# Trigonometric Preconditioners for Block Toeplitz Systems

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**Abstract.** This paper is concerned with the solution of a system of linear equations $\mathbf{T}_{M,N}\mathbf{x} = \mathbf{b}$, where $\mathbf{T}_{M,N}$ denotes a positive definite doubly symmetric block-Toeplitz matrix with Toeplitz blocks arising from a generating function $f$ of the Wiener class. We derive optimal and Strang–type trigonometric preconditioners $\mathbf{P}_{M,N}$ of $\mathbf{T}_{M,N}$ from the Fejér and Fourier sum of $f$, respectively. Using relations between trigonometric transforms and Toeplitz matrices, we prove that for all $\varepsilon > 0$ and sufficiently large $M, N$, at most $\mathcal{O}(M) + \mathcal{O}(N)$ eigenvalues of $\mathbf{P}_{M,N}^{-1}\mathbf{T}_{M,N}$ lie outside the interval $(1-\varepsilon, 1+\varepsilon)$ such that the preconditioned conjugate gradient method converges in at most $\mathcal{O}(M) + \mathcal{O}(N)$ steps.

## §1. Introduction

Systems of linear equations

$$\mathbf{T}_{M,N}\mathbf{x} = \mathbf{b}$$

where $\mathbf{T}_{M,N}$ denotes a positive definite doubly symmetric block-Toeplitz matrix with Toeplitz blocks (BTTB matrices) arise in a variety of applications in mathematics and engineering (see [8] and the references therein). Along with stabilization techniques for direct fast and superfast Toeplitz solvers, preconditioned conjugate gradient methods (PCG–methods) have attained much attention during the last years. Two types of so-called "level–2" preconditioners are mainly exploited for linear BTTB systems, namely optimal (Cesáro) doubly circulant preconditioners [4] and more simple so-called "Strang" doubly circulant preconditioners [16]. One reason for the choice of doubly circulant preconditioners is the fact that doubly circulant matrices can be diagonalized by tensor products of Fourier matrices such that the multiplication with doubly circulant matrices requires only $\mathcal{O}(MN\log(MN))$ arithmetical operations. In this paper, we restrict our attention to *real* BTTB matrices. Here, it seems to be natural, to replace the doubly circulant matrices by

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matrices which are diagonalizable by tensor products of some orthogonal matrices. We construct "level-2" trigonometric preconditioners of BTTB matrices with respect to various trigonometric transforms and show that for doubly symmetric BTTB matrices arising from a generating function $f$ of the Wiener class the corresponding PCG-method converges in at most $\mathcal{O}(M) + \mathcal{O}(N)$ steps.
The special case of preconditioning of doubly symmetric BTTB matrices with respect to a discrete sine transform was studied in [11]. However, in many examples discrete cosine transform based preconditioning leads to better convergence results.

## §2. Trigonometric Transforms and Toeplitz Matrices

We introduce four discrete cosine transforms (DCT) and four discrete sine transforms (DST) as classified in [17]:

$$\mathrm{DCT-I}: \quad \mathbf{C}_{N+1}^{I} := \left(\frac{2}{N}\right)^{1/2} \left(\varepsilon_j^N \varepsilon_k^N \cos \frac{jk\pi}{N}\right)_{j,k=0}^{N} \in \mathbb{R}^{N+1,N+1},$$

$$\mathrm{DCT-II}: \quad \mathbf{C}_{N}^{II} := \left(\frac{2}{N}\right)^{1/2} \left(\varepsilon_j^N \cos \frac{j(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

$$\mathrm{DCT-III}: \quad \mathbf{C}_{N}^{III} := \left(\mathbf{C}_{N}^{II}\right)' \in \mathbb{R}^{N,N},$$

$$\mathrm{DCT-IV}: \quad \mathbf{C}_{N}^{IV} := \left(\frac{2}{N}\right)^{1/2} \left(\cos \frac{(2j+1)(2k+1)\pi}{4N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

$$\mathrm{DST-I}: \quad \mathbf{S}_{N-1}^{I} := \left(\frac{2}{N}\right)^{1/2} \left(\sin \frac{(j+1)(k+1)\pi}{N}\right)_{j,k=0}^{N-2} \in \mathbb{R}^{N-1,N-1},$$

$$\mathrm{DST-II}: \quad \mathbf{S}_{N}^{II} := \left(\frac{2}{N}\right)^{1/2} \left(\varepsilon_{j+1}^N \sin \frac{(j+1)(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

$$\mathrm{DST-III}: \quad \mathbf{S}_{N}^{III} := \left(\mathbf{S}_{N}^{II}\right)' \in \mathbb{R}^{N,N},$$

$$\mathrm{DST-IV}: \quad \mathbf{S}_{N}^{IV} := \left(\frac{2}{N}\right)^{1/2} \left(\cos \frac{(2j+1)(2k+1)\pi}{4N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

where $\varepsilon_k^N := 2^{-1/2}$ $(k = 0, N)$ and $\varepsilon_k^N := 1$ otherwise. We refer to the corresponding transforms as *trigonometric transforms*. It is well-known that the above matrices are orthogonal and that the multiplication of such matrix with a vector takes only $\mathcal{O}(N \log N)$ arithmetical operations. There exist implementations of fast algorithms for the multiplication of above sine and cosine matrices with a vector, for example a C-implementation based on [15] and [1].
Moreover, we use the slightly modified DCT–I and DST–I matrices

$$\tilde{\mathbf{C}}_{N+1}^{I} := \left((\varepsilon_k^N)^2 \cos \frac{jk\pi}{N}\right)_{j,k=0}^{N}, \; \tilde{\mathbf{S}}_{N-1}^{I} := \left(\sin \frac{jk\pi}{N}\right)_{j,k=1}^{N-1}$$

and the slightly modified DCT–III and DST–III matrices

$$\tilde{\mathbf{C}}_N^{III} := \left( (\varepsilon_k^N)^2 \ \cos \frac{(2j+1)k\pi}{2N} \right)_{j,k=0}^{N-1} ,$$
$$\tilde{\mathbf{S}}_N^{III} := \left( (\varepsilon_{k+1}^N)^2 \sin \frac{(2j+1)(k+1)\pi}{2N} \right)_{j,k=0}^{N-1} .$$

It holds that

$$\tilde{\mathbf{C}}_{N+1}^{I} \, \tilde{\mathbf{C}}_{N+1}^{I} = \frac{N}{2} \, \mathbf{I}_{N+1} \, .$$

Let

$$\mathbf{Z}'_{N,1} := \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{N,N+1} \, , \ \mathbf{Z}'_{N,2} := \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{N,N+1} \, ,$$

$$\mathbf{R}'_N := \begin{pmatrix} 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{N-1,N+1} \, .$$

