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Circulant Preconditioners for Indefinite Toeplitz Systems

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Abstract.

In [2, 6, 8, 15, 16], circulant preconditioners were proposed for ill-conditioned Hermitian Toeplitz matrices generated by 2π-periodic continuous functions with zeros of *even* order. It was shown that the spectra of the preconditioned matrices are uniformly bounded except for a finite number of outliers and therefore the conjugate gradient method, when applied to solving these circulant preconditioned systems, converges very quickly. In this paper, we consider indefinite Toeplitz matrices generated by 2π-periodic continuous functions with zeros of *odd* order. In particular, we show that the singular values of the preconditioned matrices are essentially bounded. Numerical results are presented to illustrate the fast convergence of CGNE, MINRES and QMR methods.

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1 Introduction

An n-by-n matrix A_n with entries a_{ij} is said to be Toeplitz if $a_{ij} = a_{i-j}$. Toeplitz systems of the form $A_n\mathbf{x} = \mathbf{b}$ occur in a variety of applications in mathematics and engineering [5]. A more detailed survey is given in [13]. The fast [14] and superfast [1] Toeplitz solvers are in general numerically unstable for indefinite systems. Look-ahead methods are numerically stable, and although it may retain the $\mathcal{O}(n^2)$ complexity, it requires $\mathcal{O}(n^3)$ operations in the worst case [9].

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In this paper, we consider using preconditioned iterative methods for solving indefinite Toeplitz systems. Several successful circulant preconditioners have been introduced and analyzed; see for instance [5]. In these papers, the given Toeplitz matrix A_n is assumed to be generated by a generating function f, i.e., the diagonals of A_n are given by the Fourier coefficients of f. It was shown that if f is a positive function in the Wiener class, then these circulant preconditioned systems converge superlinearly. However, if f has zeros, the corresponding Toeplitz systems will be ill-conditioned. Tyrtyshnikov [19] has proved that the Strang and the T. Chan preconditioners both fail in this case.

In [6], Chan et al. considered circulant preconditioners for Toeplitz matrices generated by an indefinite and piecewise continuous generating function with zeros of even order (see (2.1) for the definition of the order of a zero). They showed that that the preconditioned minimal residual method requires only $\mathcal{O}(n(\log_2 n)^2)$ operations. However, their Toeplitz matrices are not generated by functions with a zero of *odd* order. To the best of our knowledge, in the literature, there is only one paper [17] that considered Toeplitz matrices generated by functions having a zero of odd order. In that paper, Serra used a kind of normal equations matrix approach to transform a given Toeplitz system $A_n\mathbf{x} = \mathbf{b}$ into a new equivalent system $(\tilde{A}_n + R)\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ in which the new generating function $\tilde{f}$ of $\tilde{A}_n$ has only a zero of even order. Here R is a correction matrix.

The main aim of this paper is to extend the results in [15] to Toeplitz matrices generated by functions with zeros of odd order. We prove that if the generating function f is given, the singular values of the circulant preconditioned matrices are uniformly bounded except for a finite number of outliers. It follows that if $\kappa_2(A_n) = O(n^\alpha)$, the CGNE method applied to the preconditioned system converges in $\mathcal{O}(\alpha \log_2 n)$ iteration steps. In summary, the proposed algorithm requires only $\mathcal{O}(n(\log_2 n)^2)$ arithmetical operations. In contrast, the complexity for solving the non-preconditioned systems will be of order $\mathcal{O}(n^{\alpha/2+1}\log_2 n)$. We are also interested in Krylov space methods like MINRES [10] and QMR [11] which do not require the normal equations. The numerical examples confirm the theoretical results and show that the MINRES and QMR methods converge very quickly.

The outline of the paper is as follows. In §2, a brief introduction on the circulant preconditioners will be given. In §3, we study the banded-Toeplitz matrix approximation. The convergence analysis will be presented in §4. Finally, numerical results are given in §5.

2 Construction of Circulant Preconditioners

Let $\mathbf{C}_{2\pi}$ be the space of all 2π-periodic continuous real-valued functions. The Fourier coefficients of a function f in $\mathbf{C}_{2\pi}$ are given by

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\theta, \quad k = 0, \pm 1, \pm 2, \cdots.$$

Clearly $a_k = \bar{a}_{-k}$ for all k. Let $A_n[f]$ be the n-by-n Hermitian Toeplitz matrix with the (i,j)th entry given by a_{i-j}, $i,j = 0,\ldots,n-1$. We will use $\mathbf{C}_{2\pi}^{+}$ to denote the space of all nonnegative functions in $\mathbf{C}_{2\pi}$ which are not identically zero. We remark that the Toeplitz matrices $A_n[f]$ generated by a nonnegative function f are Hermitian positive definite for all n, see [4, Lemma 1].

