Preconditioning of Hermitian block–Toeplitz–Toeplitz–block matrices by level–1 preconditioners

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Abstract. In this paper, we are interested in the eigenvalue distribution of sequences of preconditioned Hermitian block–Toeplitz–Toeplitz–block (BTTB) matrices of size $N_1N_2 \times N_1N_2$ if $N_1, N_2 \to \infty$. We focus on level–1 preconditioners constructed from circulant–like matrices. For some reasons we restrict our attention to BTTB matrices having the sum of nonnegative trigonometric polynomials $p(x)$ and $q(y)$ as generating function. We show that for some usual preconditioners, $\mathcal{O}(N_1)$ eigenvalues of the preconditioned matrices behave as the α–th $(0 < \alpha < 1)$ power of the reciprocal eigenvalues of the $N_1 \times N_1$ Toeplitz matrices generated by q. For example, if p and q have one zero of order 4, respectively, then the preconditioned BTTB matrices with level–1 preconditioners based on the sin-I transform have $2N_1$ eigenvalues which behave for $N_1, N_2 \to \infty$ as N_1/k $(k = 1, \ldots, N_1)$.

1. Introduction

In this paper, we are interested in the solution of Hermitian block–Toeplitz–-Toeplitz–block (BTTB) systems

$$\boldsymbol{A}_{N_1,N_2}(f)\, \boldsymbol{x} = \boldsymbol{b} \tag{1.1}$$

with nonnegative real–valued generating functions $f \in C[0, 2\pi]^2$, i.e.

$$\boldsymbol{A}_{N_1,N_2} := (\boldsymbol{A}_{r-s})_{r,s=0}^{N_1-1}, \quad \boldsymbol{A}_r := (a_{r,j-k}(f))_{j,k=0}^{N_2-1},$$

where

$$a_{r,j} := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) \mathrm{e}^{-\mathrm{i}(rx+jy)}\, \mathrm{d}x \mathrm{d}y.$$

Since f is nonnegative, the matrices $\boldsymbol{A}_{N_1,N_2}$ are positive definite [**7**]. During the last years the preconditioned conjugate gradient method (PCG–method) for the solution of (1.1) has attained much attention. The main reason for this is that each iteration step of the CG–method requires only one multiplication of $\boldsymbol{A}_{N_1,N_2}$ with a vector which can be realized with $\mathcal{O}(N_1N_2 \log(N_1N_2))$ arithmetic operations by fast Fourier transforms. The number of iteration steps depends on the distribution of the eigenvalues of $\boldsymbol{A}_{N_1,N_2}$. In particular, we recall the following result of O. Axelsson [**1**, p. 573].

1991 *Mathematics Subject Classification.* Primary 65F10; Secondary 65F15, 65T10, 65F50.

THEOREM 1.1. *Let $\boldsymbol{A}$ be a positive definite Hermitian $N \times N$ matrix, which has p and q isolated large and small eigenvalues, respectively:*

$$\begin{aligned} 0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_q \quad &< \quad a \le \lambda_{q+1} \le \ldots \lambda_{N-p} \le b \\ &< \quad \lambda_{N-p+1} \le \lambda_{N-p+2} \le \ldots \le \lambda_N \quad (0 < a < b < \infty). \end{aligned}$$

Let $\lceil x \rceil$ denote the smallest integer $\ge x$. Then the CG–method for the solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ requires at most

$$n = \left\lceil (\ln \frac{2}{\tau} + \sum_{k=1}^{q} \ln \frac{b}{\lambda_k}) \;/\; \ln \frac{1 + (\frac{a}{b})^{1/2}}{1 - (\frac{a}{b})^{1/2}} \right\rceil + p + q$$

iteration steps to achieve precision τ, i.e.

$$\frac{||\boldsymbol{x}^{(n)} - \boldsymbol{x}||_A}{||\boldsymbol{x}^{(0)} - \boldsymbol{x}||_A} \le \tau \; ,$$

where $||\boldsymbol{x}||_A := \sqrt{\boldsymbol{x}^* \boldsymbol{A}\, \boldsymbol{x}}$, $\boldsymbol{x}^*$ is the complex–conjugate transposed vector of $\boldsymbol{x}$ and $\boldsymbol{x}^{(n)}$ denotes the numerical solution after n iteration steps.

If $f \in C[0, 2\pi]^2$ is a positive function with $f_{\min} := \min\{f(x, y) : (x, y) \in [0, 2\pi]^2\}$ and $f_{\max} := \max\{f(x, y) : (x, y) \in [0, 2\pi]^2\}$, then the eigenvalues of $\boldsymbol{A}_{N_1,N_2}(f)$ are contained in $[f_{\min}, f_{\max}]$ for all $N_1, N_2 \in \mathbb{N}$ [**23**]. Thus, by Theorem 1.1, the CG–method applied to (1.1) requires only a constant number of iteration steps. Preconditioning with Hermitian positive definite matrices $\boldsymbol{M}_{N_1,N_2}$ can further decrease the number of iteration steps. Since every iteration step requires in addition the solution of a system of linear equations with coefficient matrix $\boldsymbol{M}_{N_1,N_2}$ now, this solution should take no more than $\mathcal{O}(N_1 N_2 \log(N_1 N_2))$ arithmetic operations. Therefore level–1 and level–2 preconditioners related to algebras of circulant–like matrices, i.e. matrices which are diagonalizable, up to unitary diagonal matrices, by the Fourier matrix or trigonometric matrices, were mainly used in literature [**8, 6, 18, 22**]. While in the univariate case preconditioning with suitable circulant–like matrices leads to a proper clustering of the eigenvalues of the preconditioned matrices at 1 [**21**], this is not true for the bivariate setting [**24, 25**].

In this paper, we will show a negative result for the more interesting case that $f \in C[0, 2\pi]^2$ has zeros. We restrict our attention to functions $f(x, y) := p(x) + q(y)$, where p and q are nonnegative trigonometric polynomials. Then

$$\boldsymbol{A}_{N_1,N_2}(p + q) = \boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) \; + \; \boldsymbol{A}_{N_2}(p) \otimes \boldsymbol{I}_{N_1}, \tag{1.2}$$

where $\boldsymbol{I}_{N_j}$ are the identity matrices of size $N_j \times N_j$ $(j = 1, 2)$ and

$$\boldsymbol{A}_{N_2}(p) := \left(a_{j-k}(p)\right)_{j,k=0}^{N_2} \; , \quad a_j(p) := \frac{1}{2\pi} \int_0^{2\pi} p(x) \mathrm{e}^{-\mathrm{i}jx} \, \mathrm{d}x.$$

The reason for this restriction is the following: Assume that $f \in C[0, 2\pi]^2$ has isolated zeros of order $2\theta_j$ at (x_j, y_j) $(j = 1, \ldots, k)$. Here we use the generalized definition of the order of zeros for functions of two variables given in [**19**]:

The zero (x_0, y_0) of $f(x, y)$ is of order β, if β is the smallest positive integer so that $F^{(\beta+1)}_{(x_0,y_0)}$ is continuous in a neighborhood of (x_0, y_0) and $F^{(\beta+1)}_{(x_0,y_0)}(0) \ne 0$ for all $(x, y) \ne (x_0, y_0)$, where

$$F_{(x_0,y_0)}(t) := f(x_0 + t(x - x_0), y_0 + t(y - y_0)) \quad \forall (x, y) \ne (x_0, y_0).$$

Set

$$p(x) := \prod_{j=1}^{k} \Big(2 - 2\cos(x - x_j)\Big)^{\theta_j}, \; q(y) := \prod_{j=1}^{k} \Big(2 - 2\cos(y - y_j)\Big)^{\theta_j}.$$

Based on [**23**], it was proved in [**19**] that the eigenvalues of $\boldsymbol{A}_{N_1,N_2}^{-1}(p+q)\boldsymbol{A}_{N_1,N_2}(f)$ are bounded from above and below by positive constants independent of N_1 and N_2. Since p and q are trigonometric polynomials, the matrices $\boldsymbol{A}_{N_1,N_2}(p+q)$ are banded block–Toeplitz – banded Toeplitz–block matrices. However, the sparse matrices $\boldsymbol{A}_{N_1,N_2}(p+q)$ are in general not suited as preconditioners for (1.1) since the efficient solution of a system of linear equations with coefficient matrix $\boldsymbol{A}_{N_1,N_2}(p+q)$ requires additional effort.

One possibility to solve systems of linear equations with coefficient matrix $\boldsymbol{A}_{N_1,N_2}(p+q)$ is based on the multigrid method. The convergence of the two–grid method independent of the problem size was proved for banded matrices belonging to the so–called "τ–algebra" in [**11**] and generalized to the corresponding BTTB systems in [**12**]. The convergence of the full multigrid method independent of the problem size was shown for Toeplitz systems generated by nonnegative functions with zero of order less or equal than 2 in [**5**] and enlarged to the corresponding BTTB systems in [**28**].

Furthermore, it was emphasized in [**2**] that the (direct) method of "cyclic reduction" can also be applied to the solution of BTTB systems.

