# Nonuniform fast Fourier transforms with nonequispaced spatial and frequency data and fast sinc transforms

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In this paper we study the nonuniform fast Fourier transform with nonequispaced spatial and frequency data (NNFFT) and the fast sinc transform as its application. The computation of NNFFT is mainly based on the nonuniform fast Fourier transform with nonequispaced spatial nodes and equispaced frequencies (NFFT). The NNFFT employs two compactly supported, continuous window functions. For fixed nonharmonic bandwidth, it is shown that the error of the NNFFT with two sinh-type window functions has an exponential decay with respect to the truncation parameters of the used window functions. As an important application of the NNFFT, we present the fast sinc transform. The error of the fast sinc transform is estimated, too.

*Key words*: nonuniform fast Fourier transform, NUFFT, NNFFT, nonequispaced nodes in space and frequency domain, exponential sums, fast sinc transform, error estimates, sampling.

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## 1 Introduction

The *discrete Fourier transform* (DFT) can easily be generalized to arbitrary nodes in the space domain as well as in the frequency domain (see [7, 9], [26, pp. 394–397]). Let $N \in 2\mathbb{N}$ with $N \gg 1$ and $M_1, M_2 \in 2\mathbb{N}$ be given. By $I_{M_1}$ we denote the index set $\{-\frac{M_1}{2}, 1 - \frac{M_1}{2}, \ldots, \frac{M_1}{2} - 1\}$. We consider an *exponential sum* $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ of the

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form

$$f(x) := \sum_{k \in I_{M_1}} f_k \, \mathrm{e}^{-2\pi \mathrm{i}\, N\, v_k x}\,, \quad x \in \big[-\tfrac{1}{2},\, \tfrac{1}{2}\big]\,, \tag{1.1}$$

where $f_k \in \mathbb{C}$ are given coefficients and $v_k \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $k \in I_{M_1}$, are arbitrary nodes in the frequency domain. The parameter $N \in \mathbb{N}$ is called *nonharmonic bandwidth* of the exponential sum (1.1). Note that each exponential $\exp(-2\pi \mathrm{i}\, N v_k \,\cdot)$ has a frequency smaller or equal than $\frac{N}{2}$.

We assume that a linear combination (1.1) of exponentials with bounded frequencies is given. For arbitrary nodes $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in I_{M_2}$, in the space domain, we are interested in a fast evaluation of the $M_2$ values

$$f(x_j) = \sum_{k \in I_{M_1}} f_k \, \mathrm{e}^{-2\pi \mathrm{i}\, N\, v_k x_j}\,, \quad j \in I_{M_2}\,. \tag{1.2}$$

A fast algorithm for the computation of the $M_2$ values (1.2) is called a *nonuniform fast Fourier transform with nonequispaced spatial and frequency data* (NNFFT) which was introduced by B. Elbel and G. Steidl [9]. In this approach, the rapid evaluation of NNFFT is mainly based on the use of two compactly supported, continuous window functions. As in [19] this approach is also referred to as *NFFT of type 3*.

In this paper we present new error estimates for the NNFFT. Since these estimates depend exclusively on the so-called window parameters of the NNFFT, this gives rise to an appropriate parameter choice. The outline of this paper is as follows. In Section 2, we introduce the special set $\Omega$ of continuous, even functions $\omega : \mathbb{R} \to [0,\, 1]$ with the support $[-1,\, 1]$. Choosing $\omega_1, \omega_2 \in \Omega$, we consider two window functions

$$\varphi_1(t) = \omega_1\big(\frac{N_1 t}{m_1}\big)\,, \quad \varphi_2(t) = \omega_2\big(\frac{N_2 t}{m_2}\big)\,, \quad t \in \mathbb{R}\,,$$

where $N_1 = \sigma_1 N \in 2\mathbb{N}$ with some oversampling factor $\sigma_1 > 1$ and where $m_1 \in \mathbb{N} \setminus \{1\}$ is a truncation parameter with $2m_1 \ll N_1$. Analogously, $N_2 = \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ is given with some oversampling factor $\sigma_2 > 1$ and $m_2 \in \mathbb{N} \setminus \{1\}$ is another truncation parameter with $2m_2 \ll \big(1 - \frac{1}{\sigma_1}\big)\, N_2$. For the fast, approximate computation of the values (1.2), we formulate in Algorithm 2.2 the NNFFT. In Section 3, we derive new explicit error estimates of the NNFFT with two window functions $\varphi_1$ and $\varphi_2$. In Section 4, we consider the NNFFT with two sinh-type window functions. Here we show that for fixed nonharmonic bandwidth $N$ of (1.1), the error of the related NNFFT has an exponential decay with respect to the truncation parameters $m_1$ and $m_2$. Numerical experiments illustrate the performance of our error estimates.

In Section 5, we study the approximation of the function $\mathrm{sinc}(N\pi x)$, $x \in [-1,\, 1]$, by an exponential sum. For given target accuracy $\varepsilon > 0$ and $n \ge 4N$, there exist coefficients $w_j > 0$ and frequencies $v_j \in \big(-\frac{1}{2},\, \frac{1}{2}\big)$, $j = 1\, \ldots, n$, such that for all $x \in [-1,\, 1]$,

$$\big|\mathrm{sinc}(N\pi x) - \sum_{j=1}^{n} w_j \, \mathrm{e}^{-2\pi \mathrm{i} N v_j x}\big| \le \varepsilon\,.$$

In practice, we simplify the approximation procedure of the function $\mathrm{sinc}(N\pi x)$, $x \in [-1,\, 1]$. Since for fixed $N \in \mathbb{N}$, it holds

$$\mathrm{sinc}(N\pi x) = \frac{1}{2} \int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\,\mathrm{d}t\,, \quad x \in \mathbb{R}\,,$$

we can apply the Clenshaw–Curtis quadrature with the Chebyshev points $z_k = \cos\frac{k\pi}{n} \in \big[-1,\, 1\big]$, $k = 0\,\ldots, n$, where $n \in \mathbb{N}$ fulfills $n \geq 4N$. Then the function $\mathrm{sinc}(N\pi x)$, $x \in [-1,\, 1]$, can be approximated by the exponential sum

$$\sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N z_k x} \tag{1.3}$$

with explicitly known coefficients $w_k > 0$ which satisfy the condition $\sum_{k=0}^{n} w_k = 1$. An interesting signal processing application of the NNFFT is presented in the last Section 6. If a signal $h : \big[-\frac{1}{2},\, \frac{1}{2}\big] \to \mathbb{C}$ is to be reconstructed from its nonuniform samples at $a_k \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, then $h$ is often modeled as linear combination of shifted sinc functions

$$h(x) = \sum_{k \in I_{L_1}} c_k\, \mathrm{sinc}\big(N\pi\,(x - a_k)\big)\,.$$

with complex coefficients $c_k$. Hence, we present a fast, approximate computation of the *discrete* sinc *transform* (see [12, 20])

$$h(b_\ell) = \sum_{k \in I_{L_1}} c_k\, \mathrm{sinc}\big(N\pi\,(b_\ell - a_k)\big)\,, \quad \ell \in I_{L_2}\,, \tag{1.4}$$

where $b_\ell \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ can be nonequispaced. The discrete sinc transform is motivated by numerous applications in signal processing. However, since the sinc function decays slowly, it is often avoided in favor of some more local approximation. Here we prefer the approximation of the sinc function by an exponential sum (1.3). Then we obtain the fast sinc Algorithm 6.1 which is an approximate algorithm for the fast computation of the values (6.2). This fast sinc transform applies the NNFFT twice. The error of the fast sinc transform is estimated. Numerical examples are presented, too.

## 2 NNFFT

Let $\Omega$ be the set of all functions $\omega : \mathbb{R} \to [0,\, 1]$ with the following properties:

- Each function $\omega$ is even, has the support $[-1,\, 1]$, and is continuous on $\mathbb{R}$.
- Each restricted function $\omega|_{[0,\,1]}$ is decreasing with $\omega(0) = 1$.
- For each function $\omega$ its Fourier transform

  $$\hat{\omega}(v) := \int_{\mathbb{R}} \omega(x)\, \mathrm{e}^{-2\pi \mathrm{i}\, v x}\,\mathrm{d}x = 2 \int_{0}^{1} \omega(x)\, \cos(2\pi v x)\,\mathrm{d}x$$

  is positive and decreasing for all $v \in \big[0,\, \frac{m_1}{2\sigma_1}\big]$, where it holds $m_1 \in \mathbb{N} \setminus \{1\}$ and $\sigma_1 \in \big[\frac{5}{4},\, 2\big]$.

Obviously, each $\omega \in \Omega$ is of bounded variation over $[-1,\,1]$.

**Example 2.1** *By $B_{2m_1}$, we denote the centered cardinal B-spline of even order $2m_1$ with $m_1 \in \mathbb{N}$. Thus, $B_2$ is the centered hat function. We consider the spline*

$$\omega_{\mathrm{B},1}(x) := \frac{1}{B_{2m_1}(0)}\, B_{2m_1}(m_1 x)\,, \quad x \in \mathbb{R}\,,$$

*which has the support $[-1,\,1]$. Its Fourier transform reads as follows*

$$\hat{\omega}_{\mathrm{B},1}(v) = \frac{1}{m_1\, B_{2m_1}(0)}\,\Big(\mathrm{sinc}\frac{\pi v}{m_1}\Big)^{2m_1}\,, \quad v \in \mathbb{R}\,.$$

*Obviously, $\hat{\omega}_{\mathrm{B},1}(v)$ is positive and decreasing for $v \in [0,\,m_1)$. Hence, the function $\omega_{\mathrm{B},1}$ belongs to the set $\Omega$.*
*For $\sigma_1 > \frac{\pi}{3}$ and $\beta_1 = 3m_1$ with $m_1 \in \mathbb{N}\setminus\{1\}$, we consider*

$$\omega_{\mathrm{alg},1}(x) := \begin{cases} (1-x^2)^{\beta_1 - 1/2} & x \in [-1,\,1]\,,\\ 0 & x \in \mathbb{R}\setminus[-1,\,1]\,. \end{cases}$$

*By [23, p. 8], its Fourier transform reads as follows*

$$\hat{\omega}_{\mathrm{alg},1}(v) = \frac{\pi\,(2\beta_1)!}{4^{\beta_1}\,\beta_1!}\cdot \begin{cases} (\pi v)^{-\beta_1}\, J_{\beta_1}(2\pi v) & v \in \mathbb{R}\setminus\{0\}\,,\\ \frac{1}{\beta_1!} & v = 0\,, \end{cases}$$

*where $J_{\beta_1}$ denotes the* Bessel function of order $\beta_1$. *By [1, p. 370], it holds for $v \neq 0$ the equality*

$$(\pi v)^{-\beta_1}\, J_{\beta_1}(2\pi v) = \frac{1}{\beta_1!}\prod_{s=1}^{\infty}\Big(1 - \frac{4\pi^2 v^2}{j_{\beta_1,s}^2}\Big)\,,$$

*where $j_{\beta_1,s}$ denotes the $s$th positive zero of $J_{\beta_1}$. For $\beta_1 = 3m_1$, it holds $j_{\beta_1,1} > 3m_1 + \pi - \frac{1}{2}$ (see [13]). Hence, by $\sigma_1 > \frac{\pi}{3}$ we get*

$$\frac{2\pi m_1}{2\sigma_1\, j_{\beta_1,1}} < \frac{\frac{\pi}{\sigma_1}\, m_1}{3m_1 + \pi - \frac{1}{2}} < \frac{3m_1}{3m_1 + \pi - \frac{1}{2}} < 1\,.$$

*Therefore, the Fourier transform $\hat{\omega}_{\mathrm{alg},1}(v)$ is positive and decreasing for $v \in \big[0,\,\frac{m_1}{2\sigma_1}\big]$. Hence, $\omega_{\mathrm{alg},1}$ belongs to the set $\Omega$.*
*Let $\sigma_1 \in \big[\frac{5}{4},\,2\big]$ and $m_1 \in \mathbb{N}\setminus\{1\}$ be given. We consider the function*

$$\omega_{\sinh,1}(x) := \begin{cases} \frac{1}{\sinh \beta_1}\,\sinh\big(\beta_1\sqrt{1-x^2}\big) & x \in [-1,\,1]\,,\\ 0 & x \in \mathbb{R}\setminus[-1,\,1] \end{cases} \tag{2.1}$$

*with the* shape parameter

$$\beta_1 := 2\pi m_1\big(1 - \frac{1}{2\sigma_1}\big)\,.$$

*Then by [23, p. 38], its Fourier transform reads as follows*

$$\hat{\omega}_{\mathrm{sinh},1}(v) = \frac{\pi\beta_1}{\sinh\beta_1}\cdot\begin{cases}(\beta_1^2 - 4\pi^2 v^2)^{-1/2}\, I_1\big(\sqrt{\beta_1^2 - 4\pi^2 v^2}\big) & |v| < m_1\big(1 - \frac{1}{2\sigma_1}\big)\,,\\ \frac{1}{2} & v = \pm m_1\big(1 - \frac{1}{2\sigma_1}\big)\,,\\ (4\pi^2 v^2 - \beta_1^2)^{-1/2}\, J_1\big(\sqrt{4\pi^2 v^2 - \beta_1^2}\big) & |v| > m_1\big(1 - \frac{1}{2\sigma_1}\big)\,,\end{cases} \tag{2.2}$$