LEMMA 2.1. *Let $f \in \mathbf{C}_{2\pi}^{+}$. Then $A_n[f]$ are positive definite for all n.*

Conversely, if $f \in \mathbf{C}_{2\pi}$ takes both positive and negative values, then $A_n[f]$ will be indefinite. We say that θ_0 is a zero of f of *order* ν if $f(\theta_0) = 0$ and

$$f(\theta) = \frac{f^{(\nu)}(\theta_0)(\theta - \theta_0)^{\nu}}{\nu!} + \mathcal{O}((\theta - \theta_0)^{\nu+1}) \tag{2.1}$$

for all θ in that neighborhood of θ_0.

In this paper, we consider $f \in \mathbf{C}_{2\pi}$ having zeros of odd order ν_j at ϕ_j ($j = 1, \cdots, s$). Such f can be written in the general form

$$f(\theta) = \prod_{j=1}^{s} (\theta - \phi_j)_c^{\nu_j} h(\theta), \tag{2.2}$$

where $h(\theta)$ is a positive (or negative) continuous function on $[-\pi, \pi]$ (without loss of generality, we assume that h is positive). Since f is indefinite and a 2π-periodic continuous function, here $(\theta - \phi)_c$ denotes the periodic continuous extension of $(\theta - \phi)$. It is clear that $(\theta - \phi)_c^{\nu} \in \mathbf{C}_{2\pi}$ for any positive integer ν. We remark that most papers in the literature [2, 4, 6, 8, 15, 16] considered only the case where the generating functions have zeros of even order.

The systems $A_n[f]\mathbf{x} = \mathbf{b}$ will be solved by the preconditioned conjugate-gradient-type method with circulant preconditioners. It is well known that all n-by-n circulant matrices can be diagonalized by the n-by-n Fourier matrix F_n, see [5]. Therefore, a circulant matrix is uniquely determined by its set of eigenvalues. For a given function f, we define the circulant preconditioner $C_n[f]$ to be the n-by-n circulant matrix with its j-th eigenvalue given by

$$\lambda_j(C_n[f]) = f\left(2\pi j/n\right), \quad 0 \le j < n. \tag{2.3}$$

We note that $C_n[f] = F_n^* \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) F_n$, see [5]. Hence the matrix-vector multiplication $C_n^{-1}[f]\mathbf{v}$, which is required in each iteration of the preconditioned conjugate gradient method, can be done in $\mathcal{O}(n \log_2 n)$ operations by fast Fourier transforms. Clearly if f is a positive function, then $C_n[f]$ is positive definite. We also construct circulant preconditioners for the MINRES method. The use of the preconditioned MINRES avoids the transformation of the original system to the normal equations but the method requires Hermitian positive definite preconditioners. In this case, we define our circulant preconditioner to be $C_n[|f|]$. When we are given a generating function f of Toeplitz matrices, since f may have a zero of order ν_j at $\theta = \theta_j = 2\pi j/n$, the circulant matrix $C_n[f]$ will be singular. Therefore, we let $\tilde{C}_n[f]$ be the n-by-n circulant matrix with the k-th eigenvalue given by

$$\lambda_k(\tilde{C}_n[f]) = \begin{cases} f\left(\dfrac{2\pi(k+1)}{n}\right) \text{ or } f\left(\dfrac{2\pi(k-1)}{n}\right), & \text{if } \theta_j = \dfrac{2\pi k}{n} \\ f\left(\dfrac{2\pi k}{n}\right), & \text{otherwise,} \end{cases} \tag{2.4}$$

for $k = 0, \ldots, n-1$. It is easy to see that $\tilde{C}_n[f] - C_n[f]$ is a matrix of rank at most s. We remark that positive definite circulant preconditioners $\tilde{C}[|f|]$ can be defined similarly. Recently, Chan, Potts and Steidl [6] have proposed these circulant preconditioners $C_n[|f|]$ for indefinite Toeplitz systems. However, they assume that f has zeros of even order.