In this paper, we examine another method. In the univariate case it was proved that suitable circulant–like matrices $\boldsymbol{M}_N(f)$ are very good preconditioners for Toeplitz matrices $\boldsymbol{A}_N(f)$ $(f \geq 0)$ in the sense that the eigenvalues of the preconditioned matrices have a proper cluster at 1 [**21**]. If in particular, $f = p$ is a trigonometric polynomial, then the matrices $\boldsymbol{M}_N^{-1}(p)\boldsymbol{A}_N(p)$ have only a fixed number of eigenvalues (independent of N) which are not equal to 1. This recommends the use of matrices of the form

$$\boldsymbol{M}_{N_1,N_2}(p+q) := \boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) + \boldsymbol{M}_{N_2}(p) \otimes \boldsymbol{I}_{N_1}$$

with circulant–like matrices $\boldsymbol{M}_{N_2}(p)$ as preconditioners for (1.2). Systems of linear equations with coefficient matrix $\boldsymbol{M}_{N_1,N_2}(p+q)$ can be solved with $\mathcal{O}(N_1 N_2 \log N_2)$ arithmetic operations by fast Fourier transforms or fast trigonometric transforms. Moreover, by considering

$$\boldsymbol{M}_{N_1,N_2}^{-1}(p+q)\boldsymbol{A}_{N_1,N_2}(f) = \boldsymbol{M}_{N_1,N_2}^{-1}(p+q)\boldsymbol{A}_{N_1,N_2}(p+q)\;\boldsymbol{A}_{N_1,N_2}^{-1}(p+q)\boldsymbol{A}_{N_1,N_2}(f)$$

and using the technique in [**21, 9**] and the fact that the eigenvalues of $\boldsymbol{A}_{N_1,N_2}(p+q)^{-1}\boldsymbol{A}_{N_1,N_2}(f)$ are bounded from above and below by positive constants independent of N_1, N_2, we see that $\boldsymbol{M}_{N_1,N_2}(p+q)$ is a good preconditioner for $\boldsymbol{A}_{N_1,N_2}(f)$ if $\boldsymbol{M}_{N_1,N_2}(p+q)$ is a good preconditioner for $\boldsymbol{A}_{N_1,N_2}(p+q)$. For example, if the eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}(p+q)\boldsymbol{A}_{N_1,N_2}(p+q)$ lie in a fixed interval independent of N_1 and N_2 (up to some outlyers), then the same is true for the eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}(p+q)\boldsymbol{A}_{N_1,N_2}(f)$. Therefore we only consider systems of linear equations with coefficient matrices $\boldsymbol{A}_{N_1,N_2}(p+q)$. Further, we restrict our attention to polynomials p with

i) one zero x_0 of order 2; i.e. $\boldsymbol{A}_{N_2}(p)$ is a tridiagonal Toeplitz matrix,
ii) two zeros x_0, x_1 of order 2, respectively, or one zero x_0 of order 4; i.e. $\boldsymbol{A}_{N_2}(p)$ is a pentdiagonal Toeplitz matrix.

In general, one should apply numerical methods for ill–posed problems (cf. [**13**]) to solve (1.1) if p has or zeros of order > 4. See Table 3.

In this paper, we will show the following: In case i), except for a fast sin-I based direct solution method for systems of linear equations with coefficient matrix $\boldsymbol{A}_{N_1,N_2}(p+q)$, there also exist preconditioners $\boldsymbol{M}_{N_1,N_2}(p+q)$ related to the sin-II (cos-II) transform so that the eigenvalues of $\boldsymbol{M}^{-1}_{N_1,N_2}(p+q)\boldsymbol{A}_{N_1,N_2}(p+q)$ are bounded from above and below by positive constants independent of N_1 and N_2. We will show that this result can not be achieved with (ω-) circulant preconditioners $\boldsymbol{M}_{N_1,N_2}(p+q)$ related to the discrete Fourier transform. In particular, if q has also one zero of order 2, then N_1 eigenvalues of $\boldsymbol{M}^{-1}_{N_1,N_2}(p+q)\boldsymbol{A}_{N_1,N_2}(p+q)$ behave for $N_1, N_2 \to \infty$ as N_1/k $(k = 1, \ldots, N_1)$.

In case ii), we suggest a suitable sin-I transform based preconditioner if $x_1 = x_0 - \pi$ and show that the eigenvalues of the preconditioned matrices are bounded from above and below by positive constants independent of N_1 and N_2. If $x_1 \neq x_0 - \pi$ also sin-I transform based preconditioners fail in the following sense: If p has one zero of order 4 and $\mu_{N_1,k}$ $(k = 1, \ldots, N_1)$ are the eigenvalues of $\boldsymbol{A}_{N_1}(q)$, then $2N_1$ eigenvalues $\lambda^{e/o}_{N_1,k}$ $(k = 1, \ldots, N_1)$ of the preconditioned matrices behave for $N_1, N_2 \to \infty$ as $\mu^{-1/4}_{N_1,k}$, i.e. if q has also one zero of order 4 as N_1/k $(k = 1, \ldots, N_1)$.

This paper is organized as follows: Section 2 provides the basic notation and some results concerning the relation of trigonometric transforms and Toeplitz matrices. In Section 3, we deal with case i), while we are interested in the setting ii) in Section 4.

2. Preliminaries

According to [**29**], we introduce the following matrices

$$\begin{aligned}
\boldsymbol{S}^{I}_{N-1} &:= \left(\frac{2}{N}\right)^{1/2} \left(\sin \frac{j\,k\,\pi}{N}\right)^{N-1}_{j,k=1} \in \mathbb{R}^{N-1,N-1}, \\
\boldsymbol{S}^{II}_{N} &:= \left(\frac{2}{N}\right)^{1/2} \left(\varepsilon^N_{j+1} \sin \frac{(j+1)(2k+1)\pi}{2N}\right)^{N-1}_{j,k=0} \in \mathbb{R}^{N,N}, \\
\boldsymbol{C}^{II}_{N} &:= \left(\frac{2}{N}\right)^{1/2} \left(\varepsilon^N_{j} \cos \frac{j(2k+1)\pi}{2N}\right)^{N-1}_{j,k=0} \in \mathbb{R}^{N,N},
\end{aligned}$$

where $\varepsilon^N_k := 1/\sqrt{2}$ $(k = 0, N)$ and $\varepsilon^N_k := 1$ otherwise and refer to the corresponding discrete trigonometric transforms as sin-I, sin-II and cos-II transforms. The above matrices are orthogonal and the vector multiplication with any of these matrices takes only $\mathcal{O}(N \log N)$ arithmetic operations provided that N is a power of 2 (cf. [**26**]). Furthermore, $\boldsymbol{S}^{II}_N$ and $\boldsymbol{C}^{II}_N$ are related by

$$\boldsymbol{J}_N \tilde{\boldsymbol{I}}_N \boldsymbol{S}^{II}_N \tilde{\boldsymbol{I}}_N \boldsymbol{J}_N = \boldsymbol{C}^{II}_N, \tag{2.1}$$

where $\boldsymbol{J}_N$ denotes the $N \times N$ counteridentity matrix and $\tilde{\boldsymbol{I}}_N := \operatorname{diag}((-1)^k)^{N-1}_{k=0}$. Discrete trigonometric transforms are related to Toeplitz and Hankel matrices [**20, 14**]. Let $\operatorname{toep}_N \boldsymbol{a}'$ $(\boldsymbol{a} \in \mathbb{C}^N)$ be the Hermitian $N \times N$ Toeplitz matrix with first row $\boldsymbol{a}'$ and $\operatorname{hank}_N \boldsymbol{a}'$ $(\boldsymbol{a} \in \mathbb{R}^N)$ the persymmetric $N \times N$ Hankel matrix with first row

$\boldsymbol{a}'$. Then we have for arbitrary $\boldsymbol{a} \in \mathbb{R}^{N-1}$ that

$$\begin{aligned} \boldsymbol{S}^{I}_{N-1} \operatorname{diag}(d^{I}_{k})_{k=1}^{N-1} \boldsymbol{S}^{I}_{N-1} &= \operatorname{toep}_{N-1}(a_0, \ldots, a_{N-2}) \\ &\quad - \operatorname{hank}_{N-1}(a_2, \ldots, a_{N-2}, 0, 0), \end{aligned} \tag{2.2}$$

where $d^{I}_{k} := 2 \sum_{j=0}^{N-2} (\varepsilon^{N}_{j})^2 a_j \cos \frac{jk\pi}{N}$ and for arbitrary $\boldsymbol{a} \in \mathbb{R}^{N}$ that

$$\begin{aligned} (\boldsymbol{S}^{II}_{N})' \operatorname{diag}(d^{II}_{k})_{k=1}^{N} \boldsymbol{S}^{II}_{N} &= \operatorname{toep}_{N}(a_0, \ldots, a_{N-1}) \\ &\quad - \operatorname{hank}_{N}(a_1, \ldots, a_{N-1}, 0), \\ (\boldsymbol{C}^{II}_{N})' \operatorname{diag}(d^{II}_{k})_{k=0}^{N-1} \boldsymbol{C}^{II}_{N} &= \operatorname{toep}_{N}(a_0, \ldots, a_{N-1}) \\ &\quad + \operatorname{hank}_{N}(a_1, \ldots, a_{N-1}, 0), \end{aligned} \tag{2.3}$$

where $d^{II}_{k} := 2 \sum_{j=0}^{N-1} (\varepsilon^{N}_{j})^2 a_j \cos \frac{jk\pi}{N}$. The algebra of matrices which can be diagonalized by the sin-I transform is called τ–algebra and plays a special rule in the preconditioning of Toeplitz matrices [**9, 10**].

Besides trigonometric transforms we also use discrete Fourier transforms given by the N–*th Fourier matrix*

$$\boldsymbol{F}_N := \left(\frac{1}{N}\right)^{1/2} (\mathrm{e}^{-2\pi \mathrm{i} jk/N})_{j,k=0}^{N-1} .$$

It is well–known that ω–circulant matrices ($\omega := \mathrm{e}^{\mathrm{i}\varphi}$) can be diagonalized by $\boldsymbol{F}_N$ times a unitary diagonal matrix, i.e.

$$\boldsymbol{W}_N \boldsymbol{F}_N \operatorname{diag}(d_k)_{k=0}^{N-1} \boldsymbol{F}^*_N \boldsymbol{W}^*_N = \begin{pmatrix} a_0 & \omega a_{N-1} & \ldots & \omega a_2 & \omega a_1 \\ a_1 & a_0 & \ddots & \cdots & \omega a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{N-2} & \ddots & \ddots & \ddots & \omega a_{N-1} \\ a_{N-1} & a_{N-2} & \cdots & a_1 & a_0 \end{pmatrix}, \tag{2.4}$$

where $d_k := \sum_{j=0}^{N-1} a_j \omega^{j/N} \mathrm{e}^{2\pi \mathrm{i} jk/N}$, $\boldsymbol{W}_N := \operatorname{diag}(\omega^{-k/N})_{k=0}^{N-1}$ and $\boldsymbol{F}^*_N$ denotes the transposed complex–conjugate matrix of $\boldsymbol{F}_N$. If $\omega = 1$, then (2.4) is circulant and if $\omega = -1$, then (2.4) is skew–circulant. We refer to matrices which can be diagonalized (up to multiplication by unitary diagonal matrices) by $\boldsymbol{S}^{I}_{N-1}, \boldsymbol{S}^{II}_{N}, \boldsymbol{C}^{II}_{N}$ or $\boldsymbol{F}_N$ as *circulant–like matrices.*

In the following, we will further use the $N \times N$ matrices

$$\boldsymbol{R}_N := \operatorname{hank}_N(1, 0, \ldots, 0)\,, \ \boldsymbol{H}_N(\omega) := \operatorname{toep}_N(0, \ldots, 0, \omega).$$

Note that

$$\boldsymbol{R}_N \boldsymbol{R}_N = \boldsymbol{R}_N\,, \ \boldsymbol{R}_N \boldsymbol{H}_N = \boldsymbol{H}_N \boldsymbol{R}_N = \boldsymbol{H}_N. \tag{2.5}$$

Finally, let $g(N) \sim h(N)$ $(N \to \infty)$, if there exist constants $0 < c \leq C < \infty$ independent of N so that $c\,h(N) \leq g(N) \leq C\,h(N)$.