*where $I_1$ and $J_1$ denote the* modified Bessel function *and the* Bessel function of first order*, respectively. Using the power series expansion of $I_1$ (see [1, p. 375]), for $|v| < m_1\big(1-\frac{1}{2\sigma_1}\big)$ we obtain*

$$(\beta_1^2 - 4\pi^2 v^2)^{-1/2}\, I_1\big(\sqrt{\beta_1^2 - 4\pi^2 v^2}\big) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k k!(k+1)!}\,(\beta_1^2 - 4\pi^2 v^2)^k\,.$$

*Therefore, the Fourier transform $\hat{\omega}_{\mathrm{sinh},1}(v)$ is positive and decreasing for $v \in \big[0,\, \frac{m_1}{2\sigma_1}\big]$, since for $\sigma_1 \ge \frac{5}{4}$ it holds*

$$\frac{m_1}{2\sigma_1} < m_1\big(1 - \frac{1}{2\sigma_1}\big)\,.$$

*Hence, $\omega_{\mathrm{sinh},1}$ belongs to the set $\Omega$.* □

As known (see [9, 28]), the NNFFT can be mainly computed by an NFFT. For $\omega_1 \in \Omega$ we introduce the *window function*

$$\varphi_1(t) := \omega_1\big(\frac{N_1 t}{m_1}\big)\,, \quad t \in \mathbb{R}\,. \tag{2.3}$$

By construction, the window function (2.3) is even, has the support $\big[-\frac{m_1}{N_1},\, \frac{m_1}{N_1}\big]$, and is continuous on $\mathbb{R}$. Further, the restricted window function $\varphi_1|_{[0,\, m_1/N_1]}$ is decreasing with $\varphi_1(0) = 1$. Its Fourier transform

$$\hat{\varphi}_1(v) := \int_{\mathbb{R}} \varphi_1(t)\, \mathrm{e}^{-2\pi\mathrm{i}\, vt}\, \mathrm{d}t = 2\int_0^{m_1/N_1} \varphi_1(t)\, \cos(2\pi vt)\, \mathrm{d}t$$

is positive and decreasing for $v \in \big[0,\, N_1 - \frac{N}{2}\big)$. Thus, $\varphi_1$ is of bounded variation over $\big[-\frac{1}{2},\, \frac{1}{2}\big]$.

In the following, we denote the torus $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$ and the Banach space of continuous, 1-periodic functions by $C(\mathbb{T})$. For the window function (2.3), we denote its 1-periodization by

$$\tilde{\varphi}_1^{(1)}(x) := \sum_{k\in\mathbb{Z}} \varphi_1(x+k)\,, \quad x \in \mathbb{R}\,.$$

For fixed $N,\, M_1 \in 2\mathbb{N}$ and $N_1 = \sigma_1 N$ with $\sigma_1 > 1$, the NFFT (see [7, 8, 32] or [26, pp. 377–381]) is a fast algorithm which approximately computes the values $p(x_j)$, $j \in I_{M_1}$, of any 1-periodic trigonometric polynomial

$$p(x) := \sum_{k\in I_N} c_k\, \mathrm{e}^{2\pi\mathrm{i}\, kx}$$

at nonequispaced nodes $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in I_{M_1}$, where $c_k \in \mathbb{C}$, $k \in I_N$, are given complex coefficients. Using a linear combination of shifted versions of the 1-periodized window function $\tilde{\varphi}_1^{(1)}$, we construct a 1-periodic continuous function $s \in C(\mathbb{T})$ which well approximates $p$. Then the computation of the values $s(x_j)$, $j \in I_{M_1}$, is very easy, since $\varphi_1$ has the small support $\big[-\frac{m_1}{N_1},\, \frac{m_1}{N_1}\big]$. The computational cost of NFFT is $\mathcal{O}\big(N \log N + M_1\big)$ flops, see [7, 8, 32] or [26, pp. 377–381]. The error of the NFFT (see [29]) can be estimated by

$$\begin{aligned} \max_{j\in I_{M_1}} |s(x_j) - p(x_j)| &\le \|s-p\|_{C(\mathbb{T})} := \max_{x\in[-1/2,1/2]} |s(x) - p(x)| \\ &\le e_{\sigma_1}(\varphi_1) \sum_{n\in I_N} |c_n|\,, \end{aligned}$$

where $e_{\sigma_1}(\varphi_1)$ denotes the $C(\mathbb{T})$-*error constant* defined as

$$e_{\sigma_1}(\varphi_1) = \sup_{N\in 2\mathbb{N}} e_{\sigma_1,N}(\varphi_1) \tag{2.4}$$

with

$$e_{\sigma_1,N}(\varphi_1) := \max_{n\in I_N} \Big\| \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat{\varphi}_1(n+rN_1)}{\hat{\varphi}_1(n)}\, \mathrm{e}^{2\pi \mathrm{i}\, r\, N_1 \cdot} \Big\|_{C(\mathbb{T})}\,.$$

Note that the constants $e_{\sigma_1,N}(\varphi_1)$ are bounded with respect to $N$ (see [29, Theorem 5.1]).

For better readability, we describe the NNFFT shortly. For chosen functions $\omega_1,\, \omega_2 \in \Omega$, we form the window functions

$$\varphi_1(t) := \omega_1\big(\frac{N_1 t}{m_1}\big)\,, \quad \varphi_2(t) := \omega_2\big(\frac{N_2 t}{m_2}\big)\,, \quad t \in \mathbb{R}\,, \tag{2.5}$$

where again $N_1 = \sigma_1 N \in 2\mathbb{N}$ with some oversampling factor $\sigma_1 > 1$ and $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$ and where $N_2 := \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ with an oversampling factor $\sigma_2 > 1$ and $m_2 \in \mathbb{N}\setminus\{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$. The second window function $\varphi_2$ has the support $\big[-\frac{m_2}{N_2},\, \frac{m_2}{N_2}\big]$. We introduce the constant

$$a := 1 + \frac{2m_1}{N_1} > 1\,, \tag{2.6}$$

such that $aN_1 = N_1 + 2m_1$ and $N_2 = \sigma_2\sigma_1 a N$. Without loss of generality, we can assume that

$$v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]\,. \tag{2.7}$$

If $v_k \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, then we replace the nonharmonic bandwidth $N$ by $N^* := N + \lceil \frac{2m_1}{\sigma_1} \rceil$ and set $v_j^* := \frac{N}{N^*}\, v_j \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$ such that $N\, v_j = N^* v_j^*$.

For arbitrarily given $f_k \in \mathbb{C}$, $k \in I_{M_1}$, and $v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$, $k \in I_{M_1}$, we introduce the compactly supported, continuous auxiliary function

$$h(t) := \sum_{k\in I_{M_1}} f_k\, \varphi_1(t - v_k)\,, \quad t \in \mathbb{R}\,,$$

which has the Fourier transform

$$\begin{aligned}\hat h(Nx) &= \int_{\mathbb{R}} h(t)\,\mathrm{e}^{-2\pi\mathrm{i}\,Nxt}\,\mathrm{d}t \\ &= \sum_{k\in I_{M_1}} f_k \int_{\mathbb{R}} \varphi_1(t-v_k)\,\mathrm{e}^{-2\pi\mathrm{i}\,Nxt}\,\mathrm{d}t \qquad (2.8)\\ &= \sum_{k\in I_{M_1}} f_k\,\mathrm{e}^{-2\pi\mathrm{i}\,Nv_kx}\,\hat\varphi_1(Nx) = f(x)\,\hat\varphi_1(Nx)\,, \quad x\in\mathbb{R}\,. \qquad (2.9)\end{aligned}$$

Hence, for arbitrary nodes $x_j \in \big[-\frac{1}{2}, \frac{1}{2}\big]$, $j\in I_{M_2}$, we have

$$f(x_j) = \frac{\hat h(N\,x_j)}{\hat\varphi_1(N\,x_j)}\,, \quad j\in I_{M_2}\,.$$

Therefore, it remains to compute the values $\hat h(N\,x_j)$, $j\in I_{M_2}$, because we can precompute the values $\hat\varphi_1(N\,x_j)$, $j\in I_{M_2}$. In some cases (see Section 4), these values $\hat\varphi_1(N\,x_j)$, $j\in I_{M_2}$, are explicitly known.
For arbitrary $v_k \in \big[-\frac{1}{2a}, \frac{1}{2a}\big]$ with $k\in I_{M_1}$, we have $\varphi_1(t-v_k)=0$ for all $t < -\frac{1}{2a}-\frac{m_1}{N_1} = -\frac{a}{2}+\big(\frac{1}{2}-\frac{1}{2a}\big)$ and for all $t > \frac{1}{2a}+\frac{m_1}{N_1} = \frac{a}{2}-\big(\frac{1}{2}-\frac{1}{2a}\big)$, since $\operatorname{supp}\varphi_1 = \big[-\frac{m_1}{N_1}, \frac{m_1}{N_1}\big]$ and $\frac{1}{2}-\frac{1}{2a} > 0$. Thus, by (2.8) and

$$\operatorname{supp}\varphi_1(\cdot - v_k) \subset \Big[-\frac{a}{2}, \frac{a}{2}\Big]\,, \quad k\in I_{M_1}\,,$$

we obtain

$$\hat h(Nx) = \sum_{k\in I_{M_1}} f_k \int_{-a/2}^{a/2} \varphi_1(t-v_k)\,\mathrm{e}^{-2\pi\mathrm{i}\,Nxt}\,\mathrm{d}t\,, \quad x\in\mathbb{R}\,.$$

Then the rectangular quadrature rule leads to

$$s(Nx) := \sum_{k\in I_{M_1}} f_k\,\frac{1}{N_1} \sum_{\ell\in I_{N_1+2m_1}} \varphi_1\big(\frac{\ell}{N_1}-v_k\big)\,\mathrm{e}^{-2\pi\mathrm{i}\,\ell x/\sigma_1}\,, \quad x\in\mathbb{R}\,, \qquad (2.10)$$

which approximates $\hat h(Nx)$. Note that $\frac{\ell}{N_1} \in \big[-\frac{a}{2}, \frac{a}{2}\big]$ for each $\ell \in I_{N_1+2m_1}$ by $N_1+2m_1 = aN_1$. Changing the order of summations in (2.10), it follows that

$$s(Nx) = \sum_{\ell\in I_{N_1+2m_1}} \Big(\frac{1}{N_1} \sum_{k\in I_{M_1}} f_k\,\varphi_1\big(\frac{\ell}{N_1}-v_k\big)\Big)\,\mathrm{e}^{-2\pi\mathrm{i}\,\ell x/\sigma_1}\,. \qquad (2.11)$$

After computation of the inner sums

$$g_\ell := \frac{1}{N_1} \sum_{k\in I_{M_1}} f_k\,\varphi_1\big(\frac{\ell}{N_1}-v_k\big)\,, \quad \ell\in I_{N_1+2m_1}\,, \qquad (2.12)$$

we arrive at the following NFFT

$$s(N\,x_j) = \sum_{\ell \in I_{N_1+2m_1}} g_\ell\, \mathrm{e}^{-2\pi \mathrm{i}\, \ell x_j/\sigma_1}\,, \quad j \in I_{M_2}\,.$$

If we denote the result of this NFFT (with the 1-periodization $\tilde{\varphi}_2^{(1)}$ of the second window function $\varphi_2$ and $N_2 = \sigma_2\,(N_1 + 2m_1))$ by $s_1(N\,x_j)$, then $s_1(N\,x_j)/\hat{\varphi}_1(N\,x_j)$ is an approximate value of $f(x_j)$, $j \in I_{M_2}$. We summarize:

**Algorithm 2.2 (NNFFT)**
*Input:* $N \in \mathbb{N}$ *with* $N \gg 1$, $M_1,\, M_2 \in 2\mathbb{N}$,
$N_1 = \sigma_1\, N \in 2\mathbb{N}$ *with* $\sigma_1 > 1$, $m_1 \in \mathbb{N} \setminus \{1\}$ *with* $2m_1 \ll N_1$, $a = 1 + \frac{2m_1}{N_1}$,
$N_2 = \sigma_2\,(N_1 + 2m_1) \in 2\mathbb{N}$ *with* $\sigma_2 > 1$, $m_2 \in \mathbb{N} \setminus \{1\}$ *with* $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$,
$v_k \in \big[-\frac{1}{2a},\, \frac{1}{2a}\big]$ *for* $k \in I_{M_1}$, $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ *for* $j \in I_{M_2}$, $\varphi_1$ *and* $\varphi_2$ *are given by* (2.5).
0. *Precompute the following values*
$\hat{\varphi}_1(N\,x_j)$ *for* $j \in I_{M_2}$, $\hat{\varphi}_2\big(\frac{\ell}{a}\big)$ *for* $\ell \in I_{N_1+2m_1}$,
$\varphi_1\big(\frac{\ell}{N_1} - v_k\big)$ *for* $k \in I_{M_1}$ *and* $\ell \in I'_{N_1+2m_1}(v_k) := \{\ell \in I_{N_1+2m_1} :\, \big|\frac{\ell}{N_1} - v_k\big| < \frac{m_1}{N_1}\}$,
$\varphi_2\big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\big)$ *for* $j \in I_{M_2}$ *and* $s \in I''_{N_2}(x_j) := \{s \in I_{N_2} :\, \big|\frac{s}{N_2} - \frac{x_j}{\sigma_1}\big| < \frac{m_2}{N_2}\}$.
*Further set* $\varphi_1\big(\frac{\ell}{N_1} - v_k\big) := 0$ *for* $k \in I_{M_1}$ *and* $\ell \in I_{N_1+2m_1} \setminus I'_{N_1+2m_1}(v_k)$.
1. *For all* $\ell \in I_{N_1+2m_1}$ *compute the sums* (2.12).
2. *For all* $\ell \in I_{N_1+2m_1}$ *form the values*

$$\hat{g}_\ell := \frac{g_\ell}{\hat{\varphi}_2(\ell)}\,.$$

3. *For all* $s \in I_{N_2}$ *compute by fast Fourier transform* (FFT) *of length* $N_2$

$$h_s := \frac{1}{N_2} \sum_{\ell \in I_{N_1+2m_1}} \hat{g}_\ell\, \mathrm{e}^{-2\pi \mathrm{i}\, \ell s/N_2}\,.$$

4. *For all* $j \in I_{M_2}$ *calculate the short sums*

$$s_1(N\,x_j) := \sum_{s \in I''_{N_2}(x_j)} h_s\, \varphi_2\big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\big)\,.$$

*Output:* $s_1(Nx_j)/\hat{\varphi}_1(Nx_j)$ *approximate value of* (1.2) *for* $j \in I_{M_2}$.