By stoep $\mathbf{a}'$ we denote the symmetric Toeplitz matrix with first row $\mathbf{a}'$ and by atoep $\mathbf{a}'$ the antisymmetric Toeplitz matrix with first row $\mathbf{a}'$, where $a_0 = 0$. Similarly, let shank $\mathbf{a}'$ be the persymmetric Hankel matrix with first row $\mathbf{a}'$ and let ahank $\mathbf{a}'$ be the antipersymmetric Hankel matrix with first row $\mathbf{a}'$, where $a_{N-1} = 0$. Let diag $\mathbf{a}$ be the diagonal matrix with diagonal $\mathbf{a}$ and let $\delta(\mathbf{A}) := \mathrm{diag}(a_{k,k})_{k=0}^{N-1}$, where $a_{k,k}$ is the $(k,k)$–th entry of $\mathbf{A} \in \mathbb{R}^{N,N}$.

**Theorem 1.** *There exist the following relations between trigonometric transforms and Toeplitz–plus–Hankel matrices:*

i) DCT–I *and* DST–I:

$$\begin{aligned}
2\,\mathbf{R}'_N \mathbf{C}^I_{N+1}\,\mathbf{D}\,\mathbf{C}^I_{N+1}\mathbf{R}_N &= \mathrm{stoep}(a_0,\dots,a_{N-2}) \ + \ \mathrm{shank}(a_2,\dots,a_{N-2},0,0)\,,\\
2\,\mathbf{S}^I_{N-1}\,\mathbf{R}'_N\,\mathbf{D}\,\mathbf{R}_N\mathbf{S}^I_{N-1} &= \mathrm{stoep}(a_0,\dots,a_{N-2}) \ - \ \mathrm{shank}(a_2,\dots,a_{N-2},0,0)\,,\\
2\,\mathbf{R}'_N \mathbf{C}^I_{N+1}\,\tilde{\mathbf{D}}\,\mathbf{R}_N\mathbf{S}^I_{N-1} &= \mathrm{atoep}(0,a_1,\dots,a_{N-2}) \ + \ \mathrm{ahank}(a_2,\dots,a_{N-1},0)\,,\\
2\,\mathbf{S}^I_{N-1}\,\mathbf{R}'_N\,\tilde{\mathbf{D}}\,\mathbf{C}^I_{N+1}\mathbf{R}_N &= -\mathrm{atoep}(0,a_1,\dots,a_{N-2}) + \mathrm{ahank}(a_2,\dots,a_{N-1},0)
\end{aligned}$$

*with*

$$\begin{aligned}
\mathbf{D} := \mathrm{diag}(d_0,\dots,d_N)\,,\ \tilde{\mathbf{D}} &:= \mathrm{diag}(0,\tilde{d}_1,\dots,\tilde{d}_{N-1},0)\,,\\
(d_0,\dots,d_N)' &:= \tilde{\mathbf{C}}^I_{N+1}\,(a_0,\dots,a_{N-2},0,0)'\,,\\
(\tilde{d}_1,\dots,\tilde{d}_{N-1})' &:= \tilde{\mathbf{S}}^I_{N-1}\,(a_1,\dots,a_{N-1})'\,.
\end{aligned}$$

ii) DCT–II *and* DST–II:

$$
\begin{aligned}
2\left(\mathbf{C}_N^{II}\right)'\ \mathbf{Z}'_{N,2}\,\mathbf{D}\,\mathbf{Z}_{N,2}\mathbf{C}_N^{II} &= \mathrm{stoep}(a_0,\ldots,a_{N-1})\ +\ \mathrm{shank}(a_1,\ldots,a_{N-1},0)\ ,\\
2\left(\mathbf{S}_N^{II}\right)'\ \mathbf{Z}'_{N,1}\,\mathbf{D}\,\mathbf{Z}_{N,1}\mathbf{S}_N^{II} &= \mathrm{stoep}(a_0,\ldots,a_{N-1})\ -\ \mathrm{shank}(a_1,\ldots,a_{N-1},0)\ ,\\
2\left(\mathbf{C}_N^{II}\right)'\ \mathbf{Z}'_{N,2}\,\tilde{\mathbf{D}}\,\mathbf{Z}_{N,1}\mathbf{S}_N^{II} &= \mathrm{atoep}(0,a_1,\ldots,a_{N-1})\ +\ \mathrm{ahank}(a_1,\ldots,a_{N-1},0),\\
2\left(\mathbf{S}_N^{II}\right)'\ \mathbf{Z}'_{N,1}\,\tilde{\mathbf{D}}\,\mathbf{Z}_{N,2}\mathbf{C}_N^{II} &= -\mathrm{atoep}(0,a_1,\ldots,a_{N-1}) + \mathrm{ahank}(a_1,\ldots,a_{N-1},0)
\end{aligned}
$$

*with*

$$
\begin{aligned}
\mathbf{D} := \mathrm{diag}(d_0,\ldots,d_N)\ ,\ \tilde{\mathbf{D}} &:= \mathrm{diag}(0,\tilde{d}_1,\ldots,\tilde{d}_{N-1},0)\,,\\
(d_0,\ldots,d_N)' &:= \tilde{\mathbf{C}}_{N+1}^{I}\,(a_0,\ldots,a_{N-1},0)'\,,\\
(\tilde{d}_1,\ldots,\tilde{d}_{N-1})' &:= \tilde{\mathbf{S}}_{N-1}^{I}\,(a_1,\ldots,a_{N-1})'\,.
\end{aligned}
$$

iii) DCT–IV *and* DST–IV:

$$
\begin{aligned}
2\,\mathbf{C}_N^{IV}\,\mathbf{D}\,\mathbf{C}_N^{IV} &= \mathrm{stoep}(a_0,\ldots,a_{N-1})\ +\ \mathrm{ahank}(a_1,\ldots,a_{N-1},0)\ ,\\
2\,\mathbf{S}_N^{IV}\,\mathbf{D}\,\mathbf{S}_N^{IV} &= \mathrm{stoep}(a_0,\ldots,a_{N-1})\ -\ \mathrm{ahank}(a_1,\ldots,a_{N-1},0)\ ,\\
2\,\mathbf{C}_N^{IV}\,\tilde{\mathbf{D}}\,\mathbf{S}_N^{IV} &= \mathrm{atoep}(0,a_1,\ldots,a_{N-1})\ +\ \mathrm{shank}(a_1,\ldots,a_{N-1},0)\ ,\\
2\,\mathbf{S}_N^{IV}\,\tilde{\mathbf{D}}\,\mathbf{C}_N^{IV} &= -\,\mathrm{atoep}(0,a_1,\ldots,a_{N-1})\ +\ \mathrm{shank}(a_1,\ldots,a_{N-1},0)
\end{aligned}
$$