3. Hermitian tridiagonal Toeplitz matrices

We start with the simplest case

$$p(x) := \left(2\sin\frac{x-x_0}{2}\right)^2,$$

i.e. p has a zero of order 2 at $x_0 \in [0,\pi)$. Then $\boldsymbol{A}_{N_2}(p)$ is the Hermitian tridiagonal Toeplitz matrix

$$\boldsymbol{A}_{N_2}(p) = \mathrm{toep}_{N_2}(2, -\mathrm{e}^{\mathrm{i}x_0}, 0, \dots, 0). \tag{3.1}$$

Let q be an arbitrary trigonometric polynomial.

3.1. Direct solution by sin-I transform. Using the sin-I transform, a BTTB system with nonsingular coefficient matrix (1.2), where N_2+1 is a power of 2, can be directly solved with $\mathcal{O}(N_1N_2\log N_2)$ arithmetic operations as follows: Applying that every Hermitian tridiagonal Toeplitz matrix can be written as

$$\mathrm{toep}_{N_2}(a,b,0,\dots,0) = \boldsymbol{D}\boldsymbol{S}^I_{N_2}\,\mathrm{diag}\left(a - 2|b|\cos\frac{j\pi}{N_2+1}\right)_{j=1}^{N_2}\boldsymbol{S}^I_{N_2}\boldsymbol{D}^*$$

where $a\in\mathbb{R}$, $b\in\mathbb{C}$, $|b|\neq 0$ and $\boldsymbol{D} := \mathrm{diag}\left((-b/|b|)^k\right)_{k=0}^{N-2}$, we obtain by (3.1) that

$$\boldsymbol{A}^{-1}_{N_1,N_2} = (\boldsymbol{D}\boldsymbol{S}^I_{N_2}\otimes\boldsymbol{I}_{N_1})\,(\boldsymbol{I}_{N_2}\otimes\boldsymbol{A}_{N_1}(q) + \boldsymbol{\Lambda}_{N_2}(p)\otimes\boldsymbol{I}_{N_1})^{-1}\,(\boldsymbol{S}^I_{N_2}\boldsymbol{D}^*\otimes\boldsymbol{I}_{N_1}),$$

where $\boldsymbol{D} := \mathrm{diag}(\mathrm{e}^{-\mathrm{i}kx_0})_{k=0}^{N_2-1}$ and

$$\boldsymbol{\Lambda}_{N_2}(p) := \mathrm{diag}\left(2 - 2\cos\frac{j\pi}{N_2+1}\right)_{j=1}^{N_2} = \mathrm{diag}\left(\left(2\sin\frac{j\pi}{2(N_2+1)}\right)^2\right)_{j=1}^{N_2}.$$

Multiplication of a vector with $\boldsymbol{D}\boldsymbol{S}^I_{N_2}\otimes\boldsymbol{I}_{N_1}$ requires $\mathcal{O}(N_1N_2\log N_2)$ arithmetic operations. The matrix $\boldsymbol{I}_{N_2}\otimes\boldsymbol{A}_{N_1}(q) + \boldsymbol{\Lambda}_{N_2}(p)\otimes\boldsymbol{I}_{N_1}$ is a block diagonal matrix with N_2 banded $N_1\times N_1$ diagonal blocks. Therefore it remains to solve N_2 systems of linear equations with banded $N_1\times N_1$ coefficient matrices. This can be realized with $\mathcal{O}(N_1\,N_2)$ arithmetic operations. In summary, the solution of a linear system of equations with nonsingular Hermitian BTTB matrix (1.2), (3.1) as coefficient matrix requires only $\mathcal{O}(N_1N_2\log N_2)$ arithmetic operations.

Note that the approach works also if we replace $\boldsymbol{I}_{N_2}\otimes\boldsymbol{A}_{N_1}(q)$ by an arbitrary block diagonal matrix with banded diagonal blocks.

Although this section provides us with a fast direct solution method for tridiagonal BTTB systems, we found it worthy to consider also the PCG–method with different preconditioners, since this method was frequently applied in literature [**4, 17, 15, 16**].

3.2. Sin-II and cos-II preconditioners. Let N_2 be a power of 2. We suggest the level–1 preconditioner

$$\boldsymbol{M}_{N_1,N_2} := \boldsymbol{A}_{N_1,N_2} + \boldsymbol{R}_{N_2}\otimes\boldsymbol{I}_{N_1} = \boldsymbol{I}_{N_2}\otimes\boldsymbol{A}_{N_1}(q) + \boldsymbol{M}_{N_2}(p)\otimes\boldsymbol{I}_{N_1}, \tag{3.2}$$

where $\boldsymbol{M}_{N_2}(p) := \boldsymbol{A}_{N_2}(p) + \boldsymbol{R}_{N_2}$. By (2.3), the matrix $\boldsymbol{M}_{N_1,N_2}$ can be rewritten as

$$\begin{aligned}\boldsymbol{M}_{N_1,N_2} &= (\boldsymbol{D}(\boldsymbol{S}^{II}_{N_2})'\otimes\boldsymbol{I}_{N_1})\,(\boldsymbol{I}_{N_2}\otimes\boldsymbol{A}_{N_1}(q) + \boldsymbol{\Lambda}_{N_2}(p)\otimes\boldsymbol{I}_{N_1})\,(\boldsymbol{S}^{II}_{N_2}\boldsymbol{D}^*\otimes\boldsymbol{I}_{N_1})\\ &= (\boldsymbol{D}(\boldsymbol{S}^{II}_{N_2})'\otimes\boldsymbol{I}_{N_1})\,\boldsymbol{B}\,(\boldsymbol{S}^{II}_{N_2}\boldsymbol{D}^*\otimes\boldsymbol{I}_{N_1}),\end{aligned}$$

where $\boldsymbol{D} := \operatorname{diag}(\mathrm{e}^{-\mathrm{i}kx_0})_{k=0}^{N_2-1}$,

$$\boldsymbol{\Lambda}_{N_2}(p) := \operatorname{diag}(\lambda_j)_{j=1}^{N_2}\,, \quad \lambda_j = \lambda_{N_2,j} := \left(2\sin\frac{j\pi}{2N_2}\right)^2$$

and

$$\boldsymbol{B} := \operatorname{diag}(\boldsymbol{B}_j)_{j=1}^{N_2}\,,\ \boldsymbol{B}_j := \boldsymbol{A}_{N_1}(q) + \lambda_j \boldsymbol{I}_{N_1}\,. \tag{3.3}$$

Alternatively, by (2.1), the preconditioner $\boldsymbol{M}_{N_1,N_2}$ can be written with respect to the cos-II transform as

$$\boldsymbol{M}_{N_1,N_2} = (\tilde{\boldsymbol{D}}(\boldsymbol{C}_{N_2}^{II})' \otimes \boldsymbol{I}_{N_1})(\boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) + \tilde{\boldsymbol{\Lambda}}_{N_2}(p) \otimes \boldsymbol{I}_{N_1})(\boldsymbol{C}_{N_2}^{II}\tilde{\boldsymbol{D}}^* \otimes \boldsymbol{I}_{N_1})$$

where $\tilde{\boldsymbol{D}} := \operatorname{diag}((-\mathrm{e}^{-\mathrm{i}x_0})^k)_{k=0}^{N_2-1}$ and

$$\tilde{\boldsymbol{\Lambda}}_{N_2}(p) := \operatorname{diag}(\tilde{\lambda}_j)_{j=0}^{N_2-1}\,,\ \tilde{\lambda}_j = \tilde{\lambda}_{N_2,j} := \left(2\cos\frac{j\pi}{2N_2}\right)^2.$$

By the method described in Section 3.1, the solution of a system of linear equations with coefficient matrix $\boldsymbol{M}_{N_1,N_2}$ can be computed with $\mathcal{O}(N_1\,N_2\log N_2)$ arithmetic operations. Moreover, we see by the following theorem and Theorem 1.1 that the corresponding PCG–method requires only a constant number of iterations independent of N_1 and N_2. Thus, the whole PCG–method needs $\mathcal{O}(N_1 N_2 \log N_2)$ arithmetic operations.

THEOREM 3.1. *Let $\boldsymbol{A}_{N_1,N_2}$ and $\boldsymbol{M}_{N_1,N_2}$ be given by* (1.2), (3.1) *and* (3.2)*, respectively. Assume that $\boldsymbol{A}_{N_1}(q)$ is nonnegative definite. Then $N_1(N_2-2)$ eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are equal to* 1 *and the other $2\,N_1$ eigenvalues are contained in* $[\frac{1}{2}, 1)$.