In Step 4 of Algorithm 2.2 we use the assumption $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$ such that

$$\frac{1}{2\sigma_1} + \frac{m_2}{N_2} \le \frac{1}{2}\,.$$

Then for all $j \in I_{M_2}$ and $s \in I_{N_2}$, it holds

$$\tilde{\varphi}_2^{(1)}\big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\big) = \varphi_2\big(\frac{x_j}{\sigma_1} - \frac{s}{N_2}\big)\,.$$

The computational cost of the NNFFT is equal to $\mathcal{O}\big(N\, \log N + M_1 + M_2\big)$ flops.

## 3 Error estimates for NNFFT

Now we study the error of the NNFFT which is measured in the form

$$\max_{j\in I_{M_2}} \big|f(x_j) - \frac{s_1(N\,x_j)}{\hat\varphi_1(N\,x_j)}\big|\,, \tag{3.1}$$

where $f$ is a given exponential sum (1.1) and where $x_j \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$, $j\in I_{M_2}$, are arbitrary spatial nodes. We introduce the *a-periodization* of the given window function (2.3) by

$$\tilde\varphi_1^{(a)}(x) := \sum_{\ell\in\mathbb{Z}} \varphi_1(x + a\,\ell)\,, \quad x\in\mathbb{R}\,. \tag{3.2}$$

For each $x\in\mathbb{R}$, the above series (3.2) has at most one nonzero term. This can be seen as follows: For arbitrary $x\in\mathbb{R}$ there exists a unique $\ell^*\in\mathbb{Z}$ such that $x = -a\,\ell^* + r$ with a residuum $r\in\big[-\frac{a}{2},\,\frac{a}{2}\big)$. Then $\varphi_1(x+a\,\ell^*) = \varphi_1(r)$ and hence $\varphi_1(r) > 0$ for $r\in\big(-\frac{m_1}{N_1},\,\frac{m_1}{N_1}\big)$ and $\varphi_1(r) = 0$ for $r\in\big[-\frac{a}{2},\,-\frac{m_1}{N_1}\big]\cup\big[\frac{m_1}{N_1},\,\frac{a}{2}\big)$. For each $\ell\in\mathbb{Z}\setminus\{\ell^*\}$, we have

$$\varphi_1(x+a\,\ell) = \varphi_1\big(a\,(\ell-\ell^*) + r\big) = 0\,,$$

since $\big|a\,(\ell-\ell^*)+r\big| \ge \frac{a}{2} = \frac{1}{2} + \frac{m_1}{N_1} > \frac{m_1}{N_1}$. Further it holds

$$\tilde\varphi_1^{(a)}(x) = \varphi_1(x)\,,\quad x\in\big[-1-\frac{m_1}{N_1},\,1+\frac{m_1}{N_1}\big]\,.$$

By the construction of $\varphi_1$, the $a$-periodic window function (3.2) is continuous on $\mathbb{R}$ and of bounded variation over $\big[-\frac{a}{2},\,\frac{a}{2}\big]$. Then the $k$th Fourier coefficient of the $a$-periodic window function (3.2) reads as follows

$$c_k^{(a)}(\tilde\varphi_1^{(a)}) := \frac{1}{a}\int_{-a/2}^{a/2} \tilde\varphi_1^{(a)}(t)\,\mathrm{e}^{-2\pi\mathrm{i}\,kt/a}\,\mathrm{d}t = \frac{1}{a}\,\hat\varphi_1\big(\frac{k}{a}\big)\,,\quad k\in\mathbb{Z}\,. \tag{3.3}$$

By the Convergence Theorem of Dirichlet–Jordan (see [35, Vol. 1, pp. 57–58]), the $a$-periodic Fourier series of (3.2) converges uniformly on $\mathbb{R}$ and it holds

$$\tilde\varphi_1^{(a)}(x) = \sum_{k\in\mathbb{Z}} c_k^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,kx/a} = \frac{1}{a}\sum_{k\in\mathbb{Z}}\hat\varphi_1\big(\frac{k}{a}\big)\,\mathrm{e}^{2\pi\mathrm{i}\,kx/a}\,. \tag{3.4}$$

**Lemma 3.1** *Let the window function $\varphi_1$ be given by (2.3). Then for any $n\in I_N$ with $N\in 2\mathbb{N}$, the series*

$$\sum_{r\in\mathbb{Z}} c_{n+r\,(N_1+2m_1)}^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,(n+r\,(N_1+2m_1))x/a}$$

*is uniformly convergent on $\mathbb{R}$ and has the sum*

$$\frac{1}{N_1+2m_1}\sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,n\ell/(N_1+2m_1)}\,\tilde\varphi_1^{(a)}\big(x+\frac{\ell}{N_1}\big)$$

*which coincides with the rectangular quadrature rule of the integral*

$$c_n^{(a)}\big(\tilde\varphi_1^{(a)}(x+\cdot)\big) = \frac{1}{a}\int_{-a/2}^{a/2} \tilde\varphi_1^{(a)}(x+s)\,\mathrm{e}^{2\pi\mathrm{i}\,ns/a}\,\mathrm{d}s = c_n^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,nx/a}\,.$$

*Proof.* Using the uniformly convergent Fourier series (3.4), we obtain for all $n \in I_N$ that

$$\mathrm{e}^{-2\pi\mathrm{i}\,nx/a}\,\tilde\varphi_1^{(a)}(x) = \sum_{k\in\mathbb{Z}} c_k^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,(k-n)x/a} = \sum_{q\in\mathbb{Z}} c_{n+q}^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,qx/a}\,.$$

Replacing $x$ by $x+\frac{\ell}{N_1}$ with $\ell \in I_{N_1+2m_1}$, we see that by $N_1+2m_1 = a\,N_1$,

$$\mathrm{e}^{-2\pi\mathrm{i}\,n(x+\ell/N_1)/a}\,\tilde\varphi_1^{(a)}\big(x+\frac{\ell}{N_1}\big) = \sum_{q\in\mathbb{Z}} c_{n+q}^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,qx/a}\,\mathrm{e}^{2\pi\mathrm{i}\,q\ell/(N_1+2m_1)}\,.$$

Summing the above formulas for all $\ell \in I_{N_1+2m_1}$ and applying the known formula

$$\sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{2\pi\mathrm{i}\,q\ell/(N_1+2m_1)} = \begin{cases} N_1+2m_1 & q \equiv 0 \bmod (N_1+2m_1)\,,\\ 0 & q \not\equiv 0 \bmod (N_1+2m_1)\,,\end{cases}$$

we conclude that

$$\begin{aligned}&\sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,n(x+\ell/N_1)/a}\,\tilde\varphi_1^{(a)}\big(x+\frac{\ell}{N_1}\big)\\ &= (N_1+2m_1)\sum_{r\in\mathbb{Z}} c_{n+r\,(N_1+2m_1)}^{(a)}(\tilde\varphi_1^{(a)})\,\mathrm{e}^{2\pi\mathrm{i}\,r\,(N_1+2m_1)x/a}\,.\end{aligned}$$

Obviously,

$$\frac{1}{N_1+2m_1}\sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,n(x+\ell/N_1)/a}\,\tilde\varphi_1^{(a)}\big(x+\frac{\ell}{N_1}\big)$$

is the rectangular quadrature formula of the integral

$$\frac{1}{a}\int_{-a/2}^{a/2} \tilde\varphi_1^{(a)}(x+s)\,\mathrm{e}^{2\pi\mathrm{i}\,ns/a}\,\mathrm{d}s$$

with respect to the uniform grid $\big\{\frac{\ell}{N_1} : \ell \in I_{N_1+2m_1}\big\}$ of the interval $\big[-\frac{a}{2},\,\frac{a}{2}\big]$. This completes the proof. ■

By (3.3) we obtain that for $n \in I_N$,

$$\Big|\sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{c_{n+r\,(N_1+2m_1)}^{(a)}(\tilde\varphi_1^{(a)})}{c_n^{(a)}(\tilde\varphi_1^{(a)})}\,\mathrm{e}^{2\pi\mathrm{i}\,r\,(N_1+2m_1)x/a}\Big| = \Big|\sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(n/a+rN_1)}{\hat\varphi_1(n/a)}\,\mathrm{e}^{2\pi\mathrm{i}\,r\,N_1x/a}\Big|\,.$$

Now we generalize the Lemma 3.1.

**Lemma 3.2** *For arbitrary fixed* $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$ *and given window function* (2.3)*, the function*

$$\psi_1(x) := \frac{1}{N_1} \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1(x + \frac{\ell}{N_1}) \tag{3.5}$$

*is* $\frac{1}{N_1}$*-periodic, continuous on* $\mathbb{R}$*, and of bounded variation over* $\big[-\frac{1}{2},\, \frac{1}{2}\big]$*. For each* $x \in \mathbb{R}$*, the corresponding* $\frac{1}{N_1}$*-periodic Fourier series converges uniformly to* $\psi_1(x)$*, i.e.,*

$$\psi_1(x) = \sum_{r \in \mathbb{Z}} \hat{\varphi}_1\big(\frac{v}{a} + rN_1\big)\, \mathrm{e}^{2\pi \mathrm{i}\, r\, N_1 x}\,.$$

*Proof.* The definition (3.5) of the function $\psi_1$ is correct, since

$$\psi_1(x) = \frac{1}{N_1} \sum_{\ell \in \mathbb{Z}_{m_1,N_1}(x)} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1(x + \frac{\ell}{N_1})$$

with the finite index set $\mathbb{Z}_{m_1,N_1}(x) = \{\ell \in \mathbb{Z} : |N_1 x + \ell| < m_1\}$. If $x \in [-\frac{1}{2},\, \frac{1}{2}]$, we observe that $\mathbb{Z}_{m_1,N_1}(x) \subseteq I_{N_1+2m_1}$ and therefore

$$\psi_1(x) = \frac{1}{N_1} \sum_{\ell \in I_{N_1+2m_1}} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1(x + \frac{\ell}{N_1})\,.$$

Simple calculation shows that for each $x \in \mathbb{R}$,

$$\psi_1\big(x + \frac{1}{N_1}\big) = \frac{1}{N_1} \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{-2\pi \mathrm{i}\, v(\ell+1)/(N_1+2m_1)}\, \mathrm{e}^{-2\pi \mathrm{i}\, vx/a}\, \varphi_1(x + \frac{\ell+1}{N_1}) = \psi_1(x)\,.$$

By the construction of $\varphi_1$, the $\frac{1}{N_1}$-periodic function $\psi_1$ is continuous on $\mathbb{R}$ and of bounded variation over $\big[-\frac{1}{2},\, \frac{1}{2}\big]$. Thus, by the Convergence Theorem of Dirichlet–Jordan, the Fourier series of $\psi_1$ converges uniformly on $\mathbb{R}$ to $\psi_1$. The $r$th Fourier coefficient of $\psi_1$ reads as follows

$$\begin{aligned}
c_r^{(1/N_1)}(\psi_1) &= N_1 \int_0^{1/N_1} \psi_1(t)\, \mathrm{e}^{-2\pi \mathrm{i}\, r\, N_1 t}\, \mathrm{d}t \\
&= \sum_{\ell \in \mathbb{Z}} \mathrm{e}^{-2\pi \mathrm{i}\, v\ell/(N_1+2m_1)} \int_0^{1/N_1} \mathrm{e}^{-2\pi \mathrm{i}\, vt/a}\, \varphi_1(t + \frac{\ell}{N_1})\, \mathrm{d}t \\
&= \sum_{\ell \in \mathbb{Z}} \int_{\ell/N_1}^{(\ell+1)/N_1} \varphi_1(s)\, \mathrm{e}^{-2\pi \mathrm{i}\, (v/a + rN_1)\, s}\, \mathrm{d}s = \hat{\varphi}_1\big(\frac{v}{a} + rN_1\big)\,, \quad r \in \mathbb{Z}\,.
\end{aligned}$$

This completes the proof. ■

From Lemma 3.2 it follows immediately:

**Corollary 3.3** *Let the window function* $\varphi_1$ *be given by* (2.3)*. For all* $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ *and* $w \in \big[-\frac{N}{2a},\, \frac{N}{2a}\big]$ *it holds then*

$$\sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(w+rN_1)}{\hat\varphi_1(w)}\, \mathrm{e}^{2\pi\mathrm{i}\,(w+rN_1)\,x}$$
$$= \frac{1}{N_1\,\hat\varphi_1(w)} \sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,w\ell/N_1}\, \varphi_1\big(x+\frac{\ell}{N_1}\big) - \mathrm{e}^{2\pi\mathrm{i}\,wx}\,. \tag{3.6}$$