*with*

$$
\begin{aligned}
\mathbf{D} := \mathrm{diag}(d_0,\ldots,d_{N-1})\ ,\ \tilde{\mathbf{D}} &:= \mathrm{diag}(\tilde{d}_0,\ldots,\tilde{d}_{N-1}),\\
(d_0,\ldots,d_{N-1})' &:= \tilde{\mathbf{C}}_N^{III}\,(a_0,\ldots,a_{N-1})'\,,\\
(\tilde{d}_0,\ldots,\tilde{d}_{N-1})' &:= \tilde{\mathbf{S}}_N^{III}\,(a_1,\ldots,a_{N-1},0)'\,.
\end{aligned}
$$

Typewriting this paper, we got a ps–file of a paper of G. Heinig and K. Rost [12], which contains similar results as presented in Theorem 1.

**Corollary 2.** *Let* $\mathbf{T} = \mathbf{T}_N := (t_{j-k})_{j,k=0}^{N-1}$ *be given and let* $\mathbf{C} := \mathbf{C}_N^{IV}$, $\mathbf{S} := \mathbf{S}_N^{IV}$. *Then*

$$
\mathbf{T}\ =\ \frac{1}{2}\,(\mathbf{T}+\mathbf{T}')\ +\ \frac{1}{2}\,(\mathbf{T}-\mathbf{T}') = \mathbf{CDC}\ +\ \mathbf{SDS}\ +\ \mathbf{C}\tilde{\mathbf{D}}\mathbf{S}\ -\ \mathbf{S}\tilde{\mathbf{D}}\mathbf{C},
$$

*where*

$$
\mathbf{D} := \mathrm{diag}(d_0,\ldots,d_{N-1})\ ,\ \tilde{\mathbf{D}} := \mathrm{diag}(0,\tilde{d}_1,\ldots,\tilde{d}_{N-1})\,,
$$

$$(d_0,\dots,d_{N-1})' := \tilde{\mathbf{C}}_N^{III} \left(t_0, \frac{t_1+t_{-1}}{2},\dots,\frac{t_{N-1}+t_{-(N-1)}}{2}\right)',$$
$$(\tilde{d}_0,\dots,\tilde{d}_{N-1})' := \tilde{\mathbf{S}}_N^{III} \left(\frac{t_{-1}-t_1}{2},\dots,\frac{t_{-(N-1)}-t_{N-1}}{2},0\right)'.$$

Corollary 2, which can also be formulated for the other trigonometric transforms, provides a new method for the fast multiplication of real Toeplitz matrix with a vector that avoids the complex arithmetic which comes into the play if we exploit FFT–based methods.

We are interested in real BTTB matrices

$$\mathbf{T}_{M,N} := (\mathbf{T}_{r-s})_{r,s=0}^{M-1} \quad \text{with} \quad \mathbf{T}_r := (t_{r,j-k})_{j,k=0}^{N-1}. \tag{1}$$

If $t_{r,j} = t_{|r|,|j|}$, then $\mathbf{T}_{M,N}$ is called *doubly symmetric*. Corollary 2 can be generalized to BTTB matrices. For given doubly symmetric BTTB matrix $\mathbf{T}_{M,N}$ we obtain the representation

$$\begin{aligned}\mathbf{T}_{M,N} = {} & (\mathbf{C}_M^{IV} \otimes \mathbf{C}_N^{IV})\,\mathbf{D}\,(\mathbf{C}_M^{IV} \otimes \mathbf{C}_N^{IV}) + (\mathbf{S}_M^{IV} \otimes \mathbf{C}_N^{IV})\,\mathbf{D}\,(\mathbf{S}_M^{IV} \otimes \mathbf{C}_N^{IV}) \\ & + (\mathbf{C}_M^{IV} \otimes \mathbf{S}_N^{IV})\,\mathbf{D}\,(\mathbf{C}_M^{IV} \otimes \mathbf{S}_N^{IV}) + (\mathbf{S}_M^{IV} \otimes \mathbf{S}_N^{IV})\,\mathbf{D}\,(\mathbf{S}_M^{IV} \otimes \mathbf{S}_N^{IV})\end{aligned}$$

with $\mathbf{D} := \mathrm{diag}((\tilde{\mathbf{C}}_M^{III} \otimes \tilde{\mathbf{C}}_N^{III})\ \mathrm{col}(t_{r,j})_{j,r=0}^{N-1,M-1})$. Here col: $\mathbb{R}^{N,M} \to \mathbb{R}^{MN}$ and its inverse $\mathrm{col}^{-1}$ are defined by

$$\begin{aligned}\mathrm{col}\ (x_{j,k})_{j=0,k=0}^{N-1,M-1} &:= (x_r)_{r=0}^{MN-1} \quad \text{with} \quad x_{kN+j} := x_{j,k}, \\ \mathrm{col}^{-1}\ (x_r)_{r=0}^{MN-1} &:= (x_{j,k})_{j=0,k=0}^{N-1,M-1} \quad \text{with} \quad x_{j,k} := x_{kN+j}.\end{aligned}$$

**Algorithm 3.** (*Fast multiplication of* BTTB *matrix with a vector* )
*Input:* $\mathbf{x} \in \mathbb{R}^{MN}$, $t_{r,j} \in \mathbb{R}$ $(j = -(N-1),\dots,N-1; r = -(M-1),\dots,M-1)$.
*Output:* $\mathbf{y} := \mathbf{T}_{M,N}\mathbf{x}$ *with* $\mathbf{T}_{M,N}$ *given by* (1).
*Precomputation:*
*For* $j = -(N-1),\dots,N-1$ *compute*

$$(d_{r,j}^c)_{r=0}^{M-1} := \tilde{\mathbf{C}}_M^{III} \left(t_{0,j}, \frac{t_{-1,j}+t_{1,j}}{2},\dots,\frac{t_{-(M-1),j}+t_{M-1,j}}{2}\right)',$$
$$(d_{r,j}^s)_{r=0}^{M-1} := \tilde{\mathbf{S}}_M^{III} \left(\frac{t_{-1,j}-t_{1,j}}{2},\dots,\frac{t_{-(M-1),j}-t_{M-1,j}}{2},0\right)'.$$