PROOF. By (3.2) we see that

$$\begin{aligned} \boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2} &= \boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{M}_{N_1,N_2} + (\boldsymbol{A}_{N_1,N_2} - \boldsymbol{M}_{N_1,N_2})) \\ &= \boldsymbol{I}_{N_1 N_2} - \boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{R}_{N_2} \otimes \boldsymbol{I}_{N_1}). \end{aligned}$$

The second summand is a matrix of rank $2N_1$. Thus, $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ has $N_1(N_2-2)$ eigenvalues 1, while the other $2N_1$ eigenvalues are given by one minus the nonzero eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{R}_{N_2}\otimes\boldsymbol{I}_{N_1})$. By (2.5), the eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{R}_{N_2}\otimes\boldsymbol{I}_{N_1})$ coincide with the eigenvalues of $(\boldsymbol{R}_{N_2} \otimes \boldsymbol{I}_{N_1})\boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{R}_{N_2} \otimes \boldsymbol{I}_{N_1})$. By straightforward computation we obtain that the nonzero eigenvalues of the above matrix are given by the eigenvalues of the $2N_1 \times 2N_1$ matrix

$$\frac{2}{N_2}\begin{pmatrix} \sum\limits_{j=1}^{N_2} \boldsymbol{B}_j^{-1}(\varepsilon_j^{N_2})^2\left(\sin\frac{j\pi}{2N_2}\right)^2 & \mathrm{e}^{\mathrm{i}(N_2-1)x_0}\sum\limits_{j=1}^{N_2} \boldsymbol{B}_j^{-1}(\varepsilon_j^{N_2})^2\left(\sin\frac{j\pi}{2N_2}\right)^2(-1)^{j+1} \\ \mathrm{e}^{-\mathrm{i}(N_2-1)x_0}\sum\limits_{j=1}^{N_2} \boldsymbol{B}_j^{-1}(\varepsilon_j^{N_2})^2\left(\sin\frac{j\pi}{2N_2}\right)^2(-1)^{j+1} & \sum\limits_{j=1}^{N_2} \boldsymbol{B}_j^{-1}(\varepsilon_j^{N_2})^2\left(\sin\frac{j\pi}{2N_2}\right)^2 \end{pmatrix}. \tag{3.4}$$

By (3.3), the matrices $\boldsymbol{B}_j^{-1} (j = 1,\dots,N_2)$ can be diagonalized by the same matrix $\boldsymbol{U}$ and

$$\boldsymbol{U}^*\boldsymbol{B}_j^{-1}\boldsymbol{U} = \operatorname{diag}(\mu_k + \lambda_j)_{k=1}^{N_1} \quad (j = 1,\dots,N_2),$$

where $\mu_k = \mu_{N_1,k}$ are the eigenvalues of $\boldsymbol{A}_{N_1}(q)$. Multiplying (3.4) from the left–hand side and the right–hand side by $\mathrm{diag}(\boldsymbol{U},\boldsymbol{U})$ and $\mathrm{diag}(\boldsymbol{U}^*,\boldsymbol{U}^*)$, respectively, we see that the eigenvalues of (3.4) coincide with the eigenvalues of the N_1 matrices

$$\frac{2}{N_2}\begin{pmatrix} a_k & b_k \\ \bar{b}_k & a_k \end{pmatrix} \quad (k = 1,\ldots,N_1)$$

where

$$a_k := \sum_{j=1}^{N_2} (\varepsilon_j^{N_2})^2 \frac{\left(\sin\frac{j\pi}{2N_2}\right)^2}{\mu_k + \lambda_j}\,, \; b_k := \mathrm{e}^{\mathrm{i}(N_2-1)x_0} \sum_{j=1}^{N_2} (\varepsilon_j^{N_2})^2 \frac{\left((-1)^{j+1}\sin\frac{j\pi}{2N_2}\right)^2}{\mu_k + \lambda_j}.$$

Consequently, the $2N_1$ remaining eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are given by

$$\lambda_k^e \;:=\; 1 - \frac{4}{N_2}\sum_{j=1}^{N_2/2} (\varepsilon_{2j}^{N_2})^2 \frac{\left(\sin\frac{2j\pi}{2N_2}\right)^2}{\mu_k + \lambda_{2j}} \quad (k = 1,\ldots,N_1)\,, \tag{3.5}$$

$$\lambda_k^o \;:=\; 1 - \frac{4}{N_2}\sum_{j=1}^{N_2/2} \frac{\left(\sin\frac{2j-1}{2N_2}\right)^2}{\mu_k + \lambda_{2j-1}} \quad (k = 1,\ldots,N_1)\,, \tag{3.6}$$

and by definition of λ_j by

$$\lambda_k^e \;=\; \frac{1}{2} + \frac{1}{N_2}\sum_{j=1}^{N_2/2} (\varepsilon_{2j}^{N_2})^2 \frac{\mu_k}{\mu_k + \left(2\sin\frac{j\pi}{N_2}\right)^2} \quad (k = 1,\ldots,N_1),$$

$$\lambda_k^o \;=\; \frac{1}{2} + \frac{1}{N_2}\sum_{j=1}^{N_2/2} \frac{\mu_k}{\mu_k + \left(2\sin\frac{(2j-1)\pi}{2N_2}\right)^2} \quad (k = 1,\ldots,N_1)\,.$$

Since by assumption $\mu_k \geq 0$, we conclude that $\lambda_k^{e/o} \geq \frac{1}{2}$. Moreover, the above summands are smaller than 1 so that $\lambda_k^{e/o} \leq 1$. This completes the proof. □

Note that for symmetric matrices $\boldsymbol{A}_{N_2}(p)$ the matrix $\boldsymbol{M}_{N_2}(p) := \boldsymbol{A}_{N_2}(p) + \boldsymbol{R}_{N_2}$ coincides with the so–called Strang–type preconditioner related to the sin–II transform (cf. [**17, 21**]). Using the Strang–type preconditioner related to the cos–II transform $\tilde{\boldsymbol{M}}_{N_2}(p) := \boldsymbol{A}_{N_2}(p) - \boldsymbol{R}_{N_2}$ and

(3.7)

$$\tilde{\boldsymbol{M}}_{N_1,N_2} := (\boldsymbol{C}_{N_2}^{II})' \otimes \boldsymbol{I}_{N_1})(\boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) + \boldsymbol{\Lambda}_{N_2}(p) \otimes \boldsymbol{I}_{N_1})(\boldsymbol{C}_{N_2}^{II} \otimes \boldsymbol{I}_{N_1})\,,$$

where

$$\boldsymbol{\Lambda}_{N_2}(p) := \mathrm{diag}(\lambda_j)_{j=0}^{N_2-1} \quad , \quad \lambda_j = \lambda_{N_2,j} := \left(2\sin\frac{j\pi}{2N_2}\right)^2 ,$$

we see by similar arguments as in Theorem 3.1 and 3.3 that for $N_1, N_2 \to \infty$, $2N_1$ eigenvalues of the preconditioned matrices behave as $\mu_{N_1,k}^{-1/2}$ $(k = 1,\ldots,N_1)$, where $\mu_{N_1,k}$ are the eigenvalues of $\boldsymbol{A}_{N_1}(q)$. See Table 1.

The sums in (3.5) and (3.6) can be computed by solving appropiate finite linear difference equations:

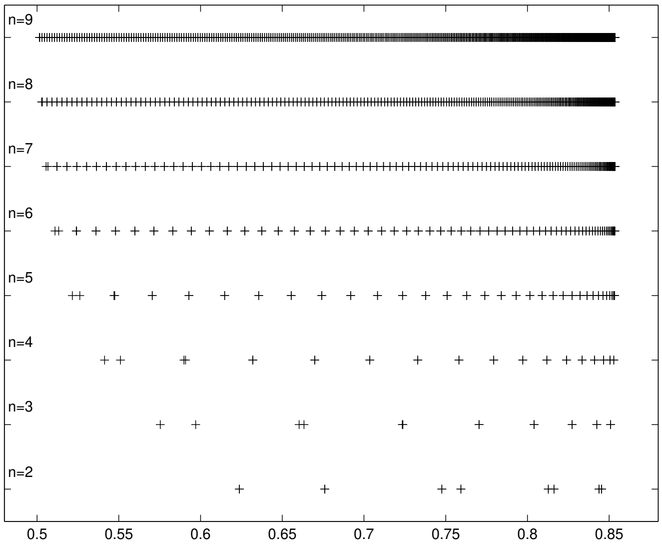

FIGURE 1. Eigenvalues $\lambda_k^{e/o}$ $(k = 1, \ldots, N)$ of $\boldsymbol{M}_{N,N}^{-1}\boldsymbol{A}_{N,N}$ with $N := 2^n$, $p(x) := (2\sin\frac{x}{2})^2$, $q(y) := (2\sin\frac{y}{2})^2$ and preconditioner (3.2).

LEMMA 3.2. *Let $N \in \mathbb{N}$ $(N \geq 4)$ be even and let $\lambda := \frac{1}{2}(\mu + 2 + \sqrt{\mu^2 + 4\mu})$, where $0 \leq \mu < \infty$. Then*

$$s^e := \frac{4}{N}\sum_{j=1}^{N/2}(\varepsilon_{2j}^N)^2 \frac{\left(\sin\frac{2j\pi}{2N}\right)^2}{\mu + \left(2\sin\frac{2j\pi}{2N}\right)^2} = \begin{cases} \frac{\lambda^{N-1}(1+\lambda)-\lambda^{1-N}(1+\lambda^{-1})+\lambda^{-1}-\lambda}{\lambda^{N-1}(1+\lambda)^2-\lambda^{1-N}(1+\lambda^{-1})^2} & \mu \neq 0, \\ \frac{1}{2} - \frac{1}{2N} & \mu = 0, \end{cases}$$

$$s^o := \frac{4}{N}\sum_{j=1}^{N/2} \frac{\left(\sin\frac{(2j-1)\pi}{2N}\right)^2}{\mu + \left(2\sin\frac{(2j-1)\pi}{2N}\right)^2} = \begin{cases} \frac{\lambda^{N-1}(1+\lambda)-\lambda^{1-N}(1+\lambda^{-1})-\lambda^{-1}+\lambda}{\lambda^{N-1}(1+\lambda)^2-\lambda^{1-N}(1+\lambda^{-1})^2} & \mu \neq 0, \\ \frac{1}{2} & \mu = 0. \end{cases}$$

By assumption on μ it is easy to check that the above sums have only values in $[0, \frac{1}{2}]$.