*Further, for all* $w \in \big[-\frac{N}{2a},\, \frac{N}{2a}\big]$*, it holds*

$$\max_{x\in[-1/2,\,1/2]} \Big|\frac{1}{N_1\,\hat\varphi_1(w)} \sum_{\ell\in I_{N_1+2m_1}} \varphi_1\big(x+\frac{\ell}{N_1}\big)\, \mathrm{e}^{-2\pi\mathrm{i}\,w\ell/N_1} - \mathrm{e}^{2\pi\mathrm{i}\,wx}\Big|$$
$$= \Big\|\sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(w+rN_1)}{\hat\varphi_1(w)}\, \mathrm{e}^{2\pi\mathrm{i}\,rN_1\,\cdot}\Big\|_{C(\mathbb{T})}\,. \tag{3.7}$$

*Proof.* As before, let $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$ be given. Substituting $w := \frac{v}{a} \in \big[-\frac{N}{2a},\, \frac{N}{2a}\big]$ and observing $N_1 + 2m_1 = aN_1$, we obtain by Lemma 3.2 that for all $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ it holds,

$$\frac{1}{N_1} \sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,w\ell/N_1}\, \mathrm{e}^{-2\pi\mathrm{i}\,w\,x}\, \varphi_1\big(x+\frac{\ell}{N_1}\big) - \hat\varphi_1(w)$$
$$= \sum_{r\in\mathbb{Z}\setminus\{0\}} \hat\varphi_1(w+rN_1)\, \mathrm{e}^{2\pi\mathrm{i}\,rN_1\,x}\,.$$

Since by assumption $\hat\varphi_1(w) > 0$ for all $w \in \big[-\frac{N}{2a},\, \frac{N}{2a}\big] \subset \big[-\frac{N}{2},\, \frac{N}{2}\big]$, it holds

$$\frac{1}{N_1\hat\varphi_1(w)} \sum_{\ell\in I_{N_1+2m_1}} \mathrm{e}^{-2\pi\mathrm{i}\,w\ell/N_1}\, \mathrm{e}^{-2\pi\mathrm{i}\,w\,x}\, \varphi_1\big(x+\frac{\ell}{N_1}\big) - 1$$
$$= \sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(w+rN_1)}{\hat\varphi_1(w)}\, \mathrm{e}^{2\pi\mathrm{i}\,rN_1\,x}\,.$$

Multiplying the above equality by the exponential $\mathrm{e}^{2\pi\mathrm{i}\,w\,x}$, this results in (3.6) and (3.7). ∎

We say that the window function $\varphi_1$ of the form (2.3) is *convenient for* NNFFT, if the *general* $C(\mathbb{T})$*-error constant*

$$E_{\sigma_1}(\varphi_1) := \sup_{N\in\mathbb{N}} E_{\sigma_1,N}(\varphi_1) \tag{3.8}$$

with

$$E_{\sigma_1,N}(\varphi_1) := \max_{v\in[-N/2,\,N/2]} \Big\|\sum_{r\in\mathbb{Z}\setminus\{0\}} \frac{\hat\varphi_1(v+rN_1)}{\hat\varphi_1(v)}\, \mathrm{e}^{2\pi\mathrm{i}\,rN_1\cdot}\Big\|_{C(\mathbb{T})} \tag{3.9}$$

fulfills the condition $E_{\sigma_1}(\varphi_1) \ll 1$ for conveniently chosen truncation parameter $m_1 \ge 2$ and oversampling factor $\sigma_1 > 1$. Obviously, the $C(\mathbb{T})$-error constant (2.4) is a "discrete" version of the general $C(\mathbb{T})$-error constant (3.8) with the property

$$e_{\sigma_1}(\varphi_1) \le E_{\sigma_1}(\varphi_1)\,. \tag{3.10}$$

Thus, Corollary 3.3 means that all complex exponentials $\mathrm{e}^{2\pi\mathrm{i}\,wx}$ with $w \in \big[-\frac{N}{2a},\, \frac{N}{2a}\big]$ and $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$ can be uniformly approximated by short linear combinations of shifted window functions, cf. [7, Theorem 2.10], if $\varphi_1$ is convenient for NNFFT.

**Theorem 3.4** *Let $\sigma_1 > 1$, $m_1 \in \mathbb{N}\setminus\{1\}$, and $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $2m_1 \ll N_1$ be given. Let $\varphi_1$ be the scaled version* (2.3) *of $\omega_1 \in \Omega$. Assume that the Fourier transform $\hat\omega_1$ fulfills the decay condition*

$$|\hat\omega_1(v)| \le \begin{cases} c_1 & |v| \in \big[m_1\big(1-\frac{1}{2\sigma_1}\big),\, m_1\big(1+\frac{1}{2\sigma_1}\big)\big]\,, \\ c_2\,|v|^{-\mu} & |v| \ge m_1\big(1+\frac{1}{2\sigma_1}\big)\,, \end{cases}$$

*with certain constants $c_1 > 0$, $c_2 > 0$, and $\mu > 1$.*
*Then the general $C(\mathbb{T})$-error constant $E_{\sigma_1}(\varphi_1)$ of the window function* (2.3) *has the upper bound*

$$E_{\sigma_1}(\varphi_1) \le \frac{1}{\hat\omega_1\big(\frac{m_1}{2\sigma_1}\big)}\Big[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^{\mu}}\big(1-\frac{1}{2\sigma_1}\big)^{1-\mu}\Big]\,. \tag{3.11}$$

*Proof.* By the scaling property of the Fourier transform, we have

$$\hat\varphi_1(v) = \int_{\mathbb{R}} \varphi_1(t)\,\mathrm{e}^{-2\pi\mathrm{i}vt}\,\mathrm{d}t = \frac{m}{N_1}\,\hat\omega_1\big(\frac{m_1 v}{N_1}\big)\,, \quad v \in \mathbb{R}\,.$$

For all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$ and $r \in \mathbb{Z}\setminus\{0,\,\pm1\}$, we obtain

$$\big|\frac{m_1 v}{N_1} + m_1 r\big| \ge m_1\big(2 - \frac{1}{2\sigma_1}\big) > m_1\big(1+\frac{1}{2\sigma_1}\big)$$

and hence

$$|\hat\varphi_1(v + rN_1)| = \frac{m_1}{N_1}\,\big|\hat\omega_1\big(\frac{m_1 v}{N_1} + m_1 r\big)\big| \le \frac{m_1\,c_2}{m_1^{\mu} N_1}\,\big|\frac{v}{N_1} + r\big|^{-\mu}\,.$$

From [29, Lemma 3.1] it follows that for fixed $u = \frac{v}{N_1} \in \big[-\frac{1}{2\sigma_1},\, \frac{1}{2\sigma_1}\big]$,

$$\sum_{r\in\mathbb{Z}\setminus\{0,\pm1\}} |u+r|^{-\mu} \le \frac{2}{\mu-1}\,\big(1-\frac{1}{2\sigma_1}\big)^{1-\mu}\,.$$

For all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$, we sustain

$$|\hat\varphi_1(v \pm N_1)| = \frac{m_1}{N_1}\,\big|\hat\omega_1\big(\frac{m_1 v}{N_1} \pm m_1\big)\big| \le \frac{m_1}{N_1}\,c_1\,,$$

since it holds

$$\big|\frac{m_1 v}{N_1} \pm m_1\big| \in \big[m_1\big(1 - \frac{1}{2\sigma_1}\big),\, m_1\big(1 + \frac{1}{2\sigma_1}\big)\big]\,.$$

Thus, for each $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$, we estimate the sum

$$\begin{aligned}
\sum_{r\in\mathbb{Z}\setminus\{0\}} |\hat\varphi_1(v + rN_1)| &\le \frac{m_1}{N_1}\Big[\big|\hat\omega_1\big(\frac{m_1 v}{N_1} - m_1\big)\big| + \big|\hat\omega_1\big(\frac{m_1 v}{N_1} + m_1\big)\big| \\
&\quad + \sum_{k\in\mathbb{Z}\setminus\{0,\pm1\}} \big|\hat\omega_1\big(\frac{m_1 v}{N_1} + m_1 r\big)\big|\Big] \\
&\le \frac{m_1}{N_1}\Big[2c_1 + \frac{c_2}{m_1^\mu} \sum_{r\in\mathbb{Z}\setminus\{0,\pm1\}} \big|\frac{v}{N_1} + r\big|^{-\mu}\Big] \\
&\le \frac{m_1}{N_1}\Big[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^\mu}\,\big(1 - \frac{1}{2\sigma_1}\big)^{1-\mu}\Big]
\end{aligned}$$

such that

$$\max_{v\in[-N/2,N/2]} \sum_{r\in\mathbb{Z}\setminus\{0\}} |\hat\varphi_1(v + rN_1)| \le \frac{m_1}{N_1}\Big[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^\mu}\,\big(1 - \frac{1}{2\sigma_1}\big)^{1-\mu}\Big]\,.$$

Now we determine the minimum of all positive values

$$\hat\varphi_1(v) = \frac{m_1}{N_1}\,\hat\omega_1\big(\frac{m_1 v}{N_1}\big)\,, \quad v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]\,.$$

Since $\frac{m_1\,|v|}{N_1} \le \frac{m_1}{2\sigma_1}$ for all $v \in \big[-\frac{N}{2},\, \frac{N}{2}\big]$, we obtain

$$\min_{v\in[-N/2,N/2]} \hat\varphi_1(v) = \frac{m_1}{N_1} \min_{v\in[-N/2,N/2]} \hat\omega_1\big(\frac{m_1 v}{N_1}\big) = \frac{m_1}{N_1}\,\hat\omega_1\big(\frac{m_1}{2\sigma_1}\big) = \hat\varphi_1\big(\frac{N}{2}\big) > 0\,.$$

Thus, we see that the constant $E_{\sigma_1,N}(\varphi_1)$ can be estimated by an upper bound which depends on $m_1$ and $\sigma_1$, but does not depend on $N$. We obtain

$$\begin{aligned}
E_{\sigma_1,N}(\varphi_1) &\le \frac{1}{\hat\varphi_1(N/2)} \max_{v\in[-N/2,N/2]} \sum_{r\in\mathbb{Z}\setminus\{0\}} |\hat\varphi_1(n + rN_1)| \\
&\le \frac{1}{\hat\omega_1\big(\frac{m_1}{2\sigma_1}\big)}\Big[2c_1 + \frac{2c_2}{(\mu-1)\,m_1^\mu}\,\big(1 - \frac{1}{2\sigma_1}\big)^{1-\mu}\Big]\,.
\end{aligned}$$

Consequently, the general $C(\mathbb{T})$-error constant $E_{\sigma_1}(\varphi_1)$ has the upper bound (3.11). By (3.10), the expression (3.11) is also an upper bound of $C(\mathbb{T})$-error constant $e_{\sigma_1}(\varphi_1)$. ∎

Now for arbitrary spatial nodes $x_j \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, $j \in I_{M_2}$, we estimate the error of the NNFFT in the form

$$\max_{j\in I_{M_2}} \big|f(x_j) - \frac{s_1(N\,x_j)}{\hat\varphi_1(N\,x_j)}\big| \le \max_{j\in I_{M_2}} \big|f(x_j) - \frac{s(N\,x_j)}{\hat\varphi_1(N\,x_j)}\big| + \max_{j\in I_{M_2}} \frac{|s(N\,x_j) - s_1(N\,x_j)|}{\hat\varphi_1(N\,x_j)}\,.$$

At first we consider

$$\max_{j\in I_{M_2}} \big|f(x_j) - \frac{s(N\,x_j)}{\hat\varphi_1(N\,x_j)}\big| \le \max_{x\in[-1/2,\,1/2]} \big|f(x) - \frac{s(N\,x)}{\hat\varphi_1(N\,x)}\big|\,.$$

From (2.9) and (2.11) it follows that for all $x \in \mathbb{R}$,

$$\begin{aligned} f(x) - \frac{s(N\,x)}{\hat\varphi_1(N\,x)} &= \frac{\hat h(N\,x) - s(N\,x)}{\hat\varphi_1(N\,x)} \\ &= \sum_{k\in I_{M_1}} f_k \Big[\mathrm e^{-2\pi\mathrm i\,N v_k x} - \frac{1}{N_1\,\hat\varphi_1(N\,x)} \sum_{\ell\in I_{N_1+2m_1}} \varphi_1\big(\frac{\ell}{N_1} - v_k\big)\,\mathrm e^{-2\pi\mathrm i\,\ell x/\sigma_1}\Big]\,. \end{aligned}$$

Thus, by (2.7), (3.7), and (3.9), we obtain the estimate

$$\max_{x\in[-1/2,\,1/2]} \big|f(x) - \frac{s(N\,x)}{\hat\varphi_1(N\,x)}\big| \le E_{\sigma_1,N}(\varphi_1) \sum_{k\in I_{M_1}} |f_k| \le E_{\sigma_1}(\varphi_1) \sum_{k\in I_{M_1}} |f_k|\,.$$