*For* $r = -(M-1),\dots,M-1$ and $\alpha \in \{c,s\}$ *compute*

$$(d_{r,j}^{\alpha,c})_{j=0}^{N-1} := \tilde{\mathbf{C}}_N^{III} \left(d_{r,0}^\alpha, \frac{d_{r,-1}^\alpha+d_{r,1}^\alpha}{2},\dots,\frac{d_{r,-(N-1)}^\alpha+d_{r,N-1}^\alpha}{2}\right)',$$
$$(d_{r,j}^{\alpha,s})_{j=0}^{N-1} := \tilde{\mathbf{S}}_N^{III} \left(\frac{d_{r,-1}^\alpha-d_{r,1}^\alpha}{2},\dots,\frac{d_{r,-(N-1)}^\alpha-d_{r,N-1}^\alpha}{2},0\right)'.$$

*Let*

$$\mathbf{X} := \mathrm{col}^{-1}\,\mathbf{x}\,,$$
$$\mathbf{D}^{\alpha,\beta} := \left(d_{r,j}^{\alpha,\beta}\right)_{j=0,r=0}^{N-1,M-1} \quad (\alpha, \beta \in \{c, s\})\,.$$

*1.* $\mathbf{X}^c := \mathbf{X}\,\mathbf{C}_M^{IV}$, $\mathbf{X}^s := \mathbf{X}\,\mathbf{S}_M^{IV}$.
*2.* $\mathbf{X}^{\alpha,c} := \mathbf{C}_N^{IV}\,\mathbf{X}^\alpha$, $\mathbf{X}^{\alpha,s} := \mathbf{S}_N^{IV}\,\mathbf{X}^\alpha$ $(\alpha \in \{c, s\})$.
*3.* $\mathbf{Z}^{\beta,\alpha} := \mathbf{C}_N^{IV}\,(\mathbf{D}^{\beta,c} \circ \mathbf{X}^{\alpha,c} + \mathbf{D}^{\beta,s} \circ \mathbf{X}^{\alpha,s}) + \mathbf{S}_N^{IV}\,(\mathbf{D}^{\beta,c} \circ \mathbf{X}^{\alpha,s} - \mathbf{D}^{\beta,s} \circ \mathbf{X}^{\alpha,c})$ $(\alpha, \beta \in \{c, s\})$. *Here* $\circ$ *denotes the componentwise matrix product.*
*4.* $\mathbf{Y} := (\mathbf{Z}^{c,c} + \mathbf{Z}^{s,s})\,\mathbf{C}_M^{IV} + (\mathbf{Z}^{c,s} - \mathbf{Z}^{s,c})\,\mathbf{S}_M^{IV}$.
*Set* $\mathbf{y} := \mathrm{col}\,\mathbf{Y}$.

Algorithm 3 requires $\mathcal{O}(MN\,\log(MN))$ real arithmetical operations.

## §3. Trigonometric Preconditioners

We are concerned with the solution of a system of linear equations

$$\mathbf{T}_{M,N}\mathbf{x} = \mathbf{b} \tag{2}$$

with doubly symmetric positive definite BTTB matrix

$$\mathbf{T} \;=\; \mathbf{T}_{M,N} \;:= \mathrm{stoep}(\mathbf{T}_0, \ldots, \mathbf{T}_{M-1})\,, \quad \mathbf{T}_j = \mathrm{stoep}(t_{j,0}, \ldots, t_{j,N-1}) \tag{3}$$

by the PCG-method. We will see that with a good preconditioner at hand, this can be realized in a fast way. There are several requirements on a preconditioner $\mathbf{P}_{M,N}$ of (2) resulting from the construction and the convergence behaviour of the CG-method as well as from the fact that the multiplication of $\mathbf{T}_{M,N}$ with a vector requires only $\mathcal{O}(MN\,\log(MN))$ arithmetical operations. Therefore, we are looking for a preconditioner with the following properties:
(P1) $\mathbf{P}_{M,N}$ is symmetric and positive definite.
(P2) For all $\varepsilon > 0$ and sufficiently large $M, N$, at most $\mathcal{O}(M) + \mathcal{O}(N)$ eigenvalues of $\mathbf{P}_{M,N}^{-1}\mathbf{T}_{M,N}$ lie outside the interval $(1 - \varepsilon, 1 + \varepsilon)$ such that the PCG-method converges in at most $\mathcal{O}(M) + \mathcal{O}(N)$ steps.
(P3) The multiplication of $\mathbf{P}_{M,N}$ with a vector can be computed with $\mathcal{O}(MN\log(MN))$ arithmetical operations.
(P4) The construction of $\mathbf{P}_{M,N}$ takes only $\mathcal{O}(MN\,\log(MN))$ arithmetical operations.
Having property (P3) in mind, a straightforward idea consists in choosing $\mathbf{P}_{M,N}$ from an algebra

$$\mathcal{A}_O := \{\mathbf{O}'\,(\mathrm{diag}\,\mathbf{d})\,\mathbf{O} : \mathbf{d} \in \mathbb{R}^{MN}\}$$

of all matrices which are diagonalizable by some orthogonal matrix $\mathbf{O} \in \mathbb{R}^{MN,MN}$, where $\mathbf{O}$ has the additional property that its fast multiplication with a vector requires only $\mathcal{O}(MN \log(MN))$ arithmetical operations. As orthogonal matrices, we will use tensor products of the trigonometric matrices of the previous section. For $\mathbf{A} \in \mathbb{R}^{MN,MN}$, the matrix $\mathbf{P}(\mathbf{A}) \in \mathcal{A}_{\mathcal{O}}$ is called an *optimal preconditioner* of $\mathbf{A}$ [4] if

$$||\mathbf{P}(\mathbf{A}) - \mathbf{A}||_F = \min\{||\mathbf{B} - \mathbf{A}||_F : \mathbf{B} \in \mathcal{A}_O\}\,.$$

Here $||\mathbf{A}||_F$ denotes the *Frobenius norm*

$$||\mathbf{A}||_F := (\sum_{j,k=0}^{MN-1} a_{jk}^2)^{1/2}\,.$$

The choice of the Frobenius norm results from the fact that this norm is induced by an inner product of $\mathbb{R}^{MN,MN}$

$$\langle \mathbf{A}, \mathbf{B} \rangle := \mathrm{tr}(\mathbf{A}'\mathbf{B}) = \sum_{j,k=0}^{MN-1} a_{j,k} b_{j,k}.$$

In particular, we have

$$||\mathbf{OAO}'||_F^2 = \mathrm{tr}(\mathbf{OA}'\mathbf{O}'\mathbf{OAO}') = \mathrm{tr}(\mathbf{A}'\mathbf{A}) = ||\mathbf{A}||_F^2\,.$$