PROOF. By (2.2) we have that

$$\begin{aligned} \boldsymbol{T} &:= \operatorname{toep}_N(\mu + 2, -1, \ldots, 0) + \operatorname{hank}_N(1, 0, \ldots, 0) \\ &= (\boldsymbol{S}_N^{II})' \operatorname{diag}\left(\mu + \left(2\sin\frac{k\pi}{2N}\right)^2\right)_{k=1}^{N} \boldsymbol{S}_N^{II} \end{aligned}$$

and further with $\boldsymbol{x} := (\alpha, 0, \ldots, 0, \beta)' \in \mathbb{R}^N$ and $\boldsymbol{y} := (1, 0, \ldots, 0)' \in \mathbb{R}^N$, $\hat{\boldsymbol{y}} := \boldsymbol{S}_N^{II}\boldsymbol{y} = \sqrt{\frac{2}{N}}\left(\varepsilon_j^N \sin\frac{j\pi}{2N}\right)_{j=1}^N$ that

$$\boldsymbol{y}'\boldsymbol{T}^{-1}\boldsymbol{x} = \frac{2}{N}\sum_{j=1}^{N}(\varepsilon_j^N)^2 \frac{(\alpha + (-1)^{j+1}\beta)\left(\sin\frac{j\pi}{2N}\right)^2}{\mu + \left(2\sin\frac{j\pi}{2N}\right)^2}, \tag{3.8}$$

which equals s^o for $\alpha = \beta = 1$ and s^e for $\alpha = -\beta = 1$.
In the following, we restrict our attention to the computation of s^o. By (3.8) and definition of $\boldsymbol{y} \in \mathbb{R}^N$ this can be done by solving the linear system

$$\boldsymbol{T}\boldsymbol{v} = \boldsymbol{x} \tag{3.9}$$

and setting $s^o := v_0$. By the structure of $\boldsymbol{T}$, the system (3.9) can be rewritten as *difference equation*.

$$-v_{k-1} + (2+\mu)v_k - v_{k+1} = 0 \qquad (k = 1, \ldots, N-2) \tag{3.10}$$

with *boundary conditions*

$$\begin{aligned} (\mu+3)v_0 - v_1 &= 1\,, \\ -v_{N-2} + (\mu+3)v_{N-1} &= 1\,. \end{aligned}$$

For $\mu \neq 0$, the equation (3.10) has the solution

$$v_k = c_1\lambda^k + c_2\lambda^{-k} \qquad (k = 0, 1, \ldots, N-1)$$

where $\lambda := \frac{1}{2}(\mu + 2 + \sqrt{\mu^2 + 4\mu}$ and for $\mu = 0$ the solution

$$v_k = c_1 + c_2 k \qquad (k = 0, 1, \ldots, N-1).$$

The constants

$$\begin{aligned} c_1 &= c_1(\lambda) = \begin{cases} \frac{(1+\lambda^{-1})\lambda^{1-N} - (1+\lambda)}{(1+\lambda^{-1})^2\lambda^{1-N} - (1+\lambda)^2\lambda^{N-1}} & \mu \neq 0, \\ \frac{1}{2} & \mu = 0, \end{cases} \\ c_2 &= c_2(\lambda) = \begin{cases} c_1(\lambda^{-1}) & \mu \neq 0, \\ 0 & \mu = 0, \end{cases} \end{aligned}$$

can be obtained from the boundary conditions by sraightforward computation. Now the assertion follows since $v_0 = c_1 + c_2$ if $\mu \neq 0$ and $v_0 = c_1$ if $\mu = 0$. □

Note the sums in the following proofs of this paper can be computed in a similar way.

Moreover, the result in [**27**, Lemma 6.1] concerning

$$\frac{1}{N}\sum_{j=0}^{N-1} \frac{e^{2\pi ijk}}{2\varphi + e^{2\pi ij} + \eta e^{-2\pi ij}}$$

can be obtained by solving appropriate finite difference equations, too.

3.3. ω–circulant preconditioners. At first glance it seems that circulant or more general ω–circulant preconditioners are better suited for the preconditioning of Hermitian complex–valued matrices than sin-II (cos-II) preconditioners. In this section, we will see that this is not the case.

As preconditioner we suggest

$$\begin{aligned} \boldsymbol{M}_{N_1,N_2} &:= \boldsymbol{A}_{N_1,N_2} + \boldsymbol{H}_{N_2}(\mathrm{e}^{-\mathrm{i}y_0}\mathrm{e}^{\mathrm{i}N_2 w_{N_2}}) \otimes \boldsymbol{I}_{N_1} \\ &= \boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) + \boldsymbol{M}_{N_2}(p) \otimes \boldsymbol{I}_{N_1}\,, \end{aligned} \tag{3.11}$$

where $\boldsymbol{M}_{N_2}(p) := \boldsymbol{A}_{N_2}(p) + \boldsymbol{H}_{N_2}(-\mathrm{e}^{-\mathrm{i}y_0}\mathrm{e}^{\mathrm{i}N_2 w_{N_2}})$. By (2.4), the matrix $\boldsymbol{M}_{N_1,N_2}$ can be written as

$$\begin{aligned} \boldsymbol{M}_{N_1,N_2} &= (\boldsymbol{W}_{N_2}\boldsymbol{F}_{N_2} \otimes \boldsymbol{I}_{N_1})(\boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) + \boldsymbol{\Lambda}_{N_2}(p) \otimes \boldsymbol{I}_{N_1})(\bar{\boldsymbol{F}}_{N_2}\bar{\boldsymbol{W}}_{N_2} \otimes \boldsymbol{I}_{N_1}), \\ &= (\boldsymbol{W}_{N_2}\boldsymbol{F}_{N_2} \otimes \boldsymbol{I}_{N_1})\boldsymbol{B}(\bar{\boldsymbol{F}}_{N_2}\bar{\boldsymbol{W}}_{N_2} \otimes \boldsymbol{I}_{N_1}), \end{aligned}$$

where

$$\boldsymbol{\Lambda}_{N_2}(p) := \operatorname{diag}(\lambda_j)_{j=0}^{N_2-1} \quad , \quad \lambda_j = \lambda_{N_2,j} := \left(2\sin\left(\frac{2\pi j}{2N_2} - \frac{y_0 - w_{N_2}}{2}\right)\right)^2$$

and

$$\boldsymbol{B} := \operatorname{diag}(\boldsymbol{B}_j)_{j=0}^{N_2-1} \quad , \quad \boldsymbol{B}_j := \boldsymbol{A}_{N_1}(q) + \lambda_j \boldsymbol{I}_{N_1}.$$

Since the values w_{N_2} were only included to keep the eigenvalues λ_j ($j = 0, \ldots, N_2 - 1$) positive, we can restrict our attention to the following cases

$$w_{N_2} := \begin{cases} y_0 + \frac{\pi}{N_2} & \text{if} \quad y_0 = \frac{2\pi}{2^n}\mathbb{Z} \quad \text{for some} \quad n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\boldsymbol{M}_{N_1,N_2}(p)$ is skew–circulant in the first case and circulant in the second case. By the following theorem, we will see that the eigenvalues of the preconditioned matrices $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are in general not bounded from above if $N_1, N_2 \to \infty$.

THEOREM 3.3. *Let $\boldsymbol{A}_{N_1,N_2}$ and $\boldsymbol{M}_{N_1,N_2}$ be given by* (1.2), (3.1) *and* (3.11), *respectively. Assume that $\boldsymbol{A}_{N_2}(q)$ has nonnegative eigenvalues $\mu_{N_1,k}$ ($k = 1, \ldots, N_1$). Then $N_1(N_2-2)$ eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are equal to* 1. *The other $2N_1$ eigenvalues are bounded from below by $\frac{1}{2}$; N_1 eigenvalues are bounded from above by a constant independent of N_1, N_2 and the other N_1 eigenvalues behave for $N_1, N_2 \to \infty$ as $\mu_{N_1,k}^{-1/2}$ ($k = 1, \ldots, N_1$).*

PROOF. The proof follows the same lines as the proof of Theorem 3.1. Since

$$\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2} = \boldsymbol{I} + \boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{H}_{N_2}(\mathrm{e}^{-\mathrm{i}y_0}\mathrm{e}^{\mathrm{i}N_2 w_{N_2}}) \otimes \boldsymbol{I}_{N_1}),$$

it remains to consider the nonzero eigenvalues of the second summand. By (2.5), these eigenvalues coincide with the eigenvalues of

$$\begin{aligned} & (\boldsymbol{R}_{N_2} \otimes \boldsymbol{I}_{N_1})\boldsymbol{M}_{N_1,N_2}^{-1}(\boldsymbol{H}_{N_2} \otimes \boldsymbol{I}_{N_1}) \\ = \; & (\boldsymbol{R}_{N_2} \otimes \boldsymbol{I}_{N_1})(\boldsymbol{W}_{N_2}\boldsymbol{F}_{N_2} \otimes \boldsymbol{I}_{N_1})\boldsymbol{B}^{-1}(\bar{\boldsymbol{F}}_{N_2}\bar{\boldsymbol{W}}_{N_2} \otimes \boldsymbol{I}_{N_1})(\boldsymbol{H}_{N_2} \otimes \boldsymbol{I}_{N_1}) \, . \end{aligned}$$