Now we show that for $\varphi_2(t) := \omega_2\big(\frac{N_2 t}{m_2}\big)$ and $N_2 = \sigma_2\,(N_1 + 2m_1)$ it holds

$$\max_{x\in[-1/2,\,1/2]} |s(N\,x) - s_1(N\,x)| \le E_{\sigma_2}(\varphi_2) \sum_{\ell\in I_{N_1+2m_1}} |g_\ell|\,. \tag{3.12}$$

By construction, the functions $s$ and $s_1$ can be represented in the form

$$\begin{aligned} s(Nx) &= \sum_{\ell\in I_{N_1+2m_1}} g_\ell\,\mathrm e^{-2\pi\mathrm i\ell x/\sigma_1}\,, \\ s_1(Nx) &= \sum_{s\in I_{N_2}} h_s\,\tilde\varphi_2^{(1)}\big(\frac{x}{\sigma_1} - \frac{s}{N_2}\big)\,, \quad x\in\mathbb{R}\,, \end{aligned}$$

where $\tilde\varphi_2^{(1)}$ denotes the 1-periodization of the second window function $\varphi_2$ and

$$h_s := \frac{1}{N_2} \sum_{\ell\in I_{N_1+2m_1}} \frac{g_\ell}{\hat\varphi_2(\ell)}\,\mathrm e^{-2\pi\mathrm i\ell s/N_2}\,.$$

Substituting $t = \frac{x}{\sigma_1}$, it follows that

$$\begin{aligned} s(N_1 t) &= \sum_{\ell\in I_{N_1+2m_1}} g_\ell\,\mathrm e^{-2\pi\mathrm i\ell t}\,, \\ s_1(N_1 t) &= \sum_{s\in I_{N_2}} h_s\,\tilde\varphi_2^{(1)}\big(t - \frac{s}{N_2}\big)\,, \quad t\in\mathbb{R}\,, \end{aligned}$$

are 1-periodic functions. By [29, Lemma 2.3], we conclude

$$\max_{t\in[-1/2,\,1/2]} |s(N_1 t) - s_1(N_1 t)| \le\ e_{\sigma_2}(\varphi_2) \sum_{\ell\in I_{N_1+2m_1}} |g_\ell| \le E_{\sigma_2}(\varphi_2) \sum_{\ell\in I_{N_1+2m_1}} |g_\ell|\,,$$

where the general $C(\mathbb{T})$-error constant $E_{\sigma_2}(\varphi_2)$ defined similar to (3.8) has an analogous property (3.10). Since $x = \sigma_1\, t$, we obtain that

$$\max_{t\in[-1/2,\,1/2]} |s(N_1 t) - s_1(N_1 t)| = \max_{x\in[-\sigma_1/2,\sigma_1/2]} |s(Nx) - s_1(Nx)| \le E_{\sigma_2}(\varphi_2) \sum_{\ell\in I_{N_1+2m_1}} |g_\ell|\,,$$

such that (3.12) is shown. Note that for $x \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$ it holds

$$s_1(Nx) = \sum_{s\in I''_{N_2}(x)} h_s\, \varphi_2\big(\frac{x}{\sigma_1} - \frac{s}{N_2}\big)$$

with the index set

$$I''_{N_2}(x) := \big\{s \in I_{N_2} :\, \big|\frac{s}{N_2} - \frac{x}{\sigma_1}\big| < \frac{m_2}{N_2}\big\}\,.$$

Further, by (2.6) and (2.12) it holds

$$\begin{aligned}\sum_{\ell\in I_{N_1+2m_1}} |g_\ell| \;&\le\; \frac{1}{N_1} \sum_{\ell\in I_{N_1+2m_1}} \sum_{k\in I_{M_1}} |f_k|\cdot 1 \le \frac{N_1+2m_1}{N_1} \sum_{k\in I_{M_1}} |f_k| \\ &=\; a \sum_{k\in I_{M_1}} |f_k|\,.\end{aligned}$$

Thus, we obtain the following error estimate for the NNFFT:

**Theorem 3.5** *Let the nonharmonic bandwidth $N \in \mathbb{N}$ with $N \gg 1$ be given. Assume that $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $\sigma_1 > 1$. For fixed $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$, let $N_2 = \sigma_2\,(N_1 + 2m_1)$ with $\sigma_2 > 1$. For $m_2 \in \mathbb{N}\setminus\{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\,N_2$, let $\varphi_1$ and $\varphi_2$ be the window functions of the form (2.5). Let $x_j \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$, $j \in I_{M_2}$, be arbitrary spatial nodes and let $f_k \in \mathbb{C}$, $k \in I_{M_1}$, be arbitrary coefficients. Let $a > 1$ be the constant (2.6).*
*Then for a given exponential sum (1.1) with arbitrary frequencies $v_k \in \big[-\frac{1}{2a},\,\frac{1}{2a}\big]$, $k \in I_{M_1}$, the error of the* NNFFT *can be estimated by*

$$\begin{aligned}\max_{j\in I_{M_2}} \big|f(x_j) - \frac{s_1(N\,x_j)}{\hat\varphi_1(N\,x_j)}\big| \;&\le\; \max_{x\in[-1/2,\,1/2]} \big|f(x) - \frac{s_1(N\,x)}{\hat\varphi_1(N\,x)}\big| \\ &\le\; \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in I_{M_1}} |f_k|\,,\end{aligned}$$

*where $E_{\sigma_j}(\varphi_j)$ for $j = 1,\,2$, are the general $C(\mathbb{T})$-error constants of the form (3.8).*

# 4 Error of NNFFT with sinh-type window functions

Let $N \in \mathbb{N}$ with $N \gg 1$ be the fixed nonharmonic bandwidth. Let $\sigma_1,\,\sigma_2 \in \big[\frac{5}{4},\,2\big]$ be given oversampling factors. Further let $N_1 = \sigma_1 N \in 2\mathbb{N}$, $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$,

and $N_2 = \sigma_2\,(N_1 + 2m_1) = \sigma_1 \sigma_2 a\, N \in 2\mathbb{N}$ be given, where $a > 1$ denotes the constant (2.6). Let $m_2 \in \mathbb{N} \setminus \{1\}$ with $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$ be given, too.
For $j = 1,\, 2$, we consider the functions

$$\omega_{\mathrm{sinh},j}(x) := \begin{cases} \frac{1}{\sinh \beta_j}\, \sinh\big(\beta_j \sqrt{1-x^2}\big) & x \in [-1,\, 1]\,, \\ 0 & x \in \mathbb{R} \setminus [-1,\, 1] \end{cases} \tag{4.1}$$

with the shape parameter

$$\beta_j := 2\pi m_j \left(1 - \frac{1}{2\sigma_j}\right) .$$

As shown in Example 2.1, both functions belong to the set $\Omega$. By scaling, for $j = 1,\, 2$, we introduce the sinh-type window functions

$$\varphi_{\mathrm{sinh},j}(t) := \omega_{\mathrm{sinh},j}\big(\frac{N_j t}{m_j}\big)\,, \quad t \in \mathbb{R}\,. \tag{4.2}$$

Applying Theorem 3.4, we obtain by the same technique as in [29, Theorem 5.6] that

$$E_{\sigma_j}\big(\varphi_{\mathrm{sinh},j}\big) \le (24 m_j^{3/2} + 10)\, \mathrm{e}^{-2\pi m_j \sqrt{1-1/\sigma_j}}\,, \quad j = 1,\, 2. \tag{4.3}$$

Now we estimate $\hat{\varphi}_{\mathrm{sinh},1}\big(\frac{N}{2}\big)$. Using the scaling property of the Fourier transform, by (2.2) we obtain

$$\begin{aligned} \hat{\varphi}_{\mathrm{sinh},1}\big(\frac{N}{2}\big) &= \frac{m_1}{N_1}\, \hat{\omega}_{\mathrm{sinh},1}\big(\frac{m_1 N}{2N_1}\big) = \frac{m_1}{N_1}\, \hat{\omega}_{\mathrm{sinh},1}\big(\frac{m_1}{2\sigma_1}\big) \\ &= \frac{\pi m_1 \beta_1}{N_1 \sinh \beta_1}\, \big(\beta_1^2 - \frac{\pi^2 m_1^2}{\sigma_1^2}\big)^{-1/2}\, I_1\Big(\sqrt{\beta_1^2 - \frac{\pi^2 m_1^2}{\sigma_1^2}}\Big) \\ &= \frac{m_1 \pi}{N_1 \sinh \beta_1}\, \big(1 - \frac{1}{2\sigma_1}\big)\big(1 - \frac{1}{\sigma_1}\big)^{-1/2}\, I_1\Big(2\pi m_1 \sqrt{1 - \frac{1}{\sigma_1}}\Big)\,, \end{aligned}$$

where we have used the equality

$$\big(\beta_1^2 - \frac{\pi^2 m_1^2}{\sigma_1^2}\big)^{1/2} = 2\pi m_1 \Big(\big(1 - \frac{1}{2\sigma_1}\big)^2 - \frac{1}{4\sigma_1^2}\Big)^{1/2} = 2\pi m_1 \sqrt{1 - \frac{1}{\sigma_1}}\,.$$

From $m_1 \ge 2$ and $\sigma_1 \ge \frac{5}{4}$, it follows that

$$2\pi m_1 \sqrt{1 - \frac{1}{\sigma_1}} \ge 4\pi \sqrt{1 - \frac{1}{\sigma_1}} \ge x_0 := \frac{4\pi}{\sqrt{5}}\,.$$

By the inequality for the modified Bessel function $I_1$ (see [29, Lemma 3.3]) it holds

$$I_1(x) \ge \sqrt{x_0}\, \mathrm{e}^{-x_0}\, I_1(x_0)\, x^{-1/2}\, \mathrm{e}^{x} > \frac{2}{5}\, x^{-1/2}\, \mathrm{e}^{x}\,, \quad x \ge x_0\,.$$

Thus, we obtain

$$\hat{\varphi}_{\mathrm{sinh},1}\big(\frac{N}{2}\big) \ge \frac{\sqrt{2 m_1 \pi}}{5 N_1\, \sinh \beta_1}\, \big(1 - \frac{1}{2\sigma_1}\big)\big(1 - \frac{1}{\sigma_1}\big)^{-3/4}\, \mathrm{e}^{2\pi m_1 \sqrt{1-1/\sigma_1}}\,.$$

By the simple inequality

$$\sinh\beta_1 < \frac{1}{2}\,\mathrm{e}^{\beta_1} = \frac{1}{2}\,\mathrm{e}^{2\pi m_1(1-1/(2\sigma_1))}\,,$$

we conclude that

$$\hat\varphi_{\sinh,1}\big(\frac{N}{2}\big) \ge \frac{2\sqrt{2m_1\pi}}{5N_1}\,\big(1-\frac{1}{2\sigma_1}\big)\big(1-\frac{1}{\sigma_1}\big)^{-3/4}\,\mathrm{e}^{2\pi m_1(\sqrt{1-1/\sigma_1}-1+1/(2\sigma_1))}$$

and hence

$$\frac{a}{\hat\varphi_{\sinh,1}(N/2)} \le \frac{5N_1\,a}{2\sqrt{2m_1\pi}}\,\big(1-\frac{1}{2\sigma_1}\big)^{-1}\big(1-\frac{1}{\sigma_1}\big)^{3/4}\,\mathrm{e}^{-2\pi m_1(\sqrt{1-1/\sigma_1}-1+1/(2\sigma_1))}\,. \tag{4.4}$$

Applying Theorem 3.5, we estimate the error of the NNFFT with two sinh-type window functions (4.2). By (4.3) and (4.4) we obtain the inequality

$$\begin{aligned} E_{\sigma_1}(\varphi_{\sinh,1}) \;&+\; \frac{a}{\hat\varphi_{\sinh,1}(N/2)}\,E_{\sigma_2}(\varphi_{\sinh,2}) \le (24m_1^{3/2}+10)\,\mathrm{e}^{-2\pi m_1\sqrt{1-1/\sigma_1}} \\ &+\; (24m_2^{3/2}+10)\,\frac{2N_1 a}{\sqrt{2m_1\pi}}\,\mathrm{e}^{2\pi m_1(1-\sqrt{1-1/\sigma_1}-1/(2\sigma_1))}\,\mathrm{e}^{-2\pi m_2\sqrt{1-1/\sigma_2}}\,, \end{aligned}$$

since it holds

$$\frac{5}{2}\,\big(1-\frac{1}{2\sigma_1}\big)^{-1}\big(1-\frac{1}{\sigma_1}\big)^{3/4} \le \frac{5}{3}\sqrt{2} < 2\,,\quad \sigma_1\in\big[\frac{5}{4},\,2\big]\,.$$

Thus, the error of NNFFT with two sinh-type window functions (4.2) has exponential decay with respect to the truncation parameters $m_1$ and $m_2$. We summarize:

