix $\boldsymbol{A}$ we can express the weighted $\ell^2_{\boldsymbol{w}}$-norm of a polynomial $f \in \boldsymbol{\Pi}_N$ by its $SO(3)$ Fourier coefficients vector $\hat{\boldsymbol{f}}$ using

$$\|\boldsymbol{A}\hat{\boldsymbol{f}}\|_2^2 = \sum_{j=0}^{M-1} \left| \sqrt{w_j} \sum_{l=0}^{N} \sum_{k,k'=-l}^{l} \sqrt{2l+1}\, \hat{f}^l_{k,k'} D^l_{k,k'}(\boldsymbol{G}_j) \right|^2 = \sum_{j=0}^{M-1} w_j\, |f(\boldsymbol{G}_j)|^2 = \|f\|^2_{\boldsymbol{w},2}. \quad (2.10)$$

Hence, due to Parseval's equality (2.2) we can formulate the problem of norm equivalences between $\|f\|_2$ and $\|f\|_{\boldsymbol{w},2}$ in terms of $SO(3)$ Fourier coefficients $\hat{\boldsymbol{f}}$, i.e. $\|\hat{\boldsymbol{f}}\|_2$ and $\|\boldsymbol{A}\hat{\boldsymbol{f}}\|_2$.

2.3 Preliminary results

We conclude this section by collecting some necessary results, which are going to be main ingredients in order to show our results in the next section. First, we have the following deterministic L^2-Marcinkiewicz-Zygmund inequality.

Theorem 2.2
Let $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \dots, M-1\}$ be a set of sampling points on $SO(3)$ with associated partition $\mathcal{R} = \{\Omega_j \subset SO(3) : j = 0, \dots, M-1\}$ and let $\varepsilon \in (0,1)$. If $NR < \frac{\varepsilon}{924}$, then we have for every polynomial $f \in \boldsymbol{\Pi}_N$

$$(1-\varepsilon)\|f\|_2^2 \le \|f\|_{\boldsymbol{w},2}^2 \le (1+\varepsilon)\|f\|_2^2 . \tag{2.11}$$

Proof. Let $f \in \boldsymbol{\Pi}_N$ be given. From $|f^2(\boldsymbol{G})| = |f(\boldsymbol{G})|^2$ for all rotations $\boldsymbol{G} \in SO(3)$ we infer that (2.11) is equivalent to

$$(1-\varepsilon)\left\|f^2\right\|_1 \le \left\|f^2\right\|_{\boldsymbol{w},1} \le (1+\varepsilon)\left\|f^2\right\|_1 .$$

Since $f^2 \in \boldsymbol{\Pi}_{2N}$, cf. (2.4), and $2NR < \frac{\varepsilon}{462}$ the assertion follows from (Schmid, 2008, Theorem 4.4). $\square$

Under a much weaker condition on the polynomial degree N we still get the following upper bound, which is essential to relax the condition in the probabilistic Marcinkiewicz-Zygmund inequality.

Lemma 2.3
Let $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \dots, M-1\}$ be a set of sampling points on $SO(3)$ with associated partition $\mathcal{R} = \{\Omega_j \subset SO(3) : j = 0, \dots, M-1\}$. If $NR \le 1$, then we have for every polynomial $f \in \boldsymbol{\Pi}_N$

$$\|f\|_{\boldsymbol{w},1} \le (1 + 1345NR)\|f\|_1 . \tag{2.12}$$

Proof. For $N = 0$ the assertion is trivially fulfilled. So we may assume $N \ge 1$. We follow exactly the proof of (Schmid, 2008, Lemma 4.3) to show that if $NR \le 1$, we have for every $\boldsymbol{H} \in SO(3)$

$$\sum_{j=0}^{M-1} \int_{\Omega_j} \left| v_N(\boldsymbol{H}^{-1}\boldsymbol{G}) - v_N(\boldsymbol{H}^{-1}\boldsymbol{G}_j) \right| \mathrm{d}\mu(\boldsymbol{G}) \le 1345NR, \tag{2.13}$$

where v_N is the reproducing kernel defined in (Schmid, 2008, (3.1)). Let $K := \lfloor \frac{\pi}{R} \rfloor$. Following the proof of (Schmid, 2008, Lemma 4.3) line by line under the condition $NR \le 1$ we end up with the estimate