For $\mathbf{B} := \mathbf{O}' \,(\mathrm{diag}\,\mathbf{d})\, \mathbf{O} \in \mathcal{A}_O$ this implies

$$||\mathbf{B} - \mathbf{A}||_F = ||\mathrm{diag}\,\mathbf{d} - \mathbf{OAO}'||_F$$

such that the optimal preconditioner of $\mathbf{A}$ is given by

$$\mathbf{P}(\mathbf{A}) = \mathbf{O}'\,\delta(\mathbf{OAO}')\,\mathbf{O}\,. \tag{4}$$

If $\mathbf{O}$ is the tensor product of any two orthogonal matrices which correspond to the DST–I, DST–II, DCT–II, DST–IV and DCT–IV, respectively, then $\mathbf{P}(\mathbf{A})$ is said to be an *optimal trigonometric preconditioner* of $\mathbf{A}$.
In the following, we restrict our attention to

$$\mathbf{O} = \mathbf{O}_{M,N} := \mathbf{C}_M^{IV} \otimes \mathbf{C}_N^{IV}\,.$$

The approach for the other transforms follows exactly the same lines. Numerical examples are included for different transforms.
By Theorem 1, $\mathcal{A}_O$ consists of block–Toeplitz–plus–Hankel matrices with Toeplitz–plus–Hankel blocks, more precisely

$$2\,\mathbf{O}\,\mathbf{D}\,\mathbf{O} = \mathrm{stoep}(\mathbf{A}_0, \ldots, \mathbf{A}_{M-1}) \;+\; \mathrm{ahank}(\mathbf{A}_1, \ldots, \mathbf{A}_{M-1}, \mathbf{0}) \tag{5}$$

where

$$2\,\mathbf{A}_r \;:=\; \mathrm{stoep}(a_{r,0}, \ldots, a_{r,N-1}) \;+\; \mathrm{ahank}(a_{r,1}, \ldots, a_{r,N-1}, 0)\,,$$

$$\mathbf{D} := \mathrm{diag}(\tilde{\mathbf{C}}_M^{III} \otimes \tilde{\mathbf{C}}_N^{III})\,\mathbf{a}\,, \qquad \mathbf{a} := \mathrm{col}\,(a_{r,k})_{k=0,r=0}^{N-1,M-1}\,.$$

The optimal trigonometric preconditioner $\mathbf{P} := \mathbf{P}(\mathbf{T}) \in \mathcal{A}_O$ of a doubly symmetric BTTB matrix $\mathbf{T}$ can be computed as follows:

**Theorem 4.** *Let a doubly symmetric* BTTB *matrix* $\mathbf{T}$ *of form* (3) *be given. Then its optimal trigonometric preconditioner* $\boldsymbol{P} \in \mathcal{A}_O$ *reads*

$$\mathbf{P} = 4\, \mathbf{O} \left( \operatorname{diag}(\tilde{\mathbf{C}}_M^{III} \otimes \tilde{\mathbf{C}}_N^{III})\, (\mathbf{w}^F \circ \mathbf{t}) \right) \mathbf{O}$$

*where* $\mathbf{t} := \operatorname{col}(t_{r,k})_{k=0,r=0}^{N-1,M-1}$ *and* $\mathbf{w}^F := \operatorname{col}\left((1-\frac{r}{M})(1-\frac{k}{N})\right)_{k=0,r=0}^{N-1,M-1}$.

**Proof:** By (4), we are interested in $\delta(\mathbf{OTO}) = \operatorname{diag}(d_r)_{r=0}^{MN-1}$.
For $k = 0, \ldots, M-1$ and $j = 0, \ldots, N-1$, we obtain by straightforward calculation that

$$d_{kN+j} = \frac{4}{MN} \sum_{u,v=0}^{M-1} \sum_{r,s=0}^{N-1} t_{|u-v|,|r-s|} \cos\frac{(2k+1)(2u+1)\pi}{4M} \cos\frac{(2j+1)(2r+1)\pi}{4N} \cdot \cos\frac{(2v+1)(2k+1)\pi}{4M} \cos\frac{(2s+1)(2j+1)\pi}{4N}.$$

Using

$$\cos x \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y),$$

we can simplify the above expression in

$$d_{kN+j} = \frac{1}{MN} \sum_{u,v=0}^{M-1} \sum_{r,s=0}^{N-1} t_{|u-v|,|r-s|} \cos\frac{(2k+1)(u-v)\pi}{2M} \cos\frac{(2j+1)(r-s)\pi}{2N},$$

since the remaining three terms vanish. Summation over all $(u,v)$ with $m := |u-v| = 0, \ldots, M-1$ leads to

$$d_{kN+j} = \frac{2}{N} \sum_{r,s=0}^{N-1} \cos\frac{(2j+1)(r-s)\pi}{2N} \sum_{m=0}^{M-1} (\varepsilon_m^M)^2 (1-\frac{m}{M}) t_{m,|r-s|} \cos\frac{(2k+1)m\pi}{2M}$$

and summation over all $(r,s)$ with $n := |r-s| = 0, \ldots, N-1$ finally to

$$d_{kN+j} = 4 \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (\varepsilon_m^M)^2 (\varepsilon_n^N)^2 (1-\frac{m}{M})(1-\frac{n}{N})\, t_{m,n} \cdot \cos\frac{(2k+1)m\pi}{2M} \cos\frac{(2j+1)n\pi}{2N}. \tag{6}$$

Consequently, we get

$$(d_r)_{r=0}^{MN-1} = 4 \left( (\tilde{\mathbf{C}}_M^{III} \otimes \tilde{\mathbf{C}}_N^{III})\, (\mathbf{w}^F \circ \mathbf{t}) \right). \quad \blacksquare$$