**Theorem 4.1** *Let the nonharmonic bandwidth $N \in 2\mathbb{N}$ with $N \gg 1$ be given. Let $N_1 = \sigma_1 N \in 2\mathbb{N}$ with $\sigma_1 \in \big[\frac{5}{4},\,2\big]$ be given. For fixed $m_1 \in \mathbb{N}\setminus\{1\}$ with $2m_1 \ll N_1$, let $N_2 = \sigma_2\,(N_1+2m_1) \in 2\mathbb{N}$ with $\sigma_2 \in \big[\frac{5}{4},\,2\big]$. For $m_2 \in \mathbb{N}\setminus\{1\}$ with $2m_2 \le \big(1-\frac{1}{\sigma_1}\big)\,N_2$, let $\varphi_{\sinh,1}$ and $\varphi_{\sinh,2}$ be the* sinh*-type window functions* (4.2). *Assume that $m_2 \ge m_1$. Let $x_j \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$, $j \in I_{M_2}$, be arbitrary spatial nodes and let $f_k \in \mathbb{C}$, $k \in I_{M_1}$, be arbitrary coefficients. Let $a > 1$ be the constant* (2.6).
*Then for the exponential sum* (1.1) *with arbitrary frequencies $v_k \in \big[-\frac{1}{2a},\,\frac{1}{2a}\big]$, $k\in I_{M_1}$, the error of the* NNFFT *with the* sinh*-type window functions* (4.2) *can be estimated in the form*

$$\max_{j\in I_{M_2}}\big|f(x_j) - \frac{s_1(N\,x_j)}{\hat\varphi_{\sinh,1}(N\,x_j)}\big| \le \max_{x\in[-1/2,\,1/2]}\big|f(x) - \frac{s_1(N\,x)}{\hat\varphi_{\sinh,1}(N\,x)}\big| \le E(\varphi_{\sinh})\sum_{k\in I_{M_1}}|f_k|$$

*with the constant*

$$\begin{aligned} E(\varphi_{\sinh}) \;&:=\; (24m_1^{3/2}+10)\,\mathrm{e}^{-2\pi m_1\sqrt{1-1/\sigma_1}} \\ &\quad + (24m_2^{3/2}+10)\,\frac{2N_1+4m_1}{\sqrt{2m_1\pi}}\,\mathrm{e}^{2\pi m_1\,(1-\sqrt{1-1/\sigma_1}-1/(2\sigma_1))}\,\mathrm{e}^{-2\pi m_2\sqrt{1-1/\sigma_2}}\,. \end{aligned}$$

Now we visualize the result of Theorem 4.1. To this end, we introduce the "relative error" (3.1), i.e.,

$$\Big( \sum_{k \in I_{M_1}} |f_k| \Big)^{-1} \max_{j \in I_{M_2}} \Big| f(x_j) - \frac{s_1(N x_j)}{\hat{\varphi}_{\mathrm{sinh},1}(N x_j)} \Big| . \tag{4.5}$$

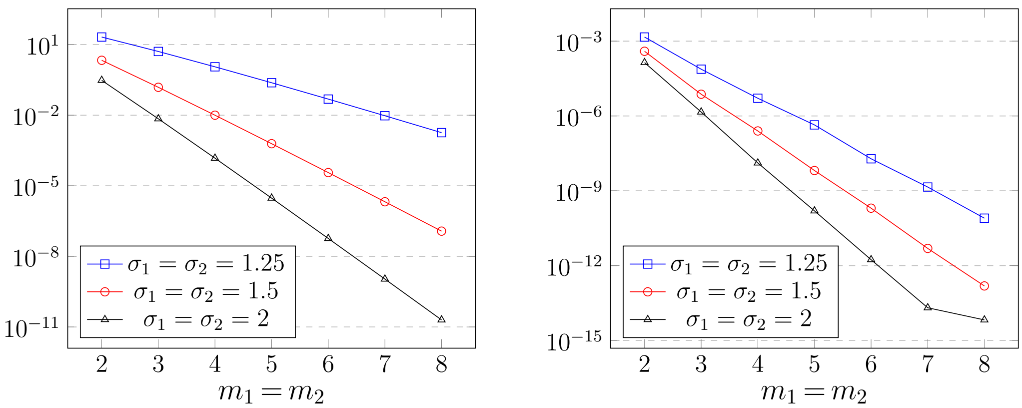
By Theorem 4.1 it holds

$$\Big( \sum_{k \in I_{M_1}} |f_k| \Big)^{-1} \max_{j \in I_{M_2}} \Big| f(x_j) - \frac{s_1(N x_j)}{\hat{\varphi}_{\mathrm{sinh},1}(N x_j)} \Big| \le E(\varphi_{\mathrm{sinh}}) .$$

For ease of presentation we consider $m_1 = m_2 \in \{2, \dots, 8\}$ and $\sigma_1 = \sigma_2 \in [\frac{5}{4}, 2]$. In this case, the constant $E(\varphi_{\mathrm{sinh}})$ reads by (4.3) as follows

$$(24 m_1^{3/2} + 10)\, \mathrm{e}^{-2\pi m_1 \sqrt{1-1/\sigma_1}} \Big( 1 + \frac{2N_1 + 4m_1}{\sqrt{2 m_1 \pi}}\, \mathrm{e}^{2\pi m_1 \,(1 - \sqrt{1-1/\sigma_1} - 1/(2\sigma_1))} \Big) . \tag{4.6}$$

Thus, we choose random nodes $x_j \in \big[ -\frac{1}{2}, \frac{1}{2} \big]$, $j \in I_{M_2}$, and $v_k \in [-\frac{1}{2a}, \frac{1}{2a}]$, $k \in I_{M_1}$, with $a = 1 + \frac{2m_1}{N_1}$ as well as random coefficients $f_k \in \mathbb{C}$, $k \in I_{M_1}$, and compute approximation (1.2) once directly and once rapidly using the NNFFT. This test is repeated one hundred times and afterwards the maximum error over all repetitions is computed. The appropriate results for the parameter choice $N = 1200$, $M_1 = 2400$ and $M_2 = 1600$ are displayed in Figure 4.1(b), while the error bound (4.6) is depicted in Figure 4.1(a). For the NNFFT we use two standard window functions, namely the Kaiser–Bessel window functions (see [30]). It can clearly be seen that the higher the truncation parameters are, the smaller the error bound (4.6) and the relative error (4.5) are.




(a) Error bound (4.6) (b) Relative NNFFT error (4.5)

Figure 4.1: Visualization of Theorem 4.1 for the NNFFT with sinh-type window functions, where $m_1 = m_2 \in \{2, \dots, 8\}$ and $\sigma_1 = \sigma_2 \in \{1.25, 1.5, 2\}$.

## 5 Approximation of sinc function by exponential sums

The exponential sum (1.1) is a linear combination of exponentials with bounded frequencies which is used in [4] for a local approximation of a bandlimited function $F$ of the form

$$F(x) := \int_{-1/2}^{1/2} w(t)\, \mathrm{e}^{-2\pi \mathrm{i}\, Ntx}\, \mathrm{d}t\,, \quad x \in \mathbb{R}\,, \tag{5.1}$$

where $w : [-\frac{1}{2},\, \frac{1}{2}] \to [0,\, \infty)$ is an integrable function with $\int_{-1/2}^{1/2} w(t)\, \mathrm{d}t > 0$. By

$$F(x) = \frac{1}{N} \int_{-N/2}^{N/2} w\big(-\frac{s}{N}\big)\, \mathrm{e}^{2\pi \mathrm{i}\, sx}\, \mathrm{d}s\,, \tag{5.2}$$

the Fourier transform of (5.1) is supported on $\big[-\frac{N}{2},\, \frac{N}{2}\big]$, i.e., the function (5.1) is bandlimited with bandwidth $N$. For $w(t) := 1$, $t \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, we obtain the famous bandlimited sinc function

$$F(x) = \mathrm{sinc}(\pi N x) := \begin{cases} \frac{\sin(\pi N x)}{\pi N x} & x \in \mathbb{R} \setminus \{0\}\,, \\ 1 & x = 0\,. \end{cases} \tag{5.3}$$

Now we show that the bandlimited sinc function (5.3) can be uniformly approximated on the interval $[-1,\, 1]$ by an exponential sum (1.1). We start with the uniform approximation of the sinc function on the interval $\big[-\frac{1}{2},\, \frac{1}{2}\big]$.

**Theorem 5.1** *Let $\varepsilon > 0$ be a given target accuracy.*
*Then for sufficiently large $n \in \mathbb{N}$ with $n \ge 2N$, there exist constants $w_j > 0$ and frequencies $v_j \in \big(-\frac{1}{2},\, \frac{1}{2}\big)$, $j = 1, \ldots, n$, such that for all $x \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$,*

$$\big|\mathrm{sinc}(\pi N x) - \sum_{j=1}^{n} w_j\, \mathrm{e}^{-2\pi \mathrm{i}\, N\, v_j x}\big| \le \varepsilon\,. \tag{5.4}$$

*Proof.* This result is a simple consequence of [4, Theorem 6.1]. Introducing $\nu := \frac{N}{n} \le \frac{1}{2}$, we obtain by substitution $\tau := -\frac{t}{2\nu}$ that

$$\mathrm{sinc}(\pi N x) = \int_{-1/2}^{1/2} \mathrm{e}^{-2\pi \mathrm{i}\, N \tau x}\, \mathrm{d}\tau = \frac{1}{2\nu} \int_{-\nu}^{\nu} \mathrm{e}^{\mathrm{i}\pi n t x}\, \mathrm{d}t\,.$$

Setting $y := nx \in \big[-\frac{n}{2},\, \frac{n}{2}\big]$, we have

$$\mathrm{sinc}(\pi \nu y) = \frac{1}{2\nu} \int_{-\nu}^{\nu} \mathrm{e}^{\mathrm{i}\pi t y}\, \mathrm{d}t\,.$$

Then from [4, Theorem 6.1] (with $d = \frac{1}{2}$), it follows the existence of $w_j > 0$ and $\Theta_j \in (-\nu,\, \nu)$, $j = 1, \ldots, n$, such that for all $y \in \big[-\frac{n}{2} - 1,\, \frac{n}{2} + 1\big]$,

$$\big|\frac{1}{2\nu} \int_{-\nu}^{\nu} \sigma(t)\, \mathrm{e}^{\mathrm{i}\pi t y}\, \mathrm{d}t - \sum_{j=1}^{n} w_j\, \mathrm{e}^{\pi \mathrm{i}\, \Theta_j y}\big| \le \varepsilon\,.$$

Hence, for all $x = \frac{y}{n} \in \big[-\frac{1}{2},\, \frac{1}{2}\big]$, we conclude that for $v_j := -\frac{\Theta_j}{2\nu} \in \big(-\frac{1}{2},\, \frac{1}{2}\big)$, $j = 1, \ldots, n$,

$$\Big|\frac{1}{2\sigma}\int_{-\nu}^{\nu} \mathrm{e}^{\mathrm{i}\pi ntx}\,\mathrm{d}t - \sum_{j=1}^{n} w_j\,\mathrm{e}^{\pi\mathrm{i}\,n\Theta_j x}\Big| = \big|\mathrm{sinc}(\pi N x) - \sum_{j=1}^{n} w_j\,\mathrm{e}^{-2\pi\mathrm{i}\,N\,v_j x}\big| \le \varepsilon\,.$$

This completes the proof. ■

Substituting the variable $x = \frac{t}{2}$, $t \in [-1,\, 1]$, and replacing the bandwidth $N$ in (5.4) by $2N$, we obtain the following uniform approximation of the sinc function (5.3) on the interval $[-1,\, 1]$ (after denoting $t$ again by $x$):

**Corollary 5.2** *Let $\varepsilon > 0$ be a given target accuracy.*
*Then for sufficiently large $n \in \mathbb{N}$ with $n \ge 4N$, there exist constants $w_j > 0$ and frequencies $v_j \in \big(-\frac{1}{2},\, \frac{1}{2}\big)$, $j = 1, \ldots, n$, such that* (5.4) *holds for all $x \in [-1,\, 1]$, i. e.,*

$$\big|\mathrm{sinc}(\pi N x) - \sum_{j=1}^{n} w_j\,\mathrm{e}^{-2\pi\mathrm{i}\,N\,v_j x}\big| \le \varepsilon\,, \quad x \in [-1,\, 1]\,. \tag{5.5}$$

The practical approximation of the function $\mathrm{sinc}(N\pi x)$ by an exponential sum on the interval $[-1, 1]$ can efficiently be realized by means of the Clenshaw–Curtis quadrature (see [33, pp. 143–153] or [26, pp. 357–364]). Using this procedure for the integrand $\frac{1}{2}\,\mathrm{e}^{-\pi\mathrm{i}Ntx}$, $t \in [-1,\, 1]$, with fixed parameter $x \in [-1,\, 1]$, the Chebyshev points $z_k = \cos\frac{k\pi}{n} \in [-1, 1]$, $k = 0, \ldots, n$, and the positive coefficients

$$w_k = \begin{cases} \frac{1}{n}\,\varepsilon_n(k)^2 \sum_{j=0}^{n/2} \varepsilon_n(2j)^2 \frac{2}{1-4j^2}\,\cos\frac{2jk\pi}{n} & n \in 2\mathbb{N}\,, \\ \frac{1}{n}\,\varepsilon_n(k)^2 \sum_{j=0}^{(n-1)/2} \varepsilon_n(2j)^2 \frac{2}{1-4j^2}\,\cos\frac{2jk\pi}{n} & n \in 2\mathbb{N}+1\,, \end{cases} \tag{5.6}$$

where it holds $\varepsilon_n(0) = \varepsilon_n(n) := \frac{\sqrt{2}}{2}$ and $\varepsilon_n(j) := 1$, $j = 1, \ldots, n-1$ (see [26, p. 359]), we obtain the following result:

$$\mathrm{sinc}(N\pi x) = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-\pi\mathrm{i}Ntx}\,\mathrm{d}t \approx \sum_{k=0}^{n} w_k\,\mathrm{e}^{-\pi\mathrm{i}Nz_k x}\,.$$

Further the coefficients fulfill the condition (see [26, p. 359])

$$\sum_{k=0}^{n} w_k = 1\,. \tag{5.7}$$

Then we receive the following error estimate.