Straightforward computation shows that the nonzero eigenvalues of this matrix are given by the eigenvalues of the $2N_1 \times 2N_1$ matrix

$$\frac{1}{N}\begin{pmatrix} \mathrm{e}^{\mathrm{i}y_0}\mathrm{e}^{-\mathrm{i}w_{N_2}} \sum\limits_{j=0}^{N_2-1} \boldsymbol{B}_j^{-1}\mathrm{e}^{-2\pi ij/N_2} & \mathrm{e}^{-\mathrm{i}y_0}\mathrm{e}^{\mathrm{i}N_2 w_{N_2}} \sum\limits_{j=0}^{N_2-1} \boldsymbol{B}_j^{-1} \\ \mathrm{e}^{\mathrm{i}y_0}\mathrm{e}^{-\mathrm{i}N_2 w_{N_2}} \sum\limits_{j=0}^{N_2-1} \boldsymbol{B}_j^{-1} & \mathrm{e}^{-\mathrm{i}y_0}\mathrm{e}^{\mathrm{i}w_N} \sum\limits_{j=0}^{N_2-1} \boldsymbol{B}_j^{-1}\mathrm{e}^{2\pi ij/N_2} \end{pmatrix}$$

or equivalently with the eigenvalues of the N_1 matrices

$$\frac{1}{N_2}\begin{pmatrix} a_k & b_k \\ \bar{b}_k & \bar{a}_k \end{pmatrix} \quad (k = 1, \ldots, N_1),$$

where

$$a_k := \mathrm{e}^{\mathrm{i}(y_0 - w_{N_2})} \sum_{j=0}^{N_2-1} \frac{\mathrm{e}^{-2\pi ij/N_2}}{\mu_k + \lambda_j}, \quad b_k := \mathrm{e}^{-\mathrm{i}(N_2 w_{N_2} - y_0)} \sum_{j=0}^{N_2-1} \frac{1}{\mu_k + \lambda_j}$$

and $\mu_k = \mu_{N_1,k}$ are the eigenvalues of $A_{N_1}(q)$. Consequently, the remaining $2N_1$ eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are given by

$$\lambda_k^{\pm} = 1 + \frac{1}{N_2}\left(\mathrm{Re}(a_k) \pm \sqrt{|b_k|^2 - \Big(\mathrm{Im}(a_k)\Big)^2}\right) \quad (k = 1, \dots, N_1),$$

$$\begin{aligned}\mathrm{Re}(a_k) &= \sum_{j=0}^{N_2-1} \frac{\cos\Big(\frac{2\pi j}{N_2} - (y_0 - w_{N_2})\Big)}{\mu_k + \Big(2\sin(\frac{2\pi j}{2N_2} - \frac{y_0 - w_{N_2}}{2})\Big)^2},\\ \mathrm{Im}(a_k) &= \sum_{j=0}^{N_2-1} \frac{\sin\Big(\frac{2\pi j}{N_2} - (y_0 - w_{N_2})\Big)}{\mu_k + \Big(2\sin(\frac{2\pi j}{2N_2} - \frac{y_0 - w_{N_2}}{2})\Big)^2}.\end{aligned}$$

Clearly, $|b_k|^2 - (\mathrm{Im}(a_k))^2 \geq 0$, so that

$$\lambda_k^{\pm} \geq 1 + \frac{1}{N_2}(\mathrm{Re}(a_k) - |b_k|). \tag{3.12}$$

We are interested in $\lambda_k^{\pm}$ as $N_2 \to \infty$. If $w_{N_2} = y_0 + \frac{\pi}{N_2}$, then it is easy to check by the symmetry of the sine function that $\mathrm{Im}(a_k) = 0$ $(k = 1, \dots, N_1)$. If $w_{N_2} = 0$, then $(1/N_2)\,\mathrm{Im}(a_k)$ can be considered as Riemann sum of the integral $\int\limits_0^{2\pi} \frac{\sin x}{\mu_k + (2\sin\frac{x}{2})^2}\,\mathrm{d}x$, which becomes zero. Thus, the eigenvalues $\lambda_k^{\pm}$ behave for $N_2 \to \infty$ as

$$\begin{aligned}\lambda_k^{\pm} &\sim 1 + \frac{1}{N_2}(\mathrm{Re}(a_k) \pm |b_k|)\\ &= 1 + \frac{1}{N_2}\sum_{j=0}^{N_2-1} \frac{\cos\Big(\frac{2\pi j}{N_2} - (y_0 - w_{N_2})\Big) \pm 1}{\mu_k + (2s_j)^2},\end{aligned}$$

where $s_j := \sin\Big(\frac{2\pi j}{2N_2} - \frac{y_0 - w_{N_2}}{2}\Big)$. This can be written as

$$\lambda_k^{-} \sim \frac{1}{2} + \frac{1}{2N_2}\sum_{j=0}^{N_2-1} \frac{\mu_k}{\mu_k + 4s_j^2} \leq 1,$$

$$\lambda_k^{+} \sim \frac{1}{2} + \frac{1}{2N_2}\sum_{j=0}^{N_2-1} \frac{2 + \mu_k}{\mu_k + 4s_j^2}. \tag{3.13}$$

Thus, we see by (3.12) that all eigenvalues $\lambda_k^{\pm}$ are restricted from below by $1/2$. Further, the eigenvalues λ_k^{-} are bounded from above by a constant C independent of N_1, N_2 ($C = 1$ if $w_{N_2} = y_0 + \pi/N_2$). The sum in (3.13) is a Riemann sum of the integral

$$I_k = \int_0^{2\pi} \frac{2 + \mu_k}{\mu_k + (2\sin\frac{x}{2})^2}\,\mathrm{d}x = 2\int_0^{\pi} \frac{2 + \mu_k}{\mu_k + (2\sin\frac{x}{2})^2}\,\mathrm{d}x$$

and since $\frac{2}{\pi}t \leq \sin t \leq t$ $(t \in [0, \pi/2])$ and by substituting $y := x/\mu_k^{1/2}$, we conclude that

$$\mu_k^{-1/2} 2\int_0^{\pi\mu_k^{-1/2}} \frac{2 + \mu_k}{1 + y^2}\,\mathrm{d}y \leq I_k \leq \mu_k^{-1/2}\frac{\pi^2}{2}\int_0^{\pi\mu_k^{-1/2}} \frac{2 + \mu_k}{\pi^2/4 + y^2}\,\mathrm{d}y. \tag{3.14}$$

The integrals on the left–hand side and the right–hand side are bounded from above and below by constants independent of the behavior of $\mu_{N_1,k}$ as $N_1 \to \infty$, i.e. the integrals remain bounded even if $\mu_{N_1,k} \to 0$. Thus $I_k \sim \mu_{N_1,k}^{-1/2}$ $(k = 1, \dots, N_1)$ and we are done. □

The computations presented in Figure 2 show an excellent agreement with Theorem 3.3.

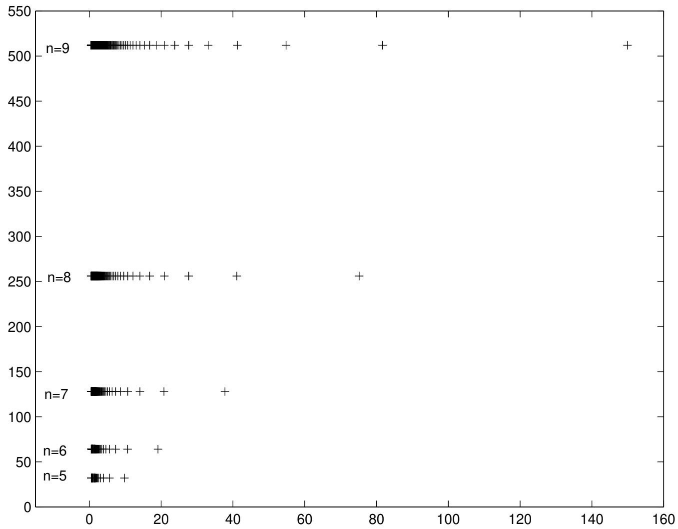

FIGURE 2. Eigenvalues $\lambda_k^{\pm}$ $(k = 1, \dots, N)$ of $\boldsymbol{M}_{N,N}^{-1}\boldsymbol{A}_{N,N}$ with $N := 2^n$, $p(x) := (2\sin\frac{x}{2})^2$, $q(y) := (2\sin\frac{y}{2})^2$ and preconditioner (3.11), where $w_N := \pi/N$.

One aim for considering the eigenvalue distribution of preconditioned BTTB matrices was the wish to explain the convergence behavior of the PCG–method. Unfortunately, the assumptions of Theorem 1.1 are only fulfilled for the sin-II (cos-II) preconditioners (3.2). In the following, we present some numerical computations also for other preconditioners.

We consider BTTB systems (1.1) of size $N_1 = N_2 = N := 2^n$ with right–hand side $\boldsymbol{b}$ consisting of N^2 ones. The PCG–method starts with the zero vector and stops if $\|\boldsymbol{r}^{(j)}\|_2/\|\boldsymbol{r}^{(0)}\|_2 < 10^{-7}$, where $\boldsymbol{r}^{(j)}$ denotes the residual vector after j iterations. Table 1 compares the number of iterations of the PCG–method for coefficient matrices (1.2), (3.1) and the following level–1 preconditioners:

$$\begin{aligned}
\boldsymbol{M}_{N,N} &= \boldsymbol{M}_{N,N}(\boldsymbol{S}_N^{II}) && \text{given by (3.2)},\\
\boldsymbol{M}_{N,N} &= \boldsymbol{M}_{N,N}(\boldsymbol{C}_N^{II}) && \text{given by (3.7)},\\
\boldsymbol{M}_{N,N} &= \boldsymbol{M}_{N,N}(F_N) && \text{given by (3.11) with } w_N := 0,\\
\boldsymbol{M}_{N,N} &= \boldsymbol{M}_{N,N}(W_N F_N) && \text{given by (3.11) with } w_N := \frac{\pi}{N}.
\end{aligned}$$

Our test matrices (1.2) correspond to the polynomials $p(x) := (2\sin\frac{x}{2})^2$, $q_1(y) := (2\sin\frac{y}{2})^2$ and $p(x) := (2\sin\frac{x}{2})^2$, $q_2(y) := (2\sin\frac{y}{2})^4$.