Let $t_{j,k} \in \mathbb{R}$ $(j,k \in \mathbb{N}_0)$ with $\sum_{j,k=0}^{\infty} |t_{j,k}| < \infty$ be given. Then the function

$$f(x,y) := 4 \sum_{j,k=0}^{\infty} \lambda_j \lambda_k \, t_{j,k} \, \cos(jx)\cos(ky) \tag{7}$$

with $\lambda_0 := 1/2$ and $\lambda_j := 1$ for $j \in \mathbb{N}$ belongs to the Wiener class. Note that $f$ is $2\pi$–periodic and even in both variables. Under these assumptions, $f$ is called *generating function* of the doubly symmetric BTTB matrix (3).
Using the Fourier sums

$$\left(s_{m,n}f\right)(x,y) := 4\sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_j \lambda_k \, t_{j,k} \, \cos(jx)\cos(ky)\,, \tag{8}$$

the Fejér mean [3] is defined by

$$\begin{aligned}\left(\sigma_{M-1,N-1}f\right)(x,y) &:= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(s_{m,n}f)(x,y)\\ &= 4\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \lambda_j\lambda_k w^F_{m,n} t_{m,n}\cos(mx)\cos(ny)\end{aligned}$$

with the corresponding *Fejér window*

$$w^F_{m,n} := \begin{cases}\left(1-\frac{m}{M}\right)\left(1-\frac{n}{N}\right) & m=0,\ldots,M-1\,; n=0,\ldots,N-1\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Defining the Fejér kernel by

$$F_{N-1}(x) := 1 + 2\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\cos(kx) \;=\; \begin{cases}\dfrac{1}{N}\left(\dfrac{\sin\frac{xN}{2}}{\sin\frac{x}{2}}\right)^2 & x \neq 0\,,\\ N & x = 0\end{cases}$$

and the *even shift* by

$$\left(\tau_{u,v}\, f\right)(x,y) := \frac{1}{4}\sum_{j,k=0}^{1} f(x+(-1)^j u, y+(-1)^k v)\,,$$

we obtain the known integral representation of the Fejér mean

$$\left(\sigma_{M-1,N-1}f\right)(x,y) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} (\tau_{u,v}f)(x,y)\,F_{M-1}(u)\,F_{N-1}(v)\,\mathrm{d}u\mathrm{d}v\,.$$

For $M,N \to \infty$, the Fejér mean $\sigma_{M-1,N-1}f$ tends uniformly to $f$ on $[0,\pi]^2$. Furthermore, we know that for all $(x,y) \in [0,\pi]^2$

$$f_{\min} \le (\sigma_{M-1,N-1}f)(x,y) \le f_{\max}\,, \tag{9}$$

where $f_{\min}$ and $f_{\max}$ denote the minimal and maximal values of $f$, respectively. By (9) and Theorem 4, we obtain:

**Corollary 5.** *Let $f$ be given by (7) and let $\mathbf{T} := \mathbf{T}_{M,N}$ be the associated doubly symmetric* BTTB *matrix* (3). *Then the eigenvalues of the optimal preconditioner $\mathbf{P} \in \mathcal{A}_O$ of $\mathbf{T}$ are special values of the Fejér mean, i.e.*

$$\operatorname{col}\left((\sigma_{M-1,N-1}f)(\frac{(2j+1)\pi}{2M}, \frac{(2k+1)\pi}{2N})\right)_{k,j=0}^{N-1,M-1} = 4\left(\tilde{\mathbf{C}}_M^{III} \otimes \tilde{\mathbf{C}}_N^{III}\right)(\mathbf{w}^F \circ \mathbf{t}).$$

*If $f$ is positive, then $\mathbf{P}$ is positive definite.*

Using the simple *rectangular window*

$$w_{m,n}^S := \begin{cases} 1 & m = 0, \dots M-1,\ n = 0, \dots N-1\,, \\ 0 & \text{otherwise,} \end{cases}$$

instead of the Fejér window in Theorem 4, we obtain the definition of the *Strang-type preconditioner* [10]

$$\mathbf{S} = \mathbf{S}(\mathbf{T}) := 4\,\mathbf{O}\left(\operatorname{diag}(\tilde{\mathbf{C}}_M^{III} \otimes \tilde{\mathbf{C}}_N^{III})(\mathbf{w}^S \circ \mathbf{t})\right)\mathbf{O}\,.$$

The eigenvalues of $\mathbf{S}$ are special values of the Fourier sum $s_{M-1,N-1}f$. For positive $f$ it could happen that not all eigenvalues of $\mathbf{S}$ are positive. Since $f$ is from the Wiener class, the Fourier sum $s_{M-1,N-1}f$ tends uniformly to $f$ on $[0,\pi]^2$. Hence, for all $\varepsilon > 0$, there exist $\mu > 0$ such that for $M, N > \mu$ all eigenvalues of $\mathbf{S}$ lie in the interval $[f_{\min} - \varepsilon, f_{\max} + \varepsilon]$. Thus, for positive $f$ and sufficiently large $M$ and $N$, the Strang–type preconditioner $\mathbf{S}$ is positive definite.

## §4. Convergence Analysis

Let $f$ be given by (7). We use $f$ as generating function of doubly symmetric BTTB matrices $\mathbf{T}_{M,N}$ of form (3). The eigenvalues of $\mathbf{T}_{M,N}$ are distributed as $f$ [13] and

$$f_{\min} \le \lambda_{\min}(\mathbf{T}_{M,N}) \le \lambda_{\max}(\mathbf{T}_{M,N}) \le f_{\max}\,,$$

where $\lambda_{\min}(\mathbf{T}_{M,N})$ and $\lambda_{\max}(\mathbf{T}_{M,N})$ denote the smallest and largest eigenvalues of $\mathbf{T}_{M,N}$, respectively. We consider again the case $\mathbf{O}_{M,N} := \mathbf{C}_M^{IV} \otimes \mathbf{C}_N^{IV}$.

**Theorem 6.** *Let $f$ in (7) be a positive function from the Wiener class with $f_{\min} \ge \gamma > 0$. Further, let $\mathbf{T}_{M,N}$ be the associated doubly symmetric* BTTB *matrix* (3). *By $\mathbf{S}_{M,N}$, $\mathbf{P}_{M,N} \in \mathcal{A}_{O_{M,N}}$, we denote the Strang-type preconditioner and the optimal trigonometric preconditioner of $\mathbf{T}_{M,N}$, respectively. Then, for all $\varepsilon > 0$, and for sufficiently large $M, N$ at most $\mathcal{O}(M) + \mathcal{O}(N)$ eigenvalues of $\mathbf{S}_{M,N}^{-1}\mathbf{T}_{M,N}$ and $\mathbf{P}_{M,N}^{-1}\mathbf{T}_{M,N}$ lie outside the interval $(1-\varepsilon, 1+\varepsilon)$.*

**Proof:** 1. First we consider $\mathbf{S}_{M,N}^{-1}\mathbf{T}_{M,N}$. By (5), it holds that

$$\begin{aligned} \mathbf{S}_{M,N} - \mathbf{T}_{M,N} =& \operatorname{ahank}(\mathbf{T}_1, \dots, \mathbf{T}_{M-1}, \mathbf{0}) + \operatorname{stoep}(\mathbf{H}_0, \dots, \mathbf{H}_{M-1}) \\ &+ \operatorname{ahank}(\mathbf{H}_1, \dots, \mathbf{H}_{M-1}, \mathbf{0})\,, \end{aligned} \tag{10}$$

where $\mathbf{H}_r := \text{ahank}(t_{r,1}, \ldots, t_{r,N-1}, 0)$. Since $f$ belongs to the Wiener class, for all $\varepsilon > 0$, there exist $m, n > 0$ such that

$$\sum_{s=m+1}^{\infty} \sum_{j=0}^{\infty} |t_{s,j}| < \varepsilon\gamma/6 \quad , \quad \sum_{s=0}^{\infty} \sum_{j=n+1}^{\infty} |t_{s,j}| < \varepsilon\gamma/6 \,. \tag{11}$$