3 Approximation by Banded Toeplitz Matrices

In this section, we derive some results on the matrix approximation properties of a banded Toeplitz matrix that will be required in the subsequent section.

LEMMA 3.1. *Let $f \in \mathbf{C}_{2\pi}$ and be given by* (2.2). *Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \leq q(\theta) \equiv f(\theta) \left(\prod_{j=1}^{s} \sin^{\nu_j}(\theta - \phi_j) \right)^{-1} \leq c_2, \quad \forall \theta \in [-\pi, \pi]. \tag{3.1}$$

Moreover, q is a continuously differentiable 2π-periodic function except at $\theta = \phi_j$ $(j = 1, \cdots, s)$.

PROOF. It is easy to check that $1 \leq (\theta - \phi)_c \,/\, \sin(\theta - \phi) \leq \pi/2$, for all $\theta \in [-\pi, \pi]$. Hence the result follows be setting $c_1 = 1$ and $c_2 = (\pi/2)^s$. $\square$

We note that $2\,\sin(\theta - \phi) = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\theta}$, for all $\theta \in [-\pi, \pi]$. Therefore, $\sin(\theta - \phi)$ is a 1-th degree trigonometric polynomial in θ, and

$$A_n[\sin(\cdot - \phi)] = \text{ tridiag } \left[\frac{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi}}{2}, 0, \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}}{2} \right].$$

It follows that $A_n[\sin(\cdot - \phi)]$ is a tridiagonal Toeplitz matrix.

LEMMA 3.2. *For all even $n > 0$, $A_n[\sin(\cdot - \phi)]$ are nonsingular.*

PROOF. The Toeplitz matrix $A_n[\sin(\cdot - \phi)]$ can be diagonalized by S_n, i.e., $A_n[\sin(\cdot - \phi)] = D_n S_n \Lambda_n S_n D_n^*$, where $D_n = \text{ diag }[1, \mathrm{e}^{\mathrm{i}\phi}, \cdots, \mathrm{e}^{\mathrm{i}(n-1)\phi}]$, S_n is the modified sine transform matrix defined by

$$[S_n]_{j,k} = \sqrt{\frac{2}{n+1}}\,\mathrm{i}^{j+k+1} \sin\left(\frac{jk\pi}{n+1}\right), \quad j, k = 1, 2, \ldots, n,$$

and

$$\Lambda_n = \text{ diag } \left[-\cos\left(\frac{\pi}{n+1}\right), -\cos\left(\frac{2\pi}{n+1}\right), \cdots, -\cos\left(\frac{n\pi}{n+1}\right) \right].$$

It follows that $A_n[\sin(\cdot - \phi)]$ are nonsingular for all even n. $\square$

LEMMA 3.3. (see [3]) *Let r be a k-th degree trigonometric polynomial and $g \in \mathbf{C}_{2\pi}$. Then for all $n > 2k$, we have $A_n[rg] = A_n[r]A_n[g] + L_n(2k)$. Here $L_n(k)$ denotes an n-by-n matrix of rank at most k.*

By using Lemma 3.3, we decompose $A_n[f]$ into the product of $A_n[\sin(\cdot - \phi_j)]$ and $A_n[q]$ into the following form:

$$A_n[f] = \prod_{j=1}^{s} A_n[\sin(\cdot - \phi_j)]^{\nu_j} A_n[q] + L_n(2\tau), \quad \text{with} \quad \tau = \sum_{j=1}^{s} \nu_j. \tag{3.2}$$

4 Spectral Properties of the Preconditioned Matrices

Before we analyze the distribution of the eigenvalues and singular values of the preconditioned matrix $\tilde{C}_n[|f|]^{-1}A_n[f]$, we consider the following two lemmas.

LEMMA 4.1. (see [2]) *Let r be a k-th degree trigonometric polynomial. Then for all $n > 2k$, we have $A_n[r] = C_n[r] + L_n(2k)$.*

LEMMA 4.2. (see [5]) *Let g be a continuously differentiable 2π-periodic function except at $\theta = \phi_j$ $(j = 1, \cdots, s)$. Then there exists a positive integer N such that for $n > N$, we have $A_n[g] = C_n[g] + L_n(N) + E_n(N)$, with $\|E_n(N)\|_2 \le \frac{c_3}{N}$, where c_3 is a positive constant (independent of n).*

With Lemmas 4.1 and 4.2, we have our main theorem in this section.