The first row of each table contains the exponent n of the transform length The polynomials q_i $(i = 1, 2)$ are listed in the first column and the applied preconditioner in the second column of the table.

	n	2	3	4	5	6	7
q_1	$\boldsymbol{M}_{N,N}(\boldsymbol{S}_N^{II})$	3	5	7	8	8	8
q_1	$\boldsymbol{M}_{N,N}(\boldsymbol{C}_N^{II})$	3	5	8	10	14	18
q_1	$\boldsymbol{M}_{N,N}(\boldsymbol{F}_N)$	3	5	8	10	13	17
q_1	$\boldsymbol{M}_{N,N}(\boldsymbol{W}_N \boldsymbol{F}_N)$	3	5	7	9	10	12
q_2	$\boldsymbol{M}_{N,N}(\boldsymbol{S}_N^{II})$	3	5	7	9	9	9
q_2	$\boldsymbol{M}_{N,N}(\boldsymbol{C}_N^{II})$	3	5	9	13	24	46
q_2	$\boldsymbol{M}_{N,N}(\boldsymbol{F}_N)$	3	5	9	12	22	38
q_2	$\boldsymbol{M}_{N,N}(\boldsymbol{W}_N \boldsymbol{F}_N)$	3	5	7	12	16	20

TABLE 1. Number of iterations of the PCG–method for different preconditioners.

4. Hermitian pentdiagonal Toeplitz matrices

Now we assume that

$$p(x) := \left(2\sin\frac{x-x_0}{2}\right)^2 \left(2\sin\frac{x-x_1}{2}\right)^2 \quad (x_0, x_1 \in [0, 2\pi]),$$

i.e. p has zeros of order 2 at x_0 and x_1, respectively, if $x_0 \neq x_1$ and a zero of order 4 at x_0 if $x_0 = x_1$. Then $A_{N_2}(p)$ is the Hermitian pentdiagonal Toeplitz matrix

$$\boldsymbol{A}_{N_2}(p) = \operatorname{toep}_{N_2}\Big(4 + 2\cos(x_0 - x_1), -2\mathrm{e}^{\mathrm{i}x_0} - 2\mathrm{e}^{\mathrm{i}x_1}, \mathrm{e}^{\mathrm{i}(x_0+x_1)}, 0, \ldots, 0\Big). \tag{4.1}$$

Due to the results in Section 3, we restrict our attention to the PCG–method with sin-I based preconditioners. Let $N_2 + 1$ be a power of 2. We examine the level–1 preconditioner

$$\boldsymbol{M}_{N_1,N_2} := \boldsymbol{A}_{N_1,N_2} - \boldsymbol{R}_{N_2} \otimes \boldsymbol{I}_{N_1} \tag{4.2}$$

which, by (2.2), can be rewritten as

$$\boldsymbol{M}_{N_1,N_2} = (\boldsymbol{D}\boldsymbol{S}_{N_2}^{I} \otimes \boldsymbol{I}_{N_1})(\boldsymbol{I}_{N_2} \otimes \boldsymbol{A}_{N_1}(q) + \boldsymbol{\Lambda}_{N_2}(p) \otimes \boldsymbol{I}_{N_1})(\boldsymbol{S}_{N_2}^{I}\boldsymbol{D}^* \otimes \boldsymbol{I}_{N_1}),$$

where $\boldsymbol{D} := \operatorname{diag}\Big(\mathrm{e}^{-\mathrm{i}k\frac{x_0+x_1}{2}}\Big)_{k=0}^{N_2-1}$ and

$$\boldsymbol{\Lambda}_{N_2}(p) = \operatorname{diag}(\lambda_j)_{j=1}^{N_2}, \ \lambda_j = \lambda_{N_2,j} = \Big((2\sin\frac{j\pi}{2(N_2+1)})^2 - (2\sin\frac{x_0-x_1}{4})^2\Big)^2.$$

Consequently, a linear system with coefficient matrix $\boldsymbol{M}_{N_1,N_2}$ can be solved with $\mathcal{O}(N_1 N_2 \log N_2)$ arithmetic operations. Unfortunately, we will see in Theorem 4.1 that for every eigenvalue $\mu_k = \mu_{N_1,k}$ of $\boldsymbol{A}_{N_1}(q)$ there exist two eigenvalues $\lambda_k^{e/o}$

of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ with $\lambda_k^{e/o} \sim \mu_k^{-1/2}$ if $x_0 \neq x_1$ and $\lambda_k^{e/o} \sim \mu_k^{-1/4}$ if $x_0 = x_1$. In particular, the eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ will not fulfill the assumptions of Theorem 1.1 if the polynomial $q \geq 0$ has zeros.

THEOREM 4.1. *Let $\boldsymbol{A}_{N_1,N_2}$ and $\boldsymbol{M}_{N_1,N_2}$ be given by* (1.2), (4.1) *and* (4.2), *respectively. Assume that $\boldsymbol{A}_{N_1}(q)$ has nonnegative eigenvalues $\mu_{N_1,k}$ $(k = 1, \ldots, N_1)$. Then $N_1(N_2-2)$ eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are equal to 1. The other $2N_1$ eigenvalues are restricted from below by 1 and behave for $N_1, N_2 \to \infty$ as $\mu_{N_1,k}^{-1/4}$ $(k = 1, \ldots, N_1)$ if $x_0 = x_1$ and as $\mu_{N_1,k}^{-1/2}$ $(k = 1, \ldots, N_1)$ if $x_0 \neq x_1$.*

PROOF. Using the same ideas as in the proof of Theorem 3.1, we obtain that the interesting $2N_1$ eigenvalues of $\boldsymbol{M}_{N_1,N_2}^{-1}\boldsymbol{A}_{N_1,N_2}$ are given by

$$\begin{aligned}
\lambda_k^e &= 1 + \frac{4}{N_2+1} \sum_{j=1}^{\frac{N_2-1}{2}} \frac{(\sin\frac{2j\pi}{N_2+1})^2}{\mu_k + \left((2\sin\frac{2j\pi}{2(N_2+1)})^2 - 4s^2\right)^2} \quad (k = 1, \ldots, N_1), \\
\lambda_k^o &= 1 + \frac{4}{N_2+1} \sum_{j=1}^{\frac{N_2+1}{2}} \frac{(\sin\frac{(2j-1)\pi}{N_2+1})^2}{\mu_k + \left((2\sin\frac{(2j-1)\pi}{2(N_2+1)})^2 - 4s^2\right)^2} \quad (k = 1, \ldots, N_1),
\end{aligned}$$

where $s := \sin\frac{x_0 - x_1}{4}$. This can be rewritten as

$$\lambda_k^e = \frac{1}{2} + \frac{1}{N_2+1}\left(1 + \sum_{j=1}^{\frac{N_2-1}{2}} \frac{(1-2s^2)(s_{2j}^2 - s^2) + s^2(1-s^2) + \theta_k}{(s_{2j}^2 - s^2)^2 + \theta_k}\right), \tag{4.3}$$

$$\lambda_k^o = \frac{1}{2} + \frac{1}{N_2+1} \sum_{j=1}^{\frac{N_2+1}{2}} \frac{(1-2s^2)(s_{2j-1}^2 - s^2) + s^2(1-s^2) + \theta_k}{(s_{2j-1}^2 - s^2)^2 + \theta_k}, \tag{4.4}$$

where $s_j := \sin\frac{j\pi}{2(N_2+1)}$ and $\theta_k := \mu_k/16$. Since $\theta_k \geq 0$ and $0 < s_j^2 < 1$, we have

$$(1-2s^2)(s_{2j}^2 - s^2) + s^2(1-s^2) + \theta_k = (1-s^2)^2 s_{2j}^2 + s^4(1-s_{2j}^2) + \theta_k > 0$$

and

$$\frac{(1-2s^2)(s_{2j}^2 - s^2) + s^2(1-s^2) + \theta_k}{(s_{2j}^2 - s^2)^2 + \theta_k} = \frac{s_{2j}^2 - 2s^2 s_{2j}^2 + s^4 + \theta_k}{s_{2j}^4 - 2s^2 s_{2j}^2 + s^4 + \theta_k} > 1.$$

Thus $\lambda_k^{e/o} > 1$.

Let $x_0 = x_1$. Then

$$\begin{aligned}
\lambda_k^e &= \frac{1}{2} + \frac{1}{N_2+1}\left(1 + \sum_{j=1}^{\frac{N_2-1}{2}} \frac{s_{2j}^2 + \theta_k}{s_{2j}^4 + \theta_k}\right), \\
\lambda_k^0 &= \frac{1}{2} + \frac{1}{N_2+1}\left(1 + \sum_{j=1}^{\frac{N_2+1}{2}} \frac{s_{2j-1}^2 + \theta_k}{s_{2j-1}^4 + \theta_k}\right).
\end{aligned}$$

The above sums are Riemann sums of the integral $I_k := \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \theta_k}{\sin^4 x + \theta_k} \, \mathrm{d}x$, which can be estimated by

$$\int_0^{\frac{\pi}{2}} \frac{(\frac{2}{\pi}x)^2 + \theta_k}{x^4 + \theta_k} \, \mathrm{d}x \le I_k \le \int_0^{\frac{\pi}{2}} \frac{x^2 + \theta_k}{(\frac{2}{\pi}x)^4 + \theta_k} \, \mathrm{d}x \tag{4.5}$$

and further by substituting $y := x/\sqrt[4]{\theta_k}$ by

$$\theta_k^{-1/4} \int_0^{\frac{\pi}{2}\theta_k^{-1/4}} \frac{(\frac{2}{\pi}y)^2 + \sqrt{\theta_k}}{y^4 + 1} \, \mathrm{d}y \quad \le I_k \le \theta_k^{-1/4} \int_0^{\frac{\pi}{2}\theta_k^{-1/4}} \frac{y^2 + \sqrt{\theta_k}}{(\frac{2}{\pi}y)^4 + 1} \, \mathrm{d}y \, .$$

The integrals on the left–hand side and the right–hand side are bounded from above and below by constants independent of N_1, i.e. independent of the behavior of θ_k as $N_1 \to \infty$. Thus $I_k \sim \mu_{N_1,k}^{-1/4}$ $(k = 1, \ldots, N_1)$.