For $N > 2n$ and $M > 2m$, we decompose (10) as

$$\mathbf{S}_{M,N} - \mathbf{T}_{M,N} = \mathbf{U} + \mathbf{V}\,,$$

where

$$\begin{aligned}\mathbf{U} &:= \text{ahank}(\mathbf{0}_m, \mathbf{T}_{m+1}, \ldots, \mathbf{T}_{M-1}, \mathbf{0}) + \text{stoep}(\mathbf{H}_0^B, \ldots, \mathbf{H}_{M-1}^B) \\ &\quad + \text{ahank}(\mathbf{0}_m, \mathbf{H}_{m+1}, \ldots, \mathbf{H}_{M-1}, \mathbf{0})\,, \\ \mathbf{V} &:= \text{ahank}(\mathbf{T}_1, \ldots, \mathbf{T}_m, \mathbf{0}_{M-m}) + \text{stoep}(\mathbf{H}_0^E, \ldots, \mathbf{H}_{M-1}^E) \\ &\quad + \text{ahank}(\mathbf{H}_1, \ldots, \mathbf{H}_m, \mathbf{0}_{M-m})\end{aligned}$$

and

$$\mathbf{H}_r^B := \text{ahank}(\mathbf{0}_n, t_{r,n+1}, \ldots, t_{r,N-1}, 0)\,, \qquad \mathbf{H}_r^E := \text{ahank}(t_{r,1}, \ldots, t_{r,n}, \mathbf{0}_{N-n})\,.$$

Here $\mathbf{0}_j$ denotes either the zero vector of length $j$ or the vector of length $j$ with zero matrices of size $(N,N)$ as entries. By (11), it holds that

$$||\mathbf{U}||_2 \le ||\mathbf{U}||_1 \le \varepsilon\gamma\,.$$

Moreover, $\mathbf{V}$ is a matrix of low rank $\le 2mn + 2nM$. Next, we verify that

$$\mathbf{I} - \mathbf{S}_{M,N}^{-1}\mathbf{T}_{M,N} = \mathbf{S}_{M,N}^{-1}(\mathbf{S}_{M,N} - \mathbf{T}_{M,N}) = \mathbf{S}_{M,N}^{-1}\mathbf{U} + \mathbf{S}_{M,N}^{-1}\mathbf{V}\,,$$

where $\mathbf{S}_{M,N}^{-1}\mathbf{V}$ is of low rank $\le 2mN + 2nM$ and where by $f_{\min} \ge \gamma > 0$

$$||\mathbf{S}_{M,N}^{-1}\mathbf{U}||_2 \le ||\mathbf{S}_{M,N}^{-1}||_2 ||\mathbf{U}||_2 \le \varepsilon.$$

Hence, by Cauchy's interlace theorem [18] (applied to the matrix $\mathbf{S}_{M,N}^{-1/2}\mathbf{U}\mathbf{S}_{M,N}^{-1/2}$ $+ \mathbf{S}_{M,N}^{-1/2}\mathbf{U}\mathbf{S}_{M,N}^{-1/2}$ ) at most $2mN + 2nM$ eigenvalues of $\mathbf{I} - \mathbf{S}_{M,N}^{-1}\mathbf{T}_{M,N}$ have absolute values larger than $\varepsilon$.
2. To prove the assertion for $\mathbf{P}_{M,N}^{-1}\mathbf{T}_{M,N}$, we use that

$$\begin{aligned}\mathbf{P}_{M,N} - \mathbf{T}_{M,N} &= \mathbf{S}_{M,N} - \mathbf{T}_{M,N} - \mathbf{O}_{M,N}\,\delta\,(\mathbf{O}_{M,N}(\mathbf{S}_{M,N} - \mathbf{T}_{M,N})\mathbf{O}_{M,N})\,\mathbf{O}_{M,N} \\ &= \mathbf{U} + \mathbf{V} + \mathbf{O}_{M,N}\,\delta(\mathbf{O}_{M,N}\mathbf{U}\,\mathbf{O}_{M,N})\,\mathbf{O}_{M,N} \;+\; \mathbf{O}_{M,N}\,\delta(\mathbf{O}_{M,N}\mathbf{V}\,\mathbf{O}_{M,N})\,\mathbf{O}_{M,N}\,.\end{aligned}$$

Regarding that the entries of $\mathbf{O}_{M,N}$ have absolute values $\le 2/\sqrt{MN}$, we get for sufficiently large $M, N$ that

$$||\delta(\mathbf{O}_{M,N}\mathbf{V}\,\mathbf{O}_{M,N})||_2 \le \varepsilon\gamma\,.$$

On the other hand, we have

$$||\delta(\mathbf{O}_{M,N}\mathbf{U}\,\mathbf{O}_{M,N})||_2 \le ||\mathbf{U}||_2 \le \varepsilon\gamma$$

such that

$$\mathbf{C}_{M,N} - \mathbf{T}_{M,N} = \mathbf{V} + \tilde{\mathbf{U}}$$

with $||\tilde{\mathbf{U}}||_2 \le 3\varepsilon\gamma$. The rest of the proof follows the lines of the first part. ■

Using standard error analysis, we conclude that the CG-method converges in at most $\mathcal{O}(M) + \mathcal{O}(N)$ steps for sufficiently large $M, N$.

## §4. Numerical Results

In this section we illustrate the efficiency of trigonometric preconditioning by various numerical examples. The algorithms were realized for the optimal preconditioners with respect to the DCT–II, DST–II, DCT–IV and DST–IV, respectively. The fast computation of the preconditioners and the PCG–method were implemented in Matlab and tested on a Sun SPARCstation 20. The fast trigonometric transforms appearing both in the computation of the preconditioner and in the PCG–steps were taken from the C–implementation based on [15] and [1] by using the cmex–program.