$$\begin{aligned} & \sum_{j=0}^{M-1} \int_{\Omega_j} \left| v_N(\boldsymbol{H}^{-1}\boldsymbol{G}) - v_N(\boldsymbol{H}^{-1}\boldsymbol{G}_j) \right| \mathrm{d}\mu(\boldsymbol{G}) \\ \le \quad & 1449.1\frac{N}{K} + \frac{2}{\pi}\sum_{k=1}^{2} \int_{\frac{(k-1)\pi}{K}}^{\frac{(k+1)\pi}{K}} \int_{\frac{(k-1)\pi}{K}}^{\frac{(k+1)\pi}{K}} \left| \frac{\mathrm{d}}{\mathrm{d}t} v_N(t) \right| \sin^2\left(\frac{\theta}{2}\right) \mathrm{d}t \, \mathrm{d}\theta . \end{aligned}$$

Using (Schmid, 2008, Lemma 3.3), we get the estimate

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{k=1}^{2}\int_{\frac{(k-1)\pi}{K}}^{\frac{(k+1)\pi}{K}}\int_{\frac{(k-1)\pi}{K}}^{\frac{(k+1)\pi}{K}}\left|\frac{\mathrm{d}}{\mathrm{d}t}v_N(t)\right|\sin^2\left(\frac{\theta}{2}\right)\mathrm{d}t\,\mathrm{d}\theta\\
\leq\;& \frac{2}{\pi}\left(\int_0^{2\pi/K}\left|\frac{\mathrm{d}}{\mathrm{d}t}v_N(t)\right|\mathrm{d}t\int_0^{2\pi/K}\left(\frac{\theta}{2}\right)^2\mathrm{d}\theta+\int_{\pi/K}^{3\pi/K}\left|\frac{\mathrm{d}}{\mathrm{d}t}v_N(t)\right|\mathrm{d}t\int_{\pi/K}^{3\pi/K}\left(\frac{\theta}{2}\right)^2\mathrm{d}\theta\right)\\
\leq\;& \frac{1}{2\pi}\left(\frac{2\pi}{K}\|v_N'\|_\infty\int_0^{2\pi/K}\theta^2\mathrm{d}\theta+\frac{2\pi}{K}\|v_N'\|_\infty\int_{\pi/K}^{3\pi/K}\theta^2\mathrm{d}\theta\right)\\
=\;& \frac{1}{2\pi}\left(\frac{2\pi}{K}\|v_N'\|_\infty\left(\frac{8\pi^3}{3K^3}+\frac{26\pi^3}{3K^3}\right)\right)\\
\leq\;& \frac{34\pi^3}{3K^4}\cdot 2n\|v_N\|_\infty\leq\frac{1360\pi^3}{3}\cdot\frac{N^4}{K^4}.
\end{aligned}
$$

Since $N\neq 0$ together with $NR\leq 1$ implies $R\leq 1$, we get

$$
\frac{N}{K}\leq\frac{N}{\frac{\pi}{R}-1}=\frac{NR}{\pi-R}\leq\frac{NR}{\pi-1}\leq\frac{1}{\pi-1}.
$$

So we arrive at

$$
\begin{aligned}
&\sum_{j=0}^{M-1}\int_{\Omega_j}\left|v_N(\boldsymbol{H}^{-1}\boldsymbol{G})-v_N(\boldsymbol{H}^{-1}\boldsymbol{G}_j)\right|\mathrm{d}\mu(\boldsymbol{G})\leq 1449.1\frac{N}{K}+\frac{1360\pi^3}{3}\cdot\frac{N^4}{K^4}\\
\leq\;&\left(1449.1+\frac{1360\pi^3}{3}\left(\frac{1}{\pi-1}\right)^3\right)\frac{N}{K}\leq 2880.2\frac{N}{K}\leq\frac{2880.2}{\pi-1}NR\leq 1345NR,
\end{aligned}
$$

which shows (2.13). Finally, using the reproduction property of the kernel v_N (see Schmid (2008, Lemma 3.3 (ii))) and (2.13) we get

$$
\begin{aligned}
\Big|\|f\|_{\boldsymbol{w},1}-\|f\|_1\Big|\;\leq\;&\left(\sup_{\boldsymbol{H}\in SO(3)}\sum_{j=0}^{M-1}\int_{\Omega_j}|v_N(\boldsymbol{H}^{-1}\boldsymbol{G})-v_N(\boldsymbol{H}^{-1}\boldsymbol{G}_j)|\mathrm{d}\mu(\boldsymbol{x})\right)\cdot\|f\|_1\\
\leq\;& 1345NR\,\|f\|_1,
\end{aligned}
$$