THEOREM 4.3. *Let $f \in \mathbf{C}_{2\pi}$ and be given by (2.2). Then there exist a positive integer N such that for $n > N$, we have*

$$(\tilde{C}_n[|f|]^{-1}A_n[f])^*(\tilde{C}_n[|f|]^{-1}A_n[f]) = I_n + L_n'(3N + 6\tau) + E_n'(N),$$

with $\|E_n'(N)\|_2 \le c_4/N$, where c_4 is a positive constant (independent of n). Here both $L_n'(3N + 6\tau)$ and $E_n'(N)$ are Hermitian matrices.

PROOF. By (3.2) and Lemma 4.1, we have

$$\tilde{C}_n[|f|]^{-1}A_n[f] = \tilde{C}_n[|f|]^{-1}\prod_{j=1}^{s}\tilde{C}_n[\sin(\cdot - \phi_j)]^{\nu_j}A_n[q] + \tilde{C}_n[|f|]^{-1}L_n(3\tau).$$

We note that q is a continuously differentiable 2π-periodic function except at $\theta = \phi_j (j = 1, \cdots, s)$. By applying Lemma 4.2, we obtain

$$\begin{aligned} \tilde{C}_n[|f|]^{-1}A_n[f] \quad &= \quad \tilde{C}_n[|f|]^{-1}\prod_{j=1}^{s}\tilde{C}_n[\sin(\cdot - \phi_j)]^{\nu_j}C_n[q] + \\ &\quad\;\; \tilde{C}_n[|f|]^{-1}\prod_{j=1}^{s}\tilde{C}_n[\sin(\cdot - \phi_{j_k})]^{\nu_j}E_n(N) + L_n(N + 3\tau). \end{aligned} \tag{4.1}$$

Since $f = q\prod_{j=1}^{s}\sin^{\nu_j}(\cdot - \phi_j)$ (see Lemma 2.1), we note that

$$\tilde{C}_n[|f|]^{-1}\prod_{j=1}^{s}\tilde{C}_n[\sin(\cdot - \phi_j)]^{\nu_j}C_n[q] = C_n[p]$$

and $\tilde{C}_n[|f|]^{-1}\prod_{j=1}^{s}\tilde{C}_n[\sin(\cdot - \phi_{j_k})]^{\nu_j} = C_n[p/q]$, where

$$p(\theta) := \begin{cases} 1, & \text{for } f(\theta) \ge 0, \\ -1, & \text{for } f(\theta) < 0. \end{cases} \tag{4.2}$$

From (4.1), we get

$$\tilde{C}_n[|f|]^{-1}A_n[f] = C_n[p] + C_n[p/q]E_n(N) + L_n(N + 3\tau), \tag{4.3}$$

and hence $(\tilde{C}_n[|f|]^{-1}A_n[f])^*(\tilde{C}_n[|f|]^{-1}A_n[f]) = I_n + L'_n(3N+6\tau) + E'_n(N)$, where

$$\begin{aligned} E'_n(N) &= C_n[p]^*C_n[p/q]E_n(N) + E_n(N)^*C_n[p/q]^*C_n[p] + \\ &\quad E_n(N)^*C_n[p/q]^*C_n[p/q]E_n(N). \end{aligned}$$

Since p and q are uniformly bounded from below and above by constants independent of n, the result follows. □

We remark that the normal equations matrix $(\tilde{C}_n[|f|]^{-1}A_n[f])^*(\tilde{C}_n[|f|]^{-1}A_n[f])$ is similar to $(\tilde{C}_n[f]^{-1}A_n[f])^*(\tilde{C}_n[f]^{-1}A_n[f])$ and therefore the results for Theorem 4.3 holds for the circulant preconditioner $\tilde{C}_n[f]$ (see [7]).

According to Theorem 4.3, the singular values of the preconditioned matrices are uniformly bounded from below and above by constants independent of n except for $3N+6\tau$ outliers singular values. We note that the conjugate gradient method for normal equations depends on the singular values of the preconditioned matrix. As for the convergence rate of the CGNE method for the preconditioned system, we can show that the method will converge in at most $\mathcal{O}(\alpha \log_2 n)$ steps if the the smallest singular value of $A_n[f]$ is of order $\mathcal{O}(n^\alpha)$, see [2]. We know that $\alpha = \max_j \nu_j$, see [18]. Since each iteration requires $\mathcal{O}(n \log_2 n)$ operations (see [5]) the total complexity of the method for solving Toeplitz systems generated by f in the form (2.2) is $\mathcal{O}(n(\log_2 n)^2)$ operations. In contrast, for the original matrix $A_n[f]$, since its condition number is growing like $\mathcal{O}(n^\alpha)$, the complexity of solving the corresponding system will be of order $\mathcal{O}(n^{\alpha/2+1}\log_2 n)$. Following the lines of the proof of Theorem 4.3 and (4.3), we get the following results.