**Theorem 5.3** *Let $n \in 2\mathbb{N}$ with $n \ge 4$ be given. Let $z_k = \cos\frac{\pi k}{n}$, $k = 0, \ldots, n$, be the Chebyshev points and let $w_k$, $k = 0, \ldots, n$, denote the coefficients* (5.6).
*Then for all $x \in [-1,\, 1]$, the approximation error of $\mathrm{sinc}(N\pi x)$ can be estimated in the form*

$$\big|\mathrm{sinc}(N\pi x) - \sum_{k=0}^{n} w_k\,\mathrm{e}^{-\pi\mathrm{i}N\,z_k x}\big| \le \frac{144}{70\,(\mathrm{e}^2-1)}\,\mathrm{e}^{-n}\,\cosh\frac{\pi\,(\mathrm{e}^2-1)\,N}{2\,\mathrm{e}}\,. \tag{5.8}$$

*Proof.* Since the imaginary part of the integrand $\frac{1}{2}\,\mathrm{e}^{-\pi \mathrm{i} N t x}$, $t \in [-1,\,1]$, is odd, it holds

$$\mathrm{sinc}(N\pi x) = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-\pi \mathrm{i} N t x}\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} \cos(\pi N t x)\,\mathrm{d}t\,. \tag{5.9}$$

Now we apply the Clenshaw–Curtis quadrature to the analytic function $f(t,x) := \frac{1}{2}\cos(\pi N t x)$, $t \in [-1,1]$, with fixed parameter $x \in [-1,\,1]$. Note that it holds

$$\sum_{k=0}^{n} w_k\,\mathrm{e}^{-\pi \mathrm{i} N\, z_k x} = \sum_{k=0}^{n} w_k\,\cos(\pi N\, z_k x) + 0$$

by the symmetry properties of the Chebyshev points $z_k$ and the coefficients $w_k$, namely $z_k = -z_{n-k}$ and $w_k = w_{n-k}$, $k = 0, \ldots, n$ (see [26, p. 359]).
By $E_\rho$ with some $\rho > 1$, we denote the Bernstein ellipse defined by

$$E_\rho := \{z \in \mathbb{C} :\, \mathrm{Re}\, z = \frac{1}{2}\big(\rho + \frac{1}{\rho}\big)\,\cos t\,,\ \mathrm{Im}\, z = \frac{1}{2}\big(\rho - \frac{1}{\rho}\big)\,\sin t\,,\ t \in [0,\,2\pi)\}\,.$$

Then $E_\rho$ has the foci $-1$ and $1$. For simplicity, we choose $\rho = \mathrm{e}$.
For $z \in \mathbb{C}$ and fixed $x \in [-1,\,1]$, it holds

$$\big|\frac{1}{2}\,\cos(\pi N x z)\big| \le \frac{1}{2}\,\cosh(\pi N x\,\mathrm{Im}\, z)\,.$$

For $z \in \mathbb{C}$ with $\mathrm{Re}\, z = 0$ we have

$$\big|\frac{1}{2}\,\cos(\pi N x z)\big| = \frac{1}{2}\,\cosh(\pi N x\,\mathrm{Im}\, z)\,.$$

Hence, in the interior of the Bernstein ellipse $E_{\mathrm{e}}$, the integrand is bounded, since

$$\big|\frac{1}{2}\,\cos(\pi N x z)\big| \le \frac{1}{2}\,\cosh\frac{\pi N x\,(\mathrm{e}^2-1)}{2\,\mathrm{e}} \le \frac{1}{2}\,\cosh\frac{\pi N\,(\mathrm{e}^2-1)}{2\,\mathrm{e}}\,.$$

Thus, by [33, p. 146] we obtain the error estimate (5.8). Note that for even $n \ge 4N$ the error (5.8) is very small. For example, in the case $N = 128$ and $n \ge 512$ the error is smaller than

$$\frac{144}{70\,(\mathrm{e}^2-1)}\,\mathrm{e}^{-512}\,\cosh\frac{128\pi\,(\mathrm{e}^2-1)}{2\,\mathrm{e}} \le 1.22\cdot 10^{-18}\,.$$

This completes the proof. ■

In practice, the coefficients $w_k$ in (5.6) can be computed by a fast algorithm, the discrete cosine transform of type I (DCT–I) of length $n+1$, $n = 2^t$, (see [26, Algorithm 6.28 or Algorithm 6.35]), where the DCT–I uses the orthogonal cosine matrix of type I

$$\boldsymbol{C}_{n+1}^{\mathrm{I}} := \sqrt{\frac{2}{n}}\,\big(\varepsilon_n(j)\,\varepsilon_n(k)\,\cos\frac{jk\pi}{n}\big)_{j,k=0}^{n}\,.$$

**Algorithm 5.4 (Fast computation of the coefficients $w_k$)**
*Input:* $n = 2^t$ *with* $t \in \mathbb{N} \setminus \{1\}$, $\varepsilon_n(0) = \varepsilon_n(n) := \frac{\sqrt{2}}{2}$, $\varepsilon_n(j) := 1$ *for* $j = 1, \ldots, n-1$.

1. *Form the vector* $(a_j)_{j=0}^n$ *with* $a_{2j} := \varepsilon_n(2j)\, \frac{2}{1-4j^2}$, $j = 0, \ldots, n/2$ *and* $a_{2j+1} := 0$, $j = 0, \ldots, n/2-1$.
2. *Compute* $(\hat{a}_k)_{k=0}^n = \boldsymbol{C}_{n+1}^{\mathrm{I}} (a_j)_{j=0}^n$ *by means of* DCT − I.
3. *Form the values* $w_k := \frac{1}{\sqrt{2n}} \varepsilon_n(k)\, \hat{a}_k$, $k = 0, \ldots, n$.

*Output:* $w_k$ *in* (5.6) *for* $k = 0, \ldots, n$.

In [12] a Gauss–Legendre quadrature was applied to obtain explicit coefficients $w_k$ for given Legendre points $z_k$. Due to their error estimate the authors claimed that $n \in \mathbb{N}$, $n \geq \frac{\pi}{2}$ would be sufficient in this setting. However, the computation of the coefficients $w_k$ using our approach in Algorithm 5.4 is more effective for large $n$.
Now we visualize the result of Theorem 5.3. To this end, we compare the error constant and the maximum approximation error, cf. (5.8). To measure the accuracy we consider a fine evaluation grid $x_r = \frac{2r}{R}$, $r \in I_R$, with $R \gg N$, where $R = 3 \cdot 10^5$ is fixed. On this grid we calculate the discrete maximum error

$$\max_{r \in I_R} \Big|\mathrm{sinc}(\pi N x_r) - \sum_{k=0}^{n} w_k\, \mathrm{e}^{-\pi \mathrm{i} N z_k x_r}\Big| \tag{5.10}$$

for different bandwidths $N = 2^\ell$, $\ell = 3, \ldots, 7$. For the parameter $n$ we investigate several choices $n = \nu N$ with $\nu \in \{1, \ldots, 10\}$. We compute the coefficients $w_k$ using Algorithm 5.4. Subsequently, the approximation to the sinc function is computed by means of the NFFT. The results for error bound (5.8) are depicted in Figure 5.1(a), while the maximum error (5.10) is displayed in Figure 5.1(b). It can clearly be seen that the higher the oversampling $\nu$ is, the smaller the error bound (5.8) and the maximum error (5.10) are.

# 6 Discrete sinc transform

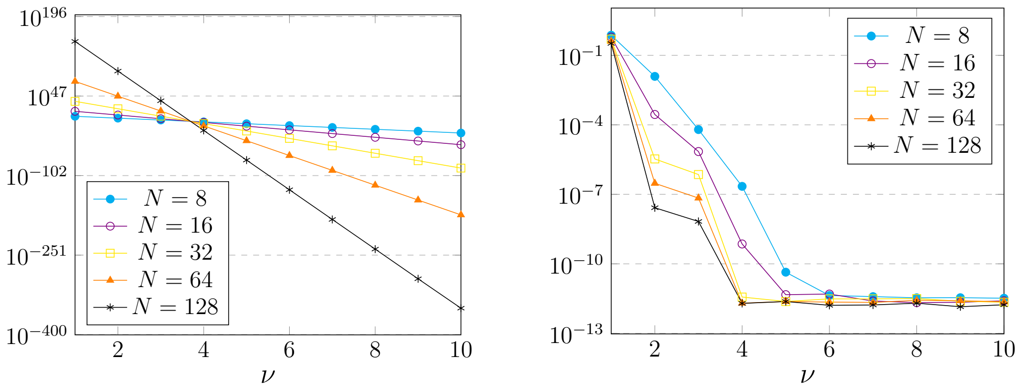
In the last section we present an interesting signal processing application of the NNFFT. If a signal $h : \big[-\frac{1}{2}, \frac{1}{2}\big] \to \mathbb{C}$ is to be reconstructed from its equispaced/nonequispaced samples at $a_k \in \big[-\frac{1}{2}, \frac{1}{2}\big]$, then $h$ is often modeled as linear combination of shifted sinc functions

$$h(x) = \sum_{k \in I_{L_1}} c_k\, \mathrm{sinc}\big(N\pi\,(x - a_k)\big)\,, \quad x \in \mathbb{R}\,, \tag{6.1}$$

with complex coefficients $c_k$. In the following, we propose a fast algorithm for the approximate computation of the *discrete* sinc *transform* (see [12, 20])

$$h(b_\ell) = \sum_{k \in I_{L_1}} c_k\, \mathrm{sinc}\big(N\pi\,(b_\ell - a_k)\big)\,, \quad \ell \in I_{L_2}\,, \tag{6.2}$$

where $b_\ell \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ can be equispaced/nonequispaced points.



(a) Error constant (5.8)


(b) Maximum error (5.10)

Figure 5.1: Visualization of Theorem 5.3 the approximation of $\mathrm{sinc}(N\pi x)$, $x \in [-1, 1]$ for different bandwidths $N = 2^\ell$, $\ell = 3, \dots, 7$, where $n = \nu N$, $\nu \in \{1, \dots, 10\}$ and Chebyshev nodes $z_k \in [-1, 1]$, $k = 0, \dots, n$.

Such a function (6.1) occurs by the application of the famous Sampling Theorem of Shannon–Whittaker–Kotelnikov (see e.g. [26, pp. 86–88]). Let $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ be bandlimited on $[-\pi L_2, \pi L_2]$ for some $L_2 > 0$, i. e., the Fourier transform of $f$ is supported on $[-\pi L_2, \pi L_2]$. Then for $N \in 2\mathbb{N}$ with $N \geq L_2$, the function $f$ is completely determined by its values $f\big(\frac{k}{N}\big)$, $k \in \mathbb{Z}$, and further $f$ can be represented in the form

$$f(x) = \sum_{k \in \mathbb{Z}} f\Big(\frac{k}{N}\Big) \,\mathrm{sinc}\Big(N\pi\Big(x - \frac{k}{N}\Big)\Big)\,, \quad x \in \mathbb{R}\,,$$

where the series converges absolutely and uniformly on $\mathbb{R}$. By truncation of this series, we obtain the linear combination of shifted sinc functions

$$\sum_{k \in I_{L_1}} f\Big(\frac{k}{N}\Big) \,\mathrm{sinc}\Big(N\pi\Big(x - \frac{k}{N}\Big)\Big)\,, \quad x \in \mathbb{R}\,,$$

which has the same form as (6.1), when $a_k$ are equispaced.

Since the naive computation of (6.2) requires $\mathcal{O}(L_1 \cdot L_2)$ arithmetic operations, the aim is to find a more efficient method for the evaluation of (6.2). Up to now, several approaches for a fast computation of the discrete sinc transform (6.2) are known. In [12], the discrete sinc transform (6.2) is realized by applying a Gauss–Legendre quadrature rule to the integral (5.9). The result can then be approximated by means of two NNFFT's with $\mathcal{O}((L_1 + L_2)\log(L_1 + L_2))$ arithmetic operations. A multilevel algorithm with $\mathcal{O}(L_2 \log(1/\delta))$ arithmetic operations is presented in [20] which is most effective for equispaced points $a_k$ and $b_\ell$ and, as the authors claim themselves, is only practical for rather large target evaluation accuracy $\delta > 0$.