and that completes the proof of the lemma. □

For further considerations in the next section we introduce some more notations. Let $\mathbb{P}(E)$ denote the probability of an event E. Furthermore, we equip $\mathbb{C}^n$ with the ℓ^2-norm $\|\boldsymbol{x}\|_2^2=\sum_{j=1}^n|x_j|^2$ and we define the complex unit sphere $\mathbb{B}^n:=\{\boldsymbol{x}\in\mathbb{C}^n:\|\boldsymbol{x}\|_2=1\}$. With these notations, we can state the following crucial lemma, which was already established in (Böttcher et al., Corollary 2.2) and is based on the paper of Böttcher and Grudsky (2003). Therein and in the remainder of this paper we denote by $\|\boldsymbol{B}\|_{\mathrm{F}}:=\sqrt{\mathrm{tr}(\boldsymbol{B}^*\boldsymbol{B})}$ the Frobenius norm of a matrix $\boldsymbol{B}\in\mathbb{C}^{m\times n}$.

Lemma 2.4
Let $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and $\|\boldsymbol{A}\|_{\mathrm{F}}^2 = n$. If $\boldsymbol{x}$ is taken at random from the uniform distribution on $\mathbb{B}^n$, then for every $\varepsilon \in (0,1)$ we have

$$\mathbb{P}\Big((1-\varepsilon)\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq (1+\varepsilon)\|\boldsymbol{x}\|_2^2\Big) \geq 1 - \frac{2\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2}{n^2\varepsilon^2}. \tag{2.14}$$

□

Let us briefly display what the use of Lemma 2.4 is. For this, let us assume for the moment that we are given a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ with $\|\boldsymbol{A}\|_2 \leq \gamma$ and $\|\boldsymbol{A}\|_F^2 = n$. Then we have

$$\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2 \leq \|\boldsymbol{A}\|_F^2\|\boldsymbol{A}^*\|_2^2 = n\|\boldsymbol{A}\|_2^2 \leq n\gamma^2$$

and so estimate (2.14) becomes

$$\mathbb{P}\Big((1-\varepsilon)\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq (1+\varepsilon)\|\boldsymbol{x}\|_2^2\Big) \geq 1 - \frac{2\gamma^2}{n\varepsilon^2} \tag{2.15}$$

for each $\varepsilon \in (0,1)$. Choosing $\varepsilon = 1/\sqrt[3]{n}$, we arrive at

$$\mathbb{P}\left(\left(1 - \frac{1}{\sqrt[3]{n}}\right)\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq \left(1 + \frac{1}{\sqrt[3]{n}}\right)\|\boldsymbol{x}\|_2^2\right) \geq 1 - \frac{2\gamma^2}{\sqrt[3]{n}}. \tag{2.16}$$

In other words we can observe that the values $\|\boldsymbol{A}\boldsymbol{x}\|_2$ are concentrated tightly close to $\|\boldsymbol{x}\|_2$ with a probability that converges to 1 as n goes to infinity, although deterministically we cannot say more than $\|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \gamma\|\boldsymbol{x}\|_2$. In the next section we will see that the weighted $SO(3)$ Fourier matrix $\boldsymbol{A} \in \mathbb{C}^{M\times d_N}$, cf. (2.9), enjoys the properties that $\|\boldsymbol{A}\|_2$ has controllable upper bounds and that $\|\boldsymbol{A}\|_F^2 = d_N$. Thus, for these matrices we get estimates like (2.15) and (2.16), which are very useful in connection with probabilistic Marcinkiewicz-Zygmund inequalities (cf. (2.10)).

3 Probabilistic Marcinkiewicz-Zygmund inequalities

In the following we always let the polynomial $f \in \boldsymbol{\Pi}_N$ be given by

$$f := \sum_{(l,k,k')\in J_N} \sqrt{2l+1}\hat{f}_{k,k'}^l D_{k,k'}^l,$$

where the $SO(3)$ Fourier coefficients vector $\hat{\boldsymbol{f}} \in \mathbb{C}^{d_N}$ is taken randomly from the uniform distribution over $\mathbb{B}^{d_N}$. To indicate that the underlying probability distribution depends on the polynomial degree N we write $\mathbb{P}_N(E)$ for the probability of an event E of random polynomials $f \in \boldsymbol{\Pi}_N$ considered above.