THEOREM 4.4. *Let $f \in \mathbf{C}_{2\pi}$ and be given by* (2.2). *Then there exist a positive integer N such that for $n > N$, we have*

$$\tilde{C}_n[|f|]^{-1}A_n[f] = C_n[p] + L''_n(N+3\tau) + E''_n(N), \quad \textit{with} \quad \|E''_n(N)\|_2 \le \frac{c_5}{N},$$

where c_5 is a positive constant (independent of n). Here p is defined as in (4.2).

THEOREM 4.5. *Let $f \in \mathbf{C}_{2\pi}$ and be given by* (2.2). *Then there exist a positive integer N such that for $n > N$, we have*

$$\tilde{C}_n[f]^{-1}A_n[f] = I_n + L'''_n(N+3\tau) + E'''_n(N), \quad \textit{with} \quad \|E'''_n(N)\|_2 \le \frac{c_6}{N},$$

where c_6 is a positive constant (*independent of n*).

According to Theorem 4.4, the eigenvalues of the preconditioned matrices $\tilde{C}_n[|f|]^{-1}A_n[f]$ are clustered around -1 and 1 (see Figures 5.1 and 5.2). In Theorem 4.5, the eigenvalues of the preconditioned matrices $\tilde{C}_n[f]^{-1}A_n[f]$ are clustered around 1 (see Figure 5.3). Therefore we expect fast convergence of the MINRES and QMR methods. In the next section, we illustrate their performance by some numerical examples.

5 Numerical Examples

In this section, we illustrate by numerical examples the effectiveness of the circulant preconditioners $\tilde{C}_n[f]$ for the CGNE and QMR methods and $\tilde{C}_n[|f|]$

for the MINRES method in solving Toeplitz systems. For comparisons, we also test the Strang and the T. Chan circulant preconditioners [5]. We solve Toeplitz systems $A_n[f]\mathbf{x} = \mathbf{b}$ by the preconditioned MINRES, QMR and CGNE methods. All the experiments are performed in MATLAB with machine precision 10^{-16}. A variant of QMR method, the symmetric quasi–minimal residual method [12], is employed to solve the indefinite system with arbitrary Hermitian, nonsingular preconditioner. We use the MATLAB-provided M-files "qmr" in our implementation. In the tests, the right-hand side vectors $\mathbf{b}$ are formed by multiplying random vectors to $A_n[f]$. The initial guess is the zero vector and the stopping criterion is $||\mathbf{r}_q||_2/||\mathbf{r}_0||_2 \le 10^{-6}$ where $\mathbf{r}_q$ is the residual vector after q iterations. Tables 5.1–5.4 show the numbers of iterations required for convergence for different choices of preconditioners. In the table, I denotes no preconditioner, S is the Strang preconditioner, T is the T. Chan preconditioner and C is our preconditioner. Iteration numbers more than 1,000 are denoted by "†". We note that S and T are not positive definite as the test functions are indefinite. In the test, when some of the eigenvalues of S and T are negative, we use their absolute values in the computation for MINRES.

The two test functions θ_c and θ_c^3 in Tables 5.1 and 5.2 respectively are indefinite functions with a single zero of orders 1 and 3 on $[-\pi, \pi]$. Thus their condition numbers of the Toeplitz matrices are growing like $\mathcal{O}(n^\gamma)$ [19], and hence the numbers of iterations required for convergence without using any preconditioners is increasing like $\mathcal{O}(n^{\gamma/2})$. We see that for $f(\theta) = \theta_c$, the number of iterations for convergence using the Strang, the T. Chan and our preconditioners are roughly constant independent of n. For $f(\theta) = \theta_c^3$, the T. Chan preconditioner fails since the number of iterations required for convergence is increasing with n. Both Strang's and our preconditioners still work very well. In Tables 5.3 and 5.4, we test the functions $(\theta - \sqrt{2})_c(\theta + \sqrt{2})_c$ and $(\theta - 1/2)_c(\theta + 1/2)_c(\theta - 2)_c(\theta + 2)_c(\theta - 3)_c(\theta + 3)_c$. They have multiple zeros of order 1. The number of iterations required for convergence for S and C are roughly constant independent of n.