As transform length we choose $N = 2^n$. The right–hand side $\mathbf{b}$ of (2) is the vector consisting of $N$ entries „1". The PCG–method starts with the zero vector and stops if $\|\mathbf{r}^{(j)}\|_2/\|\mathbf{r}^{(0)}\|_2 < 10^{-7}$, where $\mathbf{r}^{(j)}$ denotes the residual vector after $j$ iterations. We begin with symmetric Toeplitz matrices $\mathbf{T}_{1,N}$. Our test matrices correspond to the following generating functions:

(i) (see [8])

$$f(x) := x^4 + 1 \quad (-\pi \le x \le \pi)\,,$$

(ii) (see [9])

$$f(x) := x^2 \quad (-\pi \le x \le \pi)\,.$$

The second column of each table contains the number of iteration steps of the CG–method without preconditioning. The last four columns of tables 1 – 2 show the number of iterations required by the PCG–method for different optimal trigonometric preconditioners.

| $n$ | $I$ | $\mathbf{C}_N^{II}$ | $\mathbf{S}_N^{II}$ | $\mathbf{C}_N^{IV}$ | $\mathbf{S}_N^{IV}$ |
|---|---|---|---|---|---|
| 8 | 67 | 5 | 5 | 7 | 7 |
| 9 | 70 | 5 | 5 | 7 | 7 |
| 10 | 71 | 5 | 5 | 7 | 7 |
| 11 | 70 | 5 | 5 | 7 | 7 |
| 12 | 68 | 5 | 5 | 7 | 7 |
| 13 | 68 | 5 | 5 | 7 | 7 |
| 14 | 65 | 5 | 5 | 7 | 7 |

**Table 1.** Number of iterations for example (i).

| $n$ | $I$ | $\mathbf{C}_N^{II}$ | $\mathbf{S}_N^{II}$ | $\mathbf{C}_N^{IV}$ | $\mathbf{S}_N^{IV}$ |
|---|---|---|---|---|---|
| 8 | 179 | 23 | 5 | 25 | 25 |
| 9 | 375 | 29 | 5 | 33 | 33 |
| 10 | 771 | 38 | 5 | 41 | 41 |
| 11 | * | 51 | 5 | 55 | 55 |
| 12 | * | 68 | 5 | 59 | 59 |

**Table 2.** Number of iterations for example (ii).

Further extensive numerical tests with matrices from [2] and [9] lead to the following general observations:
– For symmetric Toeplitz matrices, the DST–I based PCG–method shows a similar convergence behaviour as the DST–II based PCG–method.
– The use of a Strang–type preconditioner with respect to DCT–II or DST–II yields similar results as the DCT–II or DST–II based PCG–method in all numerical tests.
– Preconditioning with respect to the DST–II and DCT–II, respectively, leads to a faster convergence of the PCG-method than preconditioning with respect to the DST–IV or DCT–IV.
Next we consider (2) with doubly symmetric BTTB matrices $\mathbf{T}_{N,N}$. By our observations, we restrict our attention to $\mathbf{O}_1 := \mathbf{C}_N^{II} \otimes \mathbf{C}_N^{II}$ and $\mathbf{O}_2 := \mathbf{S}_N^{II} \otimes \mathbf{S}_N^{II}$. Our test matrices are the four doubly symmetric BTTB matrices with the entries:
(iii) (see [6] and [11])

$$t_{j,k} := \frac{1}{(j+1)\,(k+1)^{1+0.1(j+1)}} \quad (j,k = 0,\ldots,N-1)\;,$$

(iv) (see [6] and [11])

$$t_{j,k} := \frac{1}{(j+1)^{1.1}\;+\;(k+1)^{1.1}} \quad (j,k = 0,\ldots,N-1)\;,$$

(v) (see [11] and [14])

$$t_{j,k} := 0.7^{\sqrt{j^2+k^2}} + 0.5^{\sqrt{j^2+k^2}} + 0.3^{\sqrt{j^2+k^2}} \quad (j, k = 0, \ldots, N-1)$$

(vi) and with the generating function (see [11])

$$f(x,y) := x^2 + y^2 + x^2y^2 \quad ((x,y) \in [0,\pi]^2) .$$

| $N$ | $I$ | $\mathbf{O}_1$ | $\mathbf{O}_2$ |
|---|---|---|---|
| 8 | 15 | 8 | 10 |
| 16 | 28 | 9 | 12 |
| 32 | 38 | 10 | 13 |
| 64 | 45 | 11 | 14 |
| 128 | 49 | 12 | 14 |
| 256 | 51 | 13 | 14 |
| 512 | 50 | 13 | 15 |

| $N$ | $I$ | $\mathbf{O}_1$ | $\mathbf{O}_2$ |
|---|---|---|---|
| 8 | 11 | 7 | 8 |
| 16 | 27 | 8 | 10 |
| 32 | 43 | 9 | 13 |
| 64 | 71 | 9 | 15 |
| 128 | 103 | 10 | 16 |
| 256 | 144 | 10 | 18 |
| 512 | 198 | 11 | 20 |

**Table 3.** Number of iterations for examples (iii) and (iv).

| $N$ | $I$ | $\mathbf{O}_1$ | $\mathbf{O}_2$ |
|---|---|---|---|
| 8 | 12 | 7 | 8 |
| 16 | 30 | 8 | 10 |
| 32 | 48 | 9 | 12 |
| 64 | 68 | 9 | 13 |
| 128 | 79 | 8 | 13 |
| 256 | * | 7 | 13 |
| 512 | * | 7 | 12 |

| $N$ | $I$ | $\mathbf{O}_1$ | $\mathbf{O}_2$ |
|---|---|---|---|
| 8 | 10 | 10 | 9 |
| 16 | 52 | 18 | 9 |
| 32 | 164 | 25 | 10 |
| 64 | * | 36 | 10 |
| 128 | * | 56 | 10 |
| 256 | * | 90 | 10 |
| 512 | * | 152 | 9 |

**Table 4.** Number of iterations for example (v) and (vi).

In all examples from [6], our "level–2" trigonometric preconditioners work significantly better than the "level–2" circulant preconditioners and also often better than the "level–1" circulant preconditioners. Only in example (vi), the preconditioning based on $\mathbf{O}_2$ as proposed in [11] leads to a faster convergence than the preconditioning based on $\mathbf{O}_1$. In most of our examples with BTTB matrices the $\mathbf{O}_1$ based PCG–method requires fewer iterations than the PCG–method based on $\mathbf{C}_N^{IV} \otimes \mathbf{S}_N^{IV}$ or $\mathbf{O}_2$.
Moreover, in all examples the optimal trigonometric preconditioner with respect to both $\mathbf{O}_1$ and $\mathbf{O}_2$ yields a little better convergence of the PCG–method than the Strang–type preconditioner based on $\mathbf{O}_1$ and $\mathbf{O}_2$, respectively.

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