Now we are ready to show the following probabilistic L^2-Marcinkiewicz-Zygmund inequality under a relatively mild condition on the polynomial degree.

Theorem 3.1
Let $\mathcal{X}$ be a set of sampling points on $SO(3)$ with associated partition $\mathcal{R}$. If $NR \leq 1/2$, then for every $\varepsilon \in (0,1)$ we have

$$\mathbb{P}_N\left((1-\varepsilon)\|f\|_2^2 \leq \|f\|_{\boldsymbol{w},2}^2 \leq (1+\varepsilon)\|f\|_2^2\right) \geq 1 - \frac{2(1+2690NR)}{d_N\varepsilon^2}.$$

Proof. Similarly as in the proof of Theorem 2.2 we infer from Lemma 2.3 that for every $f \in \mathbf{\Pi}_N$ we have

$$\|f\|_{\boldsymbol{w},2}^2 = \|f^2\|_{\boldsymbol{w},1} \le (1 + 1345 \cdot 2NR)\,\|f^2\|_1 = (1 + 2690NR)\,\|f\|_2^2.$$

Now we consider the weighted $SO(3)$ Fourier matrix $\boldsymbol{A} \in \mathbb{C}^{M \times d_N}$, cf. (2.9). Then, by using Parseval's equality (2.2) and the equality (2.10) we can estimate its spectral norm by

$$\|\boldsymbol{A}\|_2 = \sup_{\boldsymbol{0} \neq \hat{\boldsymbol{f}} \in \mathbb{C}^{d_N}} \frac{\|\boldsymbol{A}\hat{\boldsymbol{f}}\|_2}{\|\hat{\boldsymbol{f}}\|_2} = \sup_{0 \neq f \in \mathbf{\Pi}_N} \frac{\|f\|_{\boldsymbol{w},2}}{\|f\|_2} \le \sqrt{1 + 2690NR}. \tag{3.1}$$

Moreover, from the addition theorem (2.3) and the property $\sum_{j=0}^{M-1} w_j = 1$ of the weights w_j we deduce for the Frobenius norm of $\boldsymbol{A}$

$$\begin{aligned}
\|\boldsymbol{A}\|_{\mathrm{F}}^2 &= \operatorname{tr}(\boldsymbol{A}^*\boldsymbol{A}) = \operatorname{tr}(\hat{\boldsymbol{W}}^{\frac{1}{2}}\boldsymbol{D}^*\boldsymbol{W}\boldsymbol{D}\hat{\boldsymbol{W}}^{\frac{1}{2}}) = \sum_{(l,k,k') \in J_N} \sum_{j=0}^{M-1} w_j (2l+1) \left| D_{k,k'}^l(\boldsymbol{G}_j) \right|^2 \\
&= \sum_{j=0}^{M-1} w_j \sum_{l=0}^{N} \sum_{k,k'=-l}^{l} (2l+1) \left| D_{k,k'}^l(\boldsymbol{G}_j) \right|^2 = \sum_{j=0}^{M-1} w_j \sum_{l=0}^{N} (2l+1) U_{2l}(0) \\
&= \sum_{j=0}^{M-1} w_j \sum_{l=0}^{N} (2l+1)^2 = \sum_{j=0}^{M-1} w_j d_N = d_N.
\end{aligned}$$

So by (3.1) we can conclude

$$\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}} \le \|\boldsymbol{A}\|_{\mathrm{F}} \|\boldsymbol{A}^*\|_2 = \|\boldsymbol{A}\|_{\mathrm{F}} \|\boldsymbol{A}\|_2 \le \sqrt{d_N(1 + 2690NR)}.$$

Using Parseval's equality (2.2), equation (2.10) once again and applying Lemma 2.4 we finally get the assertion

$$\mathbb{P}_N\left((1-\varepsilon)\,\|f\|_2^2 \le \|f\|_{\boldsymbol{w},2}^2 \le (1+\varepsilon)\,\|f\|_2^2\right) \ge 1 - \frac{2\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2}{d_N^2 \varepsilon^2} \ge 1 - \frac{2(1 + 2690NR)}{d_N \varepsilon^2}. \qquad \square$$

This result can be reformulated as a condition on the partition norm R to obtain quite tight probabilistic Marcinkiewicz-Zygmund inequalities for polynomials of high degree.