To further illustrate Theorem 4.3, we give in Figures 5.1 and 5.2 the singular values of original matrices $A_n[f]$, the absolute values of the eigenvalues and the singular values of the preconditioned matrices $\tilde{C}_n^{-1}[|f|]A_n[f]$ for $f(\theta) = \theta_c$ and θ_c^3 when $n = 64, 128, 256, 512$. We see that the absolute values of the eigenvalues of the preconditioned matrices are in a small interval around 1 except for a few outliers. For both functions $f(\theta) = \theta_c$ and $f(\theta) = \theta_c^3$, we find that their smallest singular values of the preconditioned matrices $\tilde{C}_n^{-1}[|f|]A_n[f]$ decrease as n increases. This explains why the number of iterations required for convergence by using our preconditioners grow like $\log_2 n$. In Figure 5.4, we also give the spectra of the preconditioned matrices $\tilde{C}_n[f]^{-1}A_n[f]$ for θ_c and θ_c^3 when $n = 512$. We see that their spectra are also clustered around 1 except for a few outliers.

In summary, it has been shown in [15] that the circulant preconditioner $C_n[|f|]$ are effective for ill-conditioned Hermitian Toeplitz matrices $A_n[f]$ when f is a 2π-periodic continuous functions with zeros of even order. In this paper, we have further shown that this circulant preconditioner can be extended to Toeplitz matrices generated by functions with zeros of odd order.

	CGNE				MINRES				QMR			
n	I	S	T	C	I	S	T	C	I	S	T	C
32	17	7	8	7	32	14	16	16	36	9	9	9
64	36	7	8	8	72	16	16	16	97	9	9	9
128	80	9	9	9	158	16	16	16	185	9	9	9
256	173	9	9	9	344	16	16	16	385	9	9	9
512	366	9	9	9	730	16	16	16	775	9	9	9
1024	760	9	9	9	†	16	16	16	†	9	9	9

Table 5.1: Numbers of iterations for θ_c.

	CGNE				MINRES				QMR			
n	I	S	T	C	I	S	T	C	I	S	T	C
32	29	15	19	15	60	22	34	24	64	15	18	15
64	119	19	28	18	244	24	48	26	443	15	30	16
128	641	20	42	20	†	28	64	28	†	18	30	18
256	†	30	88	34	†	38	94	38	†	20	44	22
512	†	17	62	18	†	38	128	40	†	20	66	20
1024	†	10	119	10	†	40	†	40	†	23	92	23

Table 5.2: Numbers of iterations for θ_c^3.

	CGNE				MINRES				QMR			
n	I	S	T	C	I	S	T	C	I	S	T	C
32	24	9	16	9	16	7	11	7	16	6	11	6
64	68	10	22	10	34	10	15	9	35	6	15	6
128	222	12	27	12	75	12	19	11	77	7	18	6
256	701	12	35	12	160	13	25	11	164	6	19	6
512	†	12	42	13	336	11	32	11	340	6	19	6
1024	†	14	44	14	693	13	33	12	698	6	21	6

Table 5.3: Numbers of iterations for $(\theta-\sqrt{2})_c(\theta+\sqrt{2})_c$.

	CGNE				MINRES				QMR			
n	I	S	T	C	I	S	T	C	I	S	T	C
32	30	19	19	19	21	18	17	18	21	10	17	12
64	91	31	50	30	57	25	35	25	58	11	19	11
128	264	31	42	30	137	23	35	24	140	11	18	11
256	885	35	46	32	389	26	36	26	413	11	17	11
512	†	49	63	47	†	26	37	26	†	12	19	12
1024	†	58	93	55	†	27	41	27	†	12	20	12

Table 5.4: Numbers of iterations for $(\theta-1/2)_c(\theta+1/2)_c(\theta-2)_c(\theta+2)_c(\theta-3)_c(\theta+3)_c$.