Let $x_0 > x_1$ and $\alpha := \frac{x_0 - x_1}{4} \le \frac{\pi}{4}$. Then the sums in (4.3) and (4.4) are Riemann sums of the integral

$$\begin{aligned} I_k &:= \int_0^{\frac{\pi}{2}} \frac{(1-2s^2)(\sin^2 x - s^2) + s^2(1-s^2) + \theta_k}{(\sin^2 x - s^2)^2 + \theta_k} \, \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{(1-2s^2)\sin(x-\alpha)\sin(x+\alpha) + s^2(1-s^2) + \theta_k}{\sin^2(x-\alpha)\sin^2(x+\alpha) + \theta_k} \, \mathrm{d}x \, , \end{aligned}$$

which can be estimated by

$$\begin{aligned} & \int_0^{\frac{\pi}{2}-\alpha} \frac{(1-2s^2)\sin x \sin\alpha + s^2(1-s^2) + \theta_k}{\sin^2 x + \theta_k} \, \mathrm{d}x \le I_k \\ \le \; & 2\int_0^{\frac{\pi}{2}-\alpha} \frac{(1-2s^2)\sin x + s^2(1-s^2) + \theta_k}{\sin^2 x \sin^2 \alpha + \theta_k} \, \mathrm{d}x, \\ & \int_0^{\frac{\pi}{2}-\alpha} \frac{(1-2s^2)\frac{2}{\pi}x \sin\alpha + s^2(1-s^2) + \theta_k}{x^2 + \theta_k} \, \mathrm{d}x \le I_k \\ \le \; & 2\int_0^{\frac{\pi}{2}-\alpha} \frac{(1-2s^2)x + s^2(1-s^2) + \theta_k}{(\frac{2}{\pi}x)^2 \sin^2 \alpha + \theta_k} \, \mathrm{d}x, \\ & \theta_k^{-1/2} \int_0^{(\frac{\pi}{2}-\alpha)\theta_k^{-1/2}} \frac{(1-2s^2)(\frac{2}{\pi}y\,\theta_k^{1/2})\sin\alpha + s^2(1-s^2) + \theta_k}{y^2 + 1} \, \mathrm{d}y \le I_k \\ \le \; & 2\theta_k^{-1/2} \int_0^{(\frac{\pi}{2}-\alpha)\theta_k^{-1/2}} \frac{(1-2s^2)y\,\theta_k^{1/2} + s^2(1-s^2) + \theta_k}{(\frac{2}{\pi}y)^2 \sin^2\alpha + \theta_k} \, \mathrm{d}y. \end{aligned}$$

Since $s^2(1 - s^2) > 0$, the integrals on the right–hand side and left–hand side are bounded from above and below by constants independent of N_1, N_2. Consequently, $I_k \sim \mu_{N_1,k}^{-1/2}$ $(k = 1, \ldots, N_1)$.

The result in case $\frac{\pi}{4} < \alpha \le \frac{\pi}{2}$ follows in a similar way. This completes the proof. $\square$

The eigenvalue computations in Figure 3 and 4 confirm our theoretical expectations.

Table 2 compares the number of iteration steps of the PCG–method for different preconditioners. We use the same notation as in Section 3 with $N = 2^n - 1$ and the trigonometric polynomials $p(x) := (2\sin\frac{x}{2})^4$ and $q(y) := (2\sin\frac{y}{2})^4$.

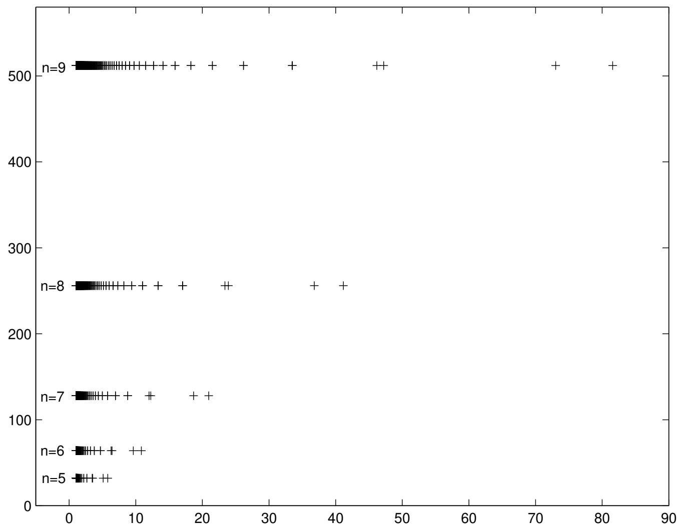

FIGURE 3. Eigenvalues $\lambda_k^{e/o}$ $(k = 1, \dots, N)$ of $\boldsymbol{M}_{N,N}^{-1}\boldsymbol{A}_{N,N}$ with $N := 2^n$, $p(x) := (2\sin\frac{x}{2})^4$, $q(y) := (2\sin\frac{y}{2})^4$ and preconditioner (4.2).

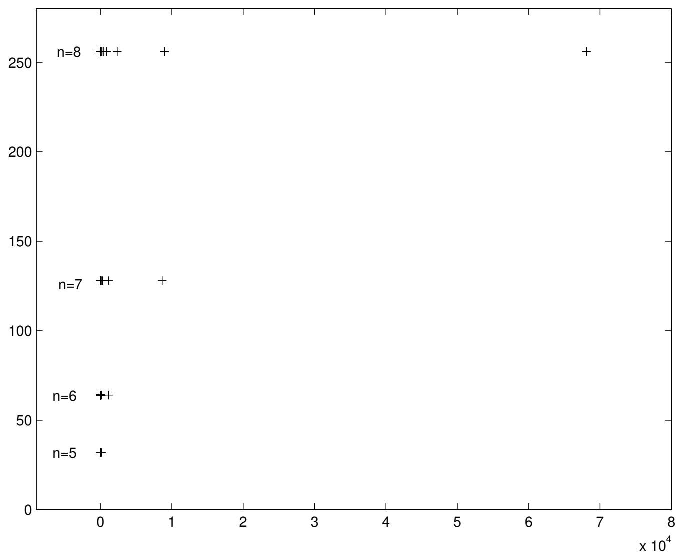

FIGURE 4. Eigenvalues $\lambda_k^{e/o}$ $(k = 1, \dots, N)$ of $\boldsymbol{M}_{N,N}^{-1}\boldsymbol{A}_{N,N}$ with $N := 2^n$, $p(x) := (2\sin\frac{x}{2})^2\,(2\sin\frac{2x-\pi}{4})^2$, $q(y) := (2\sin\frac{y}{2})^4$ and preconditioner (4.2).

In case $x_1 = x_0 - \pi$, we have that

$$\boldsymbol{A}_{N_2}(p) = \operatorname{toep}_{N_2}(2, 0, -\mathrm{e}^{2\mathrm{i}x_0}, 0, \dots, 0) \tag{4.6}$$

n	2	3	4	5	6	7
$\boldsymbol{M}_{N,N}(\boldsymbol{S}_N^I)$	3	5	7	9	12	17
$\boldsymbol{M}_{N+1,N+1}(\boldsymbol{S}_{N+1}^{II})$	4	8	10	12	16	20
$\boldsymbol{M}_{N+1,N+1}(\boldsymbol{C}_{N+1}^{II})$	4	9	16	32	65	133
$\boldsymbol{M}_{N+1,N+1}(\boldsymbol{F}_{N+1})$	3	8	14	25	51	104
$\boldsymbol{M}_{N+1,N+1}(\boldsymbol{W}_{N+1}\,\boldsymbol{F}_{N+1})$	4	9	13	22	40	89

TABLE 2. Number of iterations of the PCG–method for different preconditioners.

and even–odd ordering of the row and column blocks of A_{N_1,N_2} with corresponding permutation matrix $\boldsymbol{P}$ results in

$$\boldsymbol{P}'\,\boldsymbol{A}_{N_1,N_2}\,\boldsymbol{P} = \begin{pmatrix} \tilde{\boldsymbol{A}}_{(N_2+1)/2} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{A}}_{(N_2-1)/2} \end{pmatrix},$$

where

$$\begin{aligned} \tilde{\boldsymbol{A}}_N &:= \boldsymbol{I}_N \otimes \boldsymbol{A}_{N_1}(q) + \boldsymbol{A}_N(\tilde{p}) \otimes \boldsymbol{I}_{N_1}, \\ \tilde{p}(x) &:= \left(2\,\sin\frac{x-2x_0}{2}\right)^2 . \end{aligned}$$

The matrices $\tilde{\boldsymbol{A}}_N$ can be treated as in Section 3.1 if $N = (N_2-1)/2$ and as in Section 3.2 if $N = (N_2+1)/2$.

$\boldsymbol{A}_{N,N}$	$s=1$	$s=2$	$s=3$	$s=4$
2	1.3333e+01	5.2500e+01	1.3300e+02	2.8551e+02
3	4.6295e+01	4.9629e+02	2.8531e+03	1.1504e+04
4	1.6887e+02	5.4739e+03	8.2794e+04	7.9241e+05
5	6.4036e+02	7.0763e+04	3.3341e+06	9.2003e+07
6	2.4886e+03	1.0092e+06	1.6556e+08	1.5156e+10
7	9.8063e+03	1.5208e+07	9.2727e+09	3.0701e+12

TABLE 3. Condition number of $\boldsymbol{A}_{N,N}$ defined by (1.2) with $p(x) := (2\sin\frac{x}{2})^{2s}$, $q(y) := (2\sin\frac{y}{2})^{2s}$.

Finally, Table 3 shows the condition numbers of $\boldsymbol{A}_{N,N}$ ($N := 2^n$) for the polynomials $p = (2\sin\frac{x}{2})^{2s}$, $q = (2\sin\frac{y}{2})^{2s}$ ($s = 1,2,3,4$). Here we have used the matlab command *condest*. The underlying algorithm uses Higham's modification of Hager's method to compute a lower bound for the condition number of $\boldsymbol{A}_{N,N}$ with respect to the 1–norm. Note that the condition number of $A_N(p)$ grows as N^{2s} (cf. [**3**] and the references therein).

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