In the following, we present a new approach for a fast sinc transform (6.2), where we approximate the function $\mathrm{sinc}(N\pi x)$ by an exponential sum on the interval $[-1, 1]$ by means of the Clenshaw–Curtis quadrature as described in Section 5. Let the Chebyshev points $z_j = \cos\frac{j\pi}{n}$, $j = 0, \ldots, n$, and the coefficients $w_j$ defined by (5.6) be given. Utilizing (5.8), for arbitrary $a_k$, $b_\ell \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ we obtain the approximation

$$\operatorname{sinc}\big(N\pi(a_k - b_\ell)\big) \approx \sum_{j=0}^{n} w_j\, \mathrm{e}^{-\pi\mathrm{i}\, N z_j (a_k - b_\ell)} = \sum_{j=0}^{n} w_j\, \mathrm{e}^{-\pi\mathrm{i}\, N z_j a_k}\, \mathrm{e}^{\pi\mathrm{i}\, N z_j b_\ell}\,. \tag{6.3}$$

Inserting this approximation into (6.2) yields

$$\begin{aligned} h_\ell &:= \sum_{k\in I_{L_1}} c_k \sum_{j=0}^{n} w_j\, \mathrm{e}^{-\pi\mathrm{i}\, N z_j a_k}\, \mathrm{e}^{\pi\mathrm{i}\, N z_j b_\ell} \\ &= \sum_{j=0}^{n} w_j \Big(\sum_{k\in I_{L_1}} c_k\, \mathrm{e}^{-\pi\mathrm{i}\, N z_j a_k}\Big)\, \mathrm{e}^{\pi\mathrm{i}\, N z_j b_\ell}\,, \quad \ell \in I_{L_2}\,. \end{aligned} \tag{6.4}$$

If $\varepsilon > 0$ denotes a target accuracy, then we choose $n = 2^t$, $t \in \mathbb{N} \setminus \{1\}$ such that by Theorem 5.3 it holds

$$\frac{144}{70\,(\mathrm{e}^2 - 1)}\, \mathrm{e}^{-n} \cosh \frac{\pi\,(\mathrm{e}^2 - 1)\,N}{2\,\mathrm{e}} < \varepsilon$$

For example, in the case $\varepsilon = 10^{-8}$ we obtain $n \geq 4N$ for $N \geq 64$.
We recognize that the term inside the brackets of (6.4) is an exponential sum of the form (1.1), which can be computed by means of an NNFFT. Then the resulting outer sum is of the same form such that this can also be computed by means of an NNFFT. Thus, as in [12] we may compute the discrete sinc transform (6.2) by means of an NNFFT, a multiplication by the precomputed coefficients $w_j$ as well as another NNFFT afterwards. Hence, the fast sinc transform, which is an application of the NNFFT, can be summarized as follows.

**Algorithm 6.1 (Fast sinc transform)**
*Input:* $N \in \mathbb{N}$, $L_1$, $L_2 \in 2\mathbb{N}$ *as well as* $c_k \in \mathbb{C}$, $a_k \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ *for* $k \in I_{L_1}$, $z_j = \cos\frac{j\pi}{n}$ *with* $j = 0, \ldots, n$ *and* $n \geq 4N$.

0\. *Precompute the values* $w_j$, $j = 0, \ldots, n$, *by Algorithm* 5.4.

1\. *For all* $j = 0, \ldots, n$, *compute by* NNFFT

$$g_j := \sum_{k\in I_{L_1}} c_k\, \mathrm{e}^{-\pi\mathrm{i}\, N z_j a_k}\,, \tag{6.5}$$

*where* $\tilde{g}_j$ *is the approximate value of* $g_j$.
2\. *For all* $j = 0, \ldots, n$, *form the products*

$$\alpha_j := w_j \cdot \tilde{g}_j\,. \tag{6.6}$$

3. *For all* $\ell \in I_{L_2}$ *compute by* NNFFT

$$\hat{h}_\ell := \sum_{j=0}^{n} \alpha_j \, \mathrm{e}^{\pi \mathrm{i} \, N z_j b_\ell} \,, \tag{6.7}$$

*where* $\tilde{h}_\ell$ *is the approximate value of* $\hat{h}_\ell$.
*Output:* $\tilde{h}_\ell$ *approximate value of* (6.2) *for* $\ell \in I_{L_2}$.

If we use the same NNFFT's (with the window functions $\varphi_j$, truncation parameters $m_j$, and oversampling factors $\sigma_j$ for $j = 1, 2$), Algorithm 6.1 requires all in all

$$\mathcal{O}(N \log N + L_1 + L_2 + 2n)$$

arithmetic operations.
Considering the discrete sinc transform (6.2), we can deal with the special sums of the form

$$h\big(\frac{\ell}{N}\big) = \sum_{k \in I_{L_1}} c_k \, \mathrm{sinc}\big(N\pi\big(a_k - \frac{\ell}{N}\big)\big) \,, \quad \ell \in I_N, \tag{6.8}$$

i.e., we have given equispaced points $b_\ell = \frac{\ell}{N}$ with $L_2 = N$. In this special case, we simply obtain an adjoint NFFT instead of the NNFFT in step 3 of Algorithm 6.1. Therefore, the computational cost of Algorithm 6.1 reduces to $\mathcal{O}(N \log N + L_1 + n)$. In the case, where $a_k = \frac{k}{L_1}$, $k \in I_{L_1}$, the NNFFT in step 1 of Algorithm 6.1 naturally turns into an NFFT. Clearly, in this case the same amount of arithmetic operations is needed as in the first special case. If both sets of nodes $a_k$ and $b_\ell$ are equispaced, then the computational cost reduces even more to $\mathcal{O}(N \log N + n)$. Hence, these modifications can be included into our fast sinc transform.
Now we study the error of the fast sinc transform in Algorithm 6.1, which is measured in the form

$$\max_{\ell \in I_{L_2}} |h(b_\ell) - \tilde{h}_\ell| \,. \tag{6.9}$$

**Theorem 6.2** *Let* $N \in \mathbb{N}$ *with* $N > 1$ *be given. Let* $L_1$, $L_2 \in 2\mathbb{N}$ *be given. Assume that* $N_1 = \sigma_1 N \in 2\mathbb{N}$ *with* $\sigma_1 > 1$. *For fixed* $m_1 \in \mathbb{N} \setminus \{1\}$ *with* $2m_1 \ll N_1$, *let* $N_2 = \sigma_2\,(N_1 + 2m_1)$ *with* $\sigma_2 > 1$. *For* $m_2 \in \mathbb{N} \setminus \{1\}$ *with* $2m_2 \le \big(1 - \frac{1}{\sigma_1}\big)\, N_2$, *let* $\varphi_1$ *and* $\varphi_2$ *be the window functions of the form* (2.5). *Let* $a_k$, $b_\ell \in \big[-\frac{1}{2}, \frac{1}{2}\big]$ *with* $k \in I_{L_1}$, $\ell \in I_{L_2}$ *be arbitrary points and let* $c_k \in \mathbb{C}$, $k \in I_{L_1}$, *be arbitrary coefficients. Let* $a > 1$ *be the constant* (2.6). *For a given target accuracy* $\varepsilon > 0$, *the number* $n = 2^t$, $t \in \mathbb{N} \setminus \{1\}$, *is chosen such that*

$$\frac{144}{70\,(\mathrm{e}^2 - 1)}\, \mathrm{e}^{-n} \cosh \frac{\pi\,(\mathrm{e}^2 - 1)\,N}{2\,\mathrm{e}} < \varepsilon \,. \tag{6.10}$$

*Then the error of the fast* sinc *transform can be estimated by*

$$\max_{\ell \in I_{L_2}} \big|h(b_\ell) - \tilde{h}_\ell\big| \;\; \le \;\; \Big(\varepsilon + 2\Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat{\varphi}_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big]$$

$$+\Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]^2\Big)\sum_{k\in I_{L_1}}|c_k|\,, \qquad (6.11)$$

*where* $E_{\sigma_j}(\varphi_j)$ *for* $j = 1,\,2$, *are the general* $C(\mathbb{T})$*-error constants of the form* (3.8)*. If*

$$E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2) \ll 1\,,$$

*one can use the simplified estimate*

$$\max_{\ell\in I_{L_2}}\big|h(b_\ell) - \tilde h_\ell\big| \le \Big(\varepsilon + 2E_{\sigma_1}(\varphi_1) + \frac{2a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big)\sum_{k\in I_{L_1}}|c_k|\,.$$

*Proof.* By (6.4), the value $h_\ell$ is an approximation of $h(b_\ell)$. Since $a_k,\,b_\ell \in \big[-\frac{1}{2},\,\frac{1}{2}\big]$, it holds by (5.8) and (6.10) that

$$\Big|\mathrm{sinc}\big(\pi N(a_k - b_\ell)\big) - \sum_{j=0}^{n} w_j\,\mathrm{e}^{-\pi\mathrm{i}\,N\,z_j(a_k-b_\ell)}\Big| \le \varepsilon\,.$$

Hence, we conclude that

$$|h(b_\ell) - h_\ell| \le \varepsilon\sum_{k\in I_{L_1}}|c_k|\,, \quad \ell\in I_{L_2}\,. \qquad (6.12)$$

After step 1 of Algorithm 6.1, the error of the NNFFT (with the window functions $\varphi_1$ and $\varphi_2$) can be estimated by Theorem 3.5 in the form

$$|g_j - \tilde g_j| \le \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]\sum_{k\in I_{L_1}}|c_k|\,, \quad j = 0,\ldots,n\,.$$

Using (5.7), step 2 of Algorithm 6.1 generates the error

$$\begin{aligned}|\hat h_\ell - h_\ell| &\le \sum_{j=0}^{n} w_j\,|g_j - \tilde g_j|\\ &\le \Big(\sum_{j=0}^{n} w_j\Big)\Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]\sum_{k\in I_{L_1}}|c_k|\\ &= \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]\sum_{k\in I_{L_1}}|c_k|\,. \qquad (6.13)\end{aligned}$$

After step 3 of Algorithm 6.1, the error of the NNFFT (with the same window functions $\varphi_1$ and $\varphi_2$) can be estimated by Theorem 3.5 in the form

$$\big|\hat h_\ell - \tilde h_\ell\big| \le \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\,E_{\sigma_2}(\varphi_2)\Big]\sum_{j=0}^{n} w_j\big|\tilde g_j\big|\,, \quad \ell\in I_{L_2}\,.$$

Using the triangle inequality, we obtain

$$\begin{aligned}\big|\tilde g_j\big| &\le |g_j| + |g_j - \tilde g_j| \le \sum_{k\in I_{L_1}} |c_k| + |g_j - \tilde g_j| \\ &\le \sum_{k\in I_{L_1}} |c_k| + \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in I_{L_1}} |c_k|\,, \quad j = 0, \ldots, n\end{aligned}$$

such that by (5.7)

$$\begin{aligned}\big|\hat h_\ell - \tilde h_\ell\big| &\le \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big]\Big(\sum_{j=0}^{n} w_j\Big) \sum_{k\in I_{L_1}} |c_k| \\ &\quad + \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big]^2\Big(\sum_{j=0}^{n} w_j\Big) \sum_{k\in I_{L_1}} |c_k| \\ &= \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big] \sum_{k\in I_{L_1}} |c_k| \\ &\quad + \Big[E_{\sigma_1}(\varphi_1) + \frac{a}{\hat\varphi_1\big(\frac{N}{2}\big)}\, E_{\sigma_2}(\varphi_2)\Big]^2 \sum_{k\in I_{L_1}} |c_k|\,. \end{aligned} \tag{6.14}$$

Thus, the error of Algorithm 6.1 can be estimated by

$$\big|h(b_\ell) - \tilde h_\ell\big| \le |h(b_\ell) - h_\ell| + |h_\ell - \hat h_\ell| + \big|\hat h_\ell - \tilde h_\ell\big|\,, \quad \ell \in I_{L_2}\,.$$

From (6.12) – (6.14) it follows the estimate (6.11). ■

This is to say, the error of Algorithm 6.1 for the fast sinc transform mostly depends on the quality of its precomputation. Thus, the error of Algorithm 6.1 can easily be controlled by the error results of the NNFFT given in Theorem 3.5.

Next we verify the accuracy of our fast sinc transform in Algorithm 6.1. To this end, we choose random nodes $a_k \in \big[-\frac{1}{2}, \frac{1}{2}\big]$, equispaced points $b_\ell = \frac{\ell}{N}$ with $\ell \in I_N$, as well as random coefficients $c_k \in \mathbb{C}$, $k \in I_{L_1}$, and compute the discrete sinc transform (6.2) directly as well as its approximation (6.7) by means of the fast sinc transform. Subsequently, we compute the maximum error (6.9). Due to the randomness of the given values this test was repeated one hundred times and afterwards the maximum error over all repetitions was computed.

In this experiment we choose different bandwidths $N = 2^k$, $k = 4, \ldots, 13$, and without loss of generality we use $L_1 = \frac{N}{2}$. We apply Algorithm 6.1 using the weights $w_j$ computed by means of Algorithm 5.4 and the Chebyshev points $z_j = \cos\frac{j\pi}{n}$, $j = 0, \ldots, n$. Therefore, we only have to examine the parameter choice of $n \ge 4N$. To this end, we compare the results for several choices, namely for $n \in \{4N, 6N, 8N\}$. The appropriate results can be found in Figure 6.1. We see that for large $N$ there is almost no difference between the different choices of $n$. However, we point out that a higher choice heavily increases the computational cost of Algorithm 6.1. Therefore, it is recommended to use the smallest possible choice $n = 4N$. Compared to [12] the same approximation errors are obtained, but with a more efficient precomputation of weights.

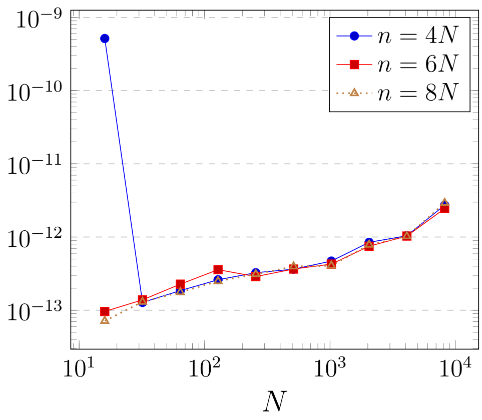



Figure 6.1: Maximum error (6.9) for different bandwidths $N = 2^k$, $k = 4, \dots, 13$, shown for several choices of $4N \le n \in \{4N, 6N, 8N\}$ when using the coefficients $w_j$ obtained by Algorithm 5.4.

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