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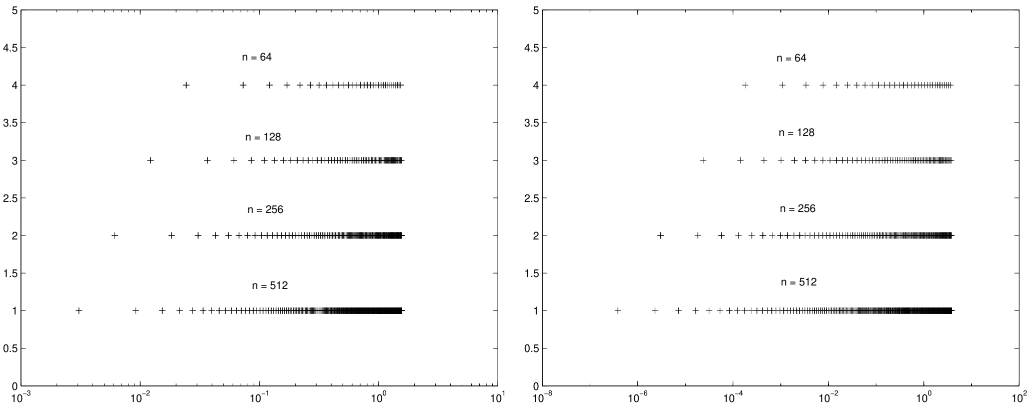
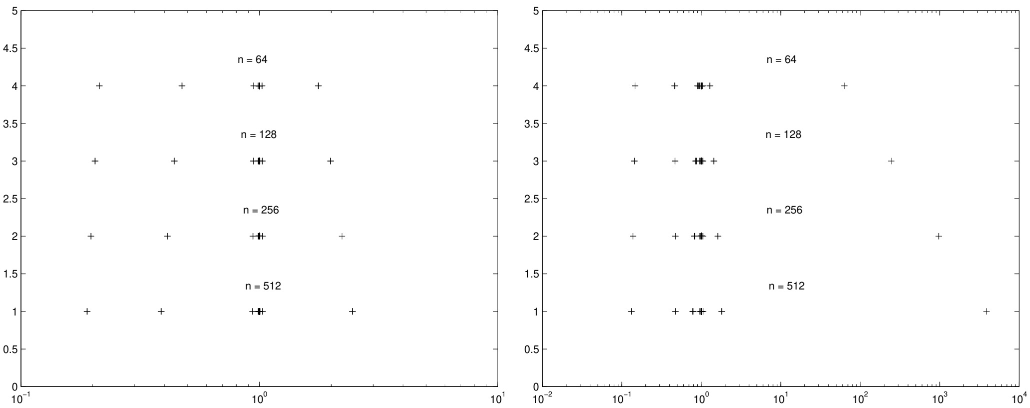
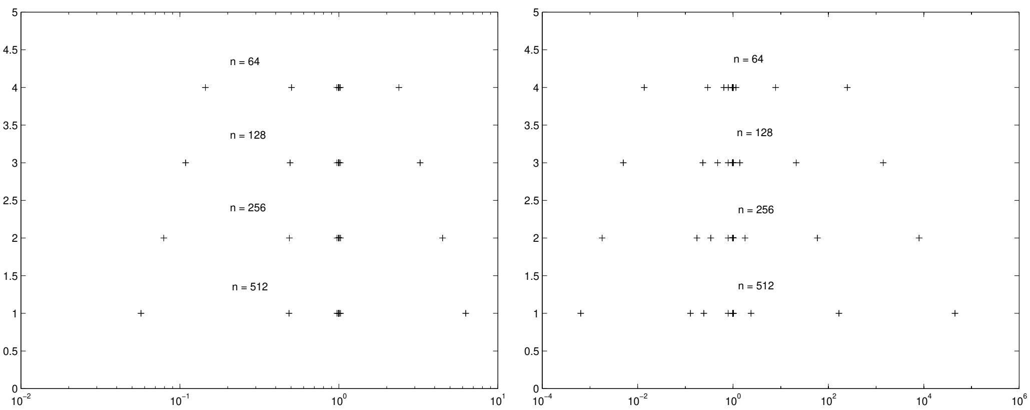


Singular values of original matrices $A_n[f]$.



Absolute values of the eigenvalues $\tilde{C}_n[|f|]^{-1}A_n[f]$.



Singular values of $\tilde{C}_n[|f|]^{-1}A_n[f]$.

Figure 5.1: Generating Functions: θ_c (left) and θ_c^3 (right).