Corollary 3.2
If for given $\varepsilon, \eta \in (0,1)$ the partition norm R of an associated partition $\mathcal{R}$ of a sampling set $\mathcal{X} \subset SO(3)$ satisfies

$$R \le \varepsilon^{\frac{5}{3}} \eta^{\frac{1}{3}} / 1667, \tag{3.2}$$

then

$$\mathbb{P}_N\left((1-\varepsilon)\,\|f\|_2^2 \le \|f\|_{\boldsymbol{w},2}^2 \le (1+\varepsilon)\,\|f\|_2^2\right) \ge 1 - \eta \tag{3.3}$$

holds for every $N \le 1/(2R)$.

Proof. For $924NR < \varepsilon$ the deterministic Marcinkiewicz-Zygmund inequality (2.11) holds and hence the probabilistic inequality (3.3) is true. So, we have to show (3.3) for $\varepsilon/(924R) \le N \le 1/(2R)$. From Theorem 3.1 we know that (3.3) is satisfied if $NR \le 1/2$ and

$$\frac{2(1 + 2690NR)}{d_N \varepsilon^2} \le \eta.$$

But the last inequality follows by the estimate $d_N \geq 4/3N^3$ from

$$N^3 - \frac{3(1+2690NR)}{2\eta\varepsilon^2} \geq 0. \tag{3.4}$$

With condition (3.2) the inequality (3.4) is satisfied if we insert $\varepsilon/(924R)$ for N. Since the left hand side in (3.4) is monotonically increasing in N for $N \geq \varepsilon/(924R)$, it follows that (3.4) holds for every $N \geq \varepsilon/(924R)$. Thus, to guarantee (3.3) we need nothing but the remaining inequality $NR \leq 1/2$. □

Remark 3.3
Decreasing the partition norm R in $o(\varepsilon^2)$ as $\varepsilon \to 0$ we can obtain an increase for both the probability in order ε and the polynomial degree N in order $1/\varepsilon^2$. In contrast the deterministic Marcinkiewicz-Zygmund inequalities, cf. Theorem 2.2, provide just an increase of the polynomial degree N in order $1/\varepsilon$. □

In order to obtain probabilistic Marcinkiewicz-Zygmund inequalities of other quality we take for the estimate of $\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2$ further properties of sampling sets $\mathcal{X} \subset SO(3)$ with associated partitions $\mathcal{R}$ into account. To this end, we follow the approach of (Böttcher et al., Theorem 4.2).

Lemma 3.4
Let for $q \in (0,\pi]$ a q-separated sampling set $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0,\dots,M-1\}$ with associated partition $\mathcal{R} = \{\Omega_j \subset SO(3) : j = 0,\dots,M-1\}$ be given. Then the matrix $\boldsymbol{A} \in \mathbb{C}^{M\times d_N}$ defined in (2.9) satisfies

$$\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2 \leq \frac{MR^6}{36\pi^2}\left(d_N^2 + 124\pi^3(N+1)^4 q^{-3}\right). \tag{3.5}$$

Proof. Due to the estimate (2.8) for the weights w_j we have

$$\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2 = \sum_{i,j=0}^{M-1} |(\boldsymbol{A}\boldsymbol{A}^*)_{i,j}|^2 = \sum_{i,j=0}^{M-1} w_i w_j \left|(\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^*)_{i,j}\right|^2 \leq \frac{R^6}{36\pi^2}\left\|\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^*\right\|_{\mathrm{F}}^2.$$

Now, we show (3.5) by estimating the Frobenius norm of $\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^*$. By virtue of the addition theorem (2.3) and the relation of the Chebyshev polynomials

$$U_{2l}(\cos(t/2)) = \frac{\sin((2l+1)t/2)}{\sin(t/2)} = \sum_{k=-l}^{l} \mathrm{e}^{\mathrm{i}kt}, \quad l \in \mathbb{N}_0,$$

to the Dirichlet kernel we obtain

$$\begin{aligned}\left\|\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^*\right\|_{\mathrm{F}}^2 &= \sum_{i,j=0}^{M-1}\left|(\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^*)_{i,j}\right|^2 = \sum_{i,j=0}^{M-1}\left|\sum_{l=0}^{N}(2l+1)U_{2l}\left(\cos\left(\frac{\mathrm{d}(\boldsymbol{G}_i,\boldsymbol{G}_j)}{2}\right)\right)\right|^2 \\ &= \sum_{i,j=0}^{M-1}\left|\sum_{l=0}^{N}(2l+1)\sum_{k=-l}^{l}\mathrm{e}^{\mathrm{i}k\mathrm{d}(\boldsymbol{G}_i,\boldsymbol{G}_j)}\right|^2 = \sum_{i,j=0}^{M-1}\left|\sum_{k=-N}^{N}\sum_{l=|k|}^{N}(2l+1)\mathrm{e}^{\mathrm{i}k\mathrm{d}(\boldsymbol{G}_i,\boldsymbol{G}_j)}\right|^2.\end{aligned}$$

Defining for $t \in [0, \pi]$ the univariate trigonometric polynomial

$$K(t) := \sum_{k=-N}^{N} \sum_{l=|k|}^{N} (2l+1) \mathrm{e}^{\mathrm{i}kt} = \sum_{k=-N}^{N} ((N+1)^2 - k^2) \mathrm{e}^{\mathrm{i}kt}$$

we have for $t \in (0, \pi]$ the estimate

$$|K(t)| = \left| \frac{1}{1-\mathrm{e}^{\mathrm{i}t}} \sum_{k=-N}^{N+1} (1-2k) \mathrm{e}^{\mathrm{i}kt} \right| \leq \frac{1}{2\sin(t/2)} \sum_{k=-N}^{N+1} |1-2k| \leq \pi (N+1)^2 t^{-1}.$$

Using the bound (2.7) for the cardinality of the shells $R_{\mathcal{X},q,m}$ we rearrange the sum and get

$$\begin{aligned}
\left\| \boldsymbol{D} \hat{\boldsymbol{W}} \boldsymbol{D}^* \right\|_{\mathrm{F}}^2 &= \sum_{i=0}^{M-1} \left(|K(0)|^2 + \sum_{\substack{j=0 \\ j \neq i}}^{M-1} |K(\mathrm{d}(\boldsymbol{G}_i, \boldsymbol{G}_j))|^2 \right) \\
&\leq M d_N^2 + \sum_{i=0}^{M-1} \sum_{m=1}^{\lfloor \pi q^{-1} \rfloor} |R_{\mathcal{X},q,m}| \max_{\boldsymbol{G} \in R_{\mathcal{X},q,m}} |K(\mathrm{d}(\boldsymbol{I}, \boldsymbol{G}))|^2 \\
&\leq M d_N^2 + M \sum_{m=1}^{\lfloor \pi q^{-1} \rfloor} (48m^2 + 48m + 28) \frac{\pi^2 (N+1)^4}{(mq)^2} \\
&\leq M d_N^2 + 124 \pi^3 M (N+1)^4 q^{-3},
\end{aligned}$$

which finishes the proof. □

Now we are ready to prove the existence of universal sampling sets $\mathcal{X} \subset SO(3)$ with associated partitions $\mathcal{R}$ as in the case of the sphere $\mathbb{S}^d$ (Böttcher et al.). That is, the Marcinkiewicz-Zygmund inequalities are satisfied with high probability and for all polynomial degrees, if the sampling set $\mathcal{X}$ is sufficiently dense and relatively uniform distributed.

Theorem 3.5

Let for $q \in (0, \pi]$ a q-separated sampling set $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \ldots, M-1\}$ with associated partition $\mathcal{R} = \{\Omega_j \subset SO(3) : j = 0, \ldots, M-1\}$ be given and let $\varepsilon \in (0,1)$, $\eta \in (0,1)$, $L \in (1, \infty)$ be fixed. Then, there exists a number $\varrho_0 = \varrho_0(\varepsilon, \eta, L) > 0$ such that

$$\mathbb{P}_N \left((1-\varepsilon) \|f\|_2^2 \leq \|f\|_{\boldsymbol{w},2}^2 \leq (1+\varepsilon) \|f\|_2^2 \right) \geq 1 - \eta \tag{3.6}$$

holds for every polynomial degree $N \geq 0$, if the uniformity condition $\frac{R}{q} \leq L$ and the density condition $R \leq \varrho_0$ are satisfied.

Proof. We apply (2.10) together with Lemma 2.4 and obtain with Lemma 3.4 the estimate for the probability

$$\begin{aligned}
\mathbb{P}_N \left(1 - \varepsilon \leq \frac{\|\boldsymbol{A}\hat{\boldsymbol{f}}\|_2^2}{\|\hat{\boldsymbol{f}}\|_2^2} \leq 1 + \varepsilon \right) &\geq 1 - \frac{2\|\boldsymbol{A}\boldsymbol{A}^*\|_{\mathrm{F}}^2}{d_N^2 \varepsilon^2} \\
&\geq 1 - \frac{MR^6}{18\pi^2\varepsilon^2} - \frac{62\pi M (N+1)^4 R^6}{9\varepsilon^2 d_N^2 q^3} \\
&\geq 1 - \frac{MR^6}{18\pi^2\varepsilon^2} - \frac{31\pi M R^6}{8\varepsilon^2 N^2 q^3}.
\end{aligned}$$

The first term of the right hand side can be bounded using inequality (2.6) for some $R \leq \varrho_1 := \varrho_1(\varepsilon, \eta, L)$ by

$$\frac{MR^6}{18\pi^2\varepsilon^2} = \frac{Mq^3}{18\pi^2\varepsilon^2}\frac{R^3}{q^3}R^3 \leq \frac{109L^3}{36\pi\varepsilon^2}R^3 \leq \frac{\eta}{2}.$$

If $NR \leq \frac{\varepsilon}{924}$, the assertion is trivially fulfilled due to the deterministic Marcinkiewicz-Zygmund inequality (2.11). Now if $NR > \frac{\varepsilon}{924}$, then again by inequality (2.6) we obtain the estimate

$$\frac{31\pi MR^6}{8q^3N^2\varepsilon^2} \leq \frac{3379\pi^2R^6(924R)^2}{16q^6\varepsilon^4} \leq \frac{2\cdot 10^9 L^6}{\varepsilon^4}R^2$$

This can be bounded for some $R \leq \varrho_2 := \varrho_2(\varepsilon, \eta, L)$ by $\frac{\eta}{2}$. So the assertion (3.6) follows for $R \leq \varrho_0 := \min(\varrho_1, \varrho_2)$. □

Remark 3.6
The consideration of probabilistic Marcinkiewicz-Zygmund inequalities for polynomials on $SO(3)$ with Fourier coefficients vectors taken from the complex sphere $\mathbb{B}^{d_N}$ is not as restrictive as it looks like. We can obtain the same probabilistic results actually for random Fourier coefficients vectors taken from the uniform distribution on the complex ball $\mathbb{G}_r^{d_N} = \left\{\boldsymbol{x} \in \mathbb{C}^{d_N} : \|x\|_2 \leq r\right\}$ of radius $r > 0$ or the Gaussian normal distribution on $\mathbb{R}^{2d_N} \cong \mathbb{C}^{d_N}$. Since these are radially symmetric probability distributions, we just use the fact that the deterministic inequalities (2.11), (2.12) are satisfied for $f \in \mathbf{\Pi}_N, \|f\|_2 = 1$ if and only if these are true for scaled polynomials λf, $\lambda \in \mathbb{C} \setminus \{0\}$. □

Finally, we present a simple example to illustrate the significant relaxation on the condition for the polynomial degree N in the probabilistic setting.

Example 3.7
Let us consider the deterministic Marcinkiewicz-Zygmund inequality (2.11) for $\varepsilon = 1/2$ and $N = 5$. If the sampling set $\mathcal{X} \subset SO(3)$ has associated partition $\mathcal{R}$ with $R \leq 1/(2\cdot 5\cdot 924) \approx 0.0001$, then for $f \in \mathbf{\Pi}_5$ it holds

$$\frac{1}{2}\|f\|_2^2 \leq \|f\|_{\boldsymbol{w},2}^2 \leq \frac{3}{2}\|f\|_2^2. \tag{3.7}$$

But Theorem 3.1 yields for such a sampling set $\mathcal{X}$ that this inequality is actually true with probability at least 0.99999991 for randomly taken $f \in \mathbf{\Pi}_{4620}$.

To get (3.7) deterministically for $N \leq 4620$ we need a sampling set $\mathcal{X}' \subset SO(3)$ with partition $\mathcal{R}'$ such that $R' \leq 1/(2\cdot 4620\cdot 924) \approx 1.1\cdot 10^{-7}$. This sampling set $\mathcal{X}'$ has approximately $(R/R')^3 \approx 10^9$ times more sampling nodes than $\mathcal{X}$, if these sets $\mathcal{X}$ and $\mathcal{X}'$ are required to have minimal cardinality